信息与计算科学丛书——84

非线性发展方程的有限差分方法

孙志忠　著

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内 容 简 介

本书针对应用科学中的 11 个重要的非线性发展方程，介绍差分求解方法的最新研究成果，包括微分方程问题解的守恒性和有界性分析、差分方法的建立、差分解的守恒性和有界性分析、差分解的存在性分析、差分解收敛性的证明、差分格式的求解等内容. 建立的差分求解格式包括非线性差分格式和线性化差分格式. 这 11 个非线性发展方程如下：Burgers 方程、正则长波方程、Korteweg-de Vries 方程、Camassa-Holm 方程、Schrödinger 方程、Kuramoto-Tsuzuki 方程、Zakharov 方程、Ginzburg-Landau 方程、Cahn-Hilliard 方程、外延增长模型方程和相场晶体模型方程.

本书可供计算数学、应用数学专业从事偏微分方程数值解法研究的研究生阅读，也可供相关学科研究人员参考.

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《信息与计算科学丛书》编委会

《信息与计算科学丛书》序

20 世纪 70 年代末，由已故著名数学家冯康先生任主编、科学出版社出版了一套《计算方法丛书》，至今已逾 30 册. 这套丛书以介绍计算数学的前沿方向和科研成果为主旨，学术水平高、社会影响大，对计算数学的发展、学术交流及人才培养起到了重要的作用.

1998 年教育部进行学科调整，将计算数学及其应用软件、信息科学、运筹控制等专业合并，定名为“信息与计算科学专业”. 为适应新形势下学科发展的需要，科学出版社将《计算方法丛书》更名为《信息与计算科学丛书》，组建了新的编委会，并于 2004 年 9 月在北京召开了第一次会议，讨论并确定了丛书的宗旨、定位及方向等问题.

新的《信息与计算科学丛书》的宗旨是面向高等学校信息与计算科学专业的高年级学生、研究生以及从事这一行业的科技工作者，针对当前的学科前沿、介绍国内外优秀的科研成果. 强调科学性、系统性及学科交叉性，体现新的研究方向. 内容力求深入浅出，简明扼要.

原《计算方法丛书》的编委和编辑人员以及多位数学家曾为丛书的出版做了大量工作，在学术界赢得了很好的声誉，在此表示衷心的感谢. 我们诚挚地希望大家一如既往地关心和支持新丛书的出版，以期为信息与计算科学在新世纪的发展起到积极的推动作用.

石钟慈

2005 年 7 月

前　言

非线性现象的研究是自然科学领域甚至社会科学领域也十分关心的问题. 由于自然界中许多的现象本质上是非线性的, 所以非线性现象引起了工程师、物理学家、数学家和许多其他领域的科学家的兴趣、关注. 在数学和物理科学里, 非线性现象是指输出的变化量不正比于输入的变化量. 很大一部分非线性现象可以用非线性偏微分方程来描述. 非线性偏微分方程的两个典型例子是流体力学中的 Navier-Stokes 方程、量子力学中的 Schrödinger 方程. 维基百科上列出的非线性偏微分方程有 118 个之多.

非线性问题最大困难之一是一般不能由已知的特解去构造新解. 例如, 线性问题, 由一族线性无关的解可以通过叠加原理构造通解. 一个非常好的例子是带有 Dirichlet 边界条件的热传导方程的解可以表示成不同频率的正弦函数的依赖于时间系数的线性组合. 叠加原理使得求解线性问题的解变得容易. 对于非线性问题找几个特解常常还是可能的, 但是试图从这几个特解出发寻找通解有很大的难度.

在科学的计算机化进程中, 科学与工程计算作为一门工具性、方法性、边缘交叉性的新学科开始了自己的新发展. 微分方程数值解法也得到了前所未有的发展.

本书选择 11 个非线性偏微分方程定解问题研究其差分解法. 这 11 个方程依次是 Burgers 方程、正则长波方程、Korteweg-de Vries 方程、Camassa-Holm 方程、Schrödinger 方程、Kuramoto-Tsuzuki 方程、Zakharov 方程、Ginzburg-Landau 方程、Cahn-Hilliard 方程、外延增长模型方程和相场晶体模型方程. 对每一个方程的定解问题, 建立了几个有效的差分格式, 对每一个差分格式证明了差分格式解的存在唯一性、守恒性和有界性、收敛性.

本书的出版得到了国家自然科学基金项目 (项目编号: 11671081, 11271068, 10871044, 10471023, 19801007) 的资助.

本书介绍的大部分内容是作者和合作者的研究成果. 在此, 对合作者表示诚挚的谢意! 由于作者的学识有限, 诚望各位专家及广大读者提供宝贵意见和建议.

Email 地址: zzsun@seu.edu.cn.

孙志忠

2017 年 8 月

目　录

第 1 章 Burgers 方程的差分方法

1.1 引　　言

Burgers 方程是描述许多物理现象的模型方程, 如流体力学、非线性声学、气体动力学、交通流动力学问题. Burgers 方程也可以作为流体动力学 Navier-Stokes 方程的简化模型. 近年来, 求解 Burgers 方程的数值方法受到科研人员的广泛关注.

考虑一维非线性 Burgers 方程初边值问题

$$u_t + uu_x = \nu u_{xx}, \quad 0 < x < L,\ 0 < t \leqslant T, \tag{1.1}$$

$$u(x,0) = \varphi(x), \quad 0 < x < L, \tag{1.2}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{1.3}$$

其中 ν 为动力黏性系数, $\varphi(x)$ 为给定函数, $\varphi(0) = \varphi(L) = 0$.

在介绍差分格式之前, 我们先用能量方法给出问题 (1.1)–(1.3) 解的先验估计式.

定理 1.1 设 $u(x,t)$ 为问题 (1.1)–(1.3) 的解. 记

$$E(t) = \int_0^L u^2(x,t)\mathrm{d}x + 2\nu \int_0^t \left[\int_0^L u_x^2(x,s)\mathrm{d}x\right] \mathrm{d}s,$$

则有

$$E(t) = E(0), \quad 0 < t \leqslant T. \tag{1.4}$$

证明 用 u 乘以 (1.1) 的两边, 可得

$$\left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x = \nu[(uu_x)_x - u_x^2].$$

将上式两边关于 x 在区间 $[0,L]$ 上积分, 并利用 (1.3), 得到

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2(x,t)\mathrm{d}t + \nu\int_0^L u_x^2(x,t)\mathrm{d}x = 0.$$

可将上式写为

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^L u^2(x,t)\mathrm{d}x + 2\nu\int_0^t \left[\int_0^L u_x^2(x,s)\mathrm{d}x\right]\mathrm{d}s\right\} = 0,$$

即

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=0,\quad 0<t\leqslant T.$$

因而

$$E(t)=E(0),\quad 0<t\leqslant T. \qquad \square$$

1.2　二层非线性差分格式

1.2.1　记号及引理

为了用差分格式求解问题 (1.1)–(1.3), 将求解区域 $[0,L]\times[0,T]$ 作剖分. 取正整数 m,n. 将 $[0,L]$ 作 m 等分, 将 $[0,T]$ 作 n 等分. 记 $h=L/m,\tau=T/n;x_i=ih,0\leqslant i\leqslant m;t_k=k\tau,0\leqslant k\leqslant n;\Omega_h=\{x_i\,|\,0\leqslant i\leqslant m\},\Omega_\tau=\{t_k\,|\,0\leqslant k\leqslant n\};\Omega_{h\tau}=\Omega_h\times\Omega_\tau$. 称在直线 $t=t_k$ 上的所有结点 $\{(x_i,t_k)\,|\,0\leqslant i\leqslant m\}$ 为第 k 层结点. 此外, 记 $x_{i+\frac12}=\dfrac12(x_i+x_{i+1}),t_{k+\frac12}=\dfrac12(t_k+t_{k+1})$.

记

$$\mathcal{U}_h=\{u\,|\,u=(u_0,u_1,\cdots,u_m)\text{为}\Omega_h\text{上的网格函数}\},$$
$$\mathring{\mathcal{U}}_h=\{u\,|\,u\in\mathcal{U}_h,u_0=u_m=0\}.$$

设 $u\in\mathcal{U}_h$, 引进如下记号:

$$\delta_x u_{i+\frac12}=\frac1h(u_{i+1}-u_i),\quad \delta_x^2u_i=\frac1{h^2}(u_{i-1}-2u_i+u_{i+1}),\quad \Delta_xu_i=\frac1{2h}(u_{i+1}-u_{i-1}).$$

易知

$$\delta_x^2u_i=\frac1h(\delta_xu_{i+\frac12}-\delta_xu_{i-\frac12}),\quad \Delta_xu_i=\frac12(\delta_xu_{i-\frac12}+\delta_xu_{i+\frac12}).$$

设 $u,v\in\mathcal{U}_h$, 引进内积、范数及半范数

$$(u,v)=h\left(\frac12u_0v_0+\sum_{i=1}^{m-1}u_iv_i+\frac12u_mv_m\right),$$

$$\|u\|_\infty=\max_{0\leqslant i\leqslant m}|u_i|,\quad \|u\|=\sqrt{h\left(\frac12u_0^2+\sum_{i=1}^{m-1}u_i^2+\frac12u_m^2\right)},$$

$$|u|_1=\sqrt{h\sum_{i=1}^{m}(\delta_xu_{i-\frac12})^2},\quad \|u\|_1=\sqrt{\|v\|^2+|u|_1^2},$$

$$|u|_2=\sqrt{h\sum_{i=1}^{m-1}(\delta_x^2u_i)^2},\quad \|u\|_2=\sqrt{\|u\|^2+|u|_1^2+|u|_2^2}.$$

如果 $\mathcal{U}_h$ 为复空间, 则相应的内积定义为

$$(u,v)=h\left(\frac{1}{2}u_0\bar{v}_0+\sum_{i=1}^{m-1}u_i\bar{v}_i+\frac{1}{2}u_m\bar{v}_m\right),$$

其中 $\bar{v}_i$ 为 v_i 的共轭.

记

$$\mathcal{S}_\tau=\{w\,|\,w=(w_0,w_1,\cdots,w_n)\text{为 }\Omega_\tau\text{ 上的网格函数}\}.$$

设 $w\in\mathcal{S}_\tau$, 引进如下记号:

$$w^{k+\frac{1}{2}}=\frac{1}{2}(w^k+w^{k+1}),\qquad w^{\bar{k}}=\frac{1}{2}(w^{k+1}+w^{k-1}),$$
$$\delta_t w^{k+\frac{1}{2}}=\frac{1}{\tau}(w^{k+1}-w^k),\quad \Delta_t w^k=\frac{1}{2\tau}(w^{k+1}-w^{k-1}).$$

易知

$$\Delta_t w^k=\frac{1}{2}(\delta_t w^{k-\frac{1}{2}}+\delta_t w^{k+\frac{1}{2}}).$$

设 $u=\{u_i^k\,|\,0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$ 为 $\Omega_{h\tau}$ 上的网格函数, 则 $v=\{u_i^k\,|\,0\leqslant i\leqslant m\}$ 为 Ω_h 上的网格函数, $w=\{u_i^k\,|\,0\leqslant k\leqslant n\}$ 为 Ω_τ 上的网格函数.

引理 1.1 ([4, 25]) (a) 设 $u,v\in\mathcal{U}_h$, 则有

$$-h\sum_{i=1}^{m-1}(\delta_x^2u_i)v_i=h\sum_{i=1}^{m}(\delta_xu_{i-\frac{1}{2}})(\delta_xv_{i-\frac{1}{2}})+(\delta_xu_{\frac{1}{2}})v_0-(\delta_xu_{m-\frac{1}{2}})v_m.$$

(b) 设 $v\in\mathring{\mathcal{U}}_h$, 则有

$$-h\sum_{i=1}^{m-1}(\delta_x^2u_i)u_i=|u|_1^2,$$
$$|u|_1^2\leqslant\|u\|\cdot|u|_2,$$
$$\|u\|_\infty\leqslant\frac{\sqrt{L}}{2}|u|_1,$$
$$\|u\|\leqslant\frac{L}{\sqrt{6}}|u|_1.$$

(c) 设 $u\in\mathring{\mathcal{U}}_h$, 则有

$$\|u\|_\infty^2\leqslant\|u\|\cdot|u|_1,$$

且对任意 $\varepsilon>0$, 有

$$\|u\|_\infty^2\leqslant\varepsilon|u|_1^2+\frac{1}{4\varepsilon}\|u\|^2.$$

(d) 设 $u \in \mathcal{U}_h$, 则

$$|u|_1^2 \leqslant \frac{4}{h^2}\|u\|^2.$$

(e) 设 $u \in \mathcal{U}_h$, 则有

$$\|u\|_\infty^2 \leqslant 2\|u\| \cdot |u|_1 + \frac{1}{L}\|u\|^2.$$

且对任意 $\varepsilon > 0$, 有

$$\|u\|_\infty^2 \leqslant \varepsilon |u|_1^2 + \left(\frac{1}{\varepsilon} + \frac{1}{L}\right)\|u\|^2.$$

证明　我们仅证明(c)和(e).

(c) 由 $u_0 = 0$, 当 $1 \leqslant i \leqslant m-1$ 时, 有

$$u_i^2 = \sum_{l=1}^{i}(u_l^2 - u_{l-1}^2) = \sum_{l=1}^{i}(u_l + u_{l-1})(u_l - u_{l-1}) = 2h\sum_{l=1}^{i} u_{l-\frac{1}{2}}\delta_x u_{l-\frac{1}{2}}.$$

因而

$$u_i^2 \leqslant 2h\sum_{l=1}^{i}\left|u_{l-\frac{1}{2}}\right| \cdot \left|\delta_x u_{l-\frac{1}{2}}\right|.$$

类似地, 注意到 $u_m = 0$, 可得

$$u_i^2 \leqslant 2h\sum_{l=i+1}^{m}\left|u_{l-\frac{1}{2}}\right| \cdot \left|\delta_x u_{l-\frac{1}{2}}\right|.$$

将以上两式相加得到

$$u_i^2 \leqslant h\sum_{l=1}^{m}\left|u_{l-\frac{1}{2}}\right| \cdot \left|\delta_x u_{l-\frac{1}{2}}\right| \leqslant \sqrt{h\sum_{l=1}^{m}\left|u_{l-\frac{1}{2}}\right|^2} \cdot \sqrt{h\sum_{l=1}^{m}\left|\delta_x u_{l-\frac{1}{2}}\right|^2} \leqslant \|u\| \cdot |u|_1.$$

容易得到

$$\|u\|_\infty^2 \leqslant \|u\| \cdot |u|_1.$$

对任意的 $\varepsilon > 0$ 有

$$\|u\|_\infty^2 \leqslant \varepsilon |u|_1^2 + \frac{1}{4\varepsilon}\|u\|^2.$$

(e) 当 $i > j$ 时,

$$\begin{aligned}
u_i^2 &= u_j^2 + \sum_{l=j+1}^{i}(u_l^2 - u_{l-1}^2) \\
&= u_j^2 + 2h\sum_{l=j+1}^{i} u_{l-\frac{1}{2}}\delta_x u_{l-\frac{1}{2}} \\
&\leqslant u_j^2 + 2h\sum_{l=j+1}^{i} \left|u_{l-\frac{1}{2}}\right| \cdot \left|\delta_x u_{l-\frac{1}{2}}\right| \\
&\leqslant u_j^2 + 2h\sum_{l=1}^{m} \left|u_{l-\frac{1}{2}}\right| \cdot \left|\delta_x u_{l-\frac{1}{2}}\right| \\
&\leqslant u_j^2 + 2\|u\| \cdot |u|_1.
\end{aligned} \tag{1.5}$$

易知上式对 $i \leqslant j$ 也是成立的.

记

$$w_j = \begin{cases} 1, & 1 \leqslant j \leqslant m-1, \\ \dfrac{1}{2}, & j = 0, m. \end{cases}$$

将 (1.5) 乘以 hw_j 并对 j 从 0 到 m 求和得到

$$h\sum_{j=0}^{m} w_j u_i^2 \leqslant h\sum_{j=0}^{m} w_j u_j^2 + 2h\sum_{j=0}^{m} w_j \|u\| \cdot |u|_1.$$

由上式易得

$$L\|u\|_\infty^2 \leqslant \|u\|^2 + 2L\|u\| \cdot |u|_1,$$

即

$$\|u\|_\infty^2 \leqslant 2\|u\| \cdot |u|_1 + \frac{1}{L}\|u\|^2.$$

对任意的 $\varepsilon > 0$, 有

$$\|u\|_\infty^2 \leqslant \varepsilon|u|_1^2 + \left(\frac{1}{\varepsilon} + \frac{1}{L}\right)\|u\|^2.$$ □

下面我们给出几个常用的数值微分公式.

引理 1.2 ([4]) 设 c, h 为给定的常数, 且 $h > 0$.

(a) 如果 $g(x) \in C^2[c-h, c+h]$, 则有

$$g(c) = \frac{1}{2}[g(c-h) + g(c+h)] - \frac{h^2}{2}g''(\xi_0), \quad c-h < \xi_0 < c+h;$$

(b) 如果 $g(x) \in C^2[c, c+h]$, 则有

$$g'(c) = \frac{1}{h}[g(c+h) - g(c)] - \frac{h}{2}g''(\xi_1), \quad c < \xi_1 < c+h;$$

(c) 如果 $g(x) \in C^2[c-h,c]$, 则有

$$g'(c) = \frac{1}{h}[g(c) - g(c-h)] + \frac{h}{2}g''(\xi_2), \quad c-h < \xi_2 < c;$$

(d) 如果 $g(x) \in C^3[c-h,c+h]$, 则有

$$g'(c) = \frac{1}{2h}[g(c+h) - g(c-h)] - \frac{h^2}{6}g'''(\xi_3), \quad c-h < \xi_3 < c+h;$$

(e) 如果 $g(x) \in C^4[c-h,c+h]$, 则有

$$g''(c) = \frac{1}{h^2}[g(c+h) - 2g(c) + g(c-h)] - \frac{h^2}{12}g^{(4)}(\xi_4), \quad c-h < \xi_4 < c+h;$$

(f) 如果 $g(x) \in C^3[c,c+h]$, 则有

$$g''(c) = \frac{2}{h}\left[\frac{g(c+h) - g(c)}{h} - g'(c)\right] - \frac{h}{3}g'''(\xi_5), \quad c < \xi_5 < c+h;$$

如果 $g(x) \in C^4[c,c+h]$, 则有

$$g''(c) = \frac{2}{h}\left[\frac{g(c+h) - g(c)}{h} - g'(c)\right] - \frac{h}{3}g'''(c) - \frac{h^2}{12}g^{(4)}(\xi_6),$$
$$c < \xi_6 < c+h;$$

(g) 如果 $g(x) \in C^3[c-h,c]$, 则有

$$g''(c) = \frac{2}{h}\left[g'(c) - \frac{g(c) - g(c-h)}{h}\right] + \frac{h}{3}g'''(\xi_7), \quad c-h < \xi_7 < c;$$

如果 $g(x) \in C^4[c-h,c]$, 则有

$$g''(c) = \frac{2}{h}\left[g'(c) - \frac{g(c) - g(c-h)}{h}\right] + \frac{h}{3}g'''(c) - \frac{h^2}{12}g^{(4)}(\xi_8),$$
$$c-h < \xi_8 < c;$$

(h) 如果 $g(x) \in C^6[c-h,c+h]$, 则有

$$\begin{aligned}&\frac{1}{12}[g''(c-h) + 10g''(c) + g''(c+h)]\\ =&\frac{1}{h^2}[g(c+h) - 2g(c) + g(c-h)] + \frac{h^4}{240}g^{(6)}(\xi_9), \quad c-h < \xi_9 < c+h.\end{aligned}$$

1.2.2 差分格式的建立

定义 $\Omega_{h\tau}$ 上的网格函数 $U=\{U_i^k\,|\,0\leqslant i\leqslant m, 0\leqslant k\leqslant n\}$, 其中

$$U_i^k=u(x_i,t_k),\quad 0\leqslant i\leqslant m,\quad 0\leqslant k\leqslant n.$$

在 $(x_i,t_{k+\frac12})$ 处考虑方程 (1.1), 有

$$u_t(x_i,t_{k+\frac12})+u(x_i,t_{k+\frac12})u_x(x_i,t_{k+\frac12})=\nu u_{xx}(x_i,t_{k+\frac12}),\\ 1\leqslant i\leqslant m-1,\quad 0\leqslant k\leqslant n-1. \tag{1.6}$$

应用引理 1.2, 有

$$u_t(x_i,t_{k+\frac12})=\delta_t U_i^{k+\frac12}+O(\tau^2); \tag{1.7}$$

$$\begin{aligned}u(x_i,t_{k+\frac12})&=\frac13[u(x_{i-1},t_{k+\frac12})+u(x_i,t_{k+\frac12})+u(x_{i+1},t_{k+\frac12})]+O(h^2)\\&=\frac13(U_{i-1}^{k+\frac12}+U_i^{k+\frac12}+U_{i+1}^{k+\frac12})+O(\tau^2+h^2),\end{aligned} \tag{1.8}$$

$$\begin{aligned}u_x(x_i,t_{k+\frac12})&=\frac12[u_x(x_i,t_k)+u_x(x_i,t_{k+1})]+O(\tau^2)\\&=\frac12(\Delta_x U_i^k+\Delta_x U_i^{k+1})+O(\tau^2+h^2)\\&=\Delta_x U_i^{k+\frac12}+O(\tau^2+h^2),\end{aligned} \tag{1.9}$$

$$\begin{aligned}u_{xx}(x_i,t_{k+\frac12})&=\frac12[u_{xx}(x_i,t_k)+u_{xx}(x_i,t_{k+1})]+O(\tau^2)\\&=\frac12(\delta_x^2 U_i^k+\delta_x^2 U_i^{k+1})+O(\tau^2+h^2)\\&=\delta_x^2 U_i^{k+\frac12}+O(\tau^2+h^2).\end{aligned} \tag{1.10}$$

将 (1.7)–(1.10) 代入 (1.6), 得到

$$\delta_t U_i^{k+\frac12}+\frac13(U_{i-1}^{k+\frac12}+U_i^{k+\frac12}+U_{i+1}^{k+\frac12})\Delta_x U_i^{k+\frac12}=\nu\delta_x^2 U_i^{k+\frac12}+R_i^{k+\frac12},\\ 1\leqslant i\leqslant m-1,\quad 0\leqslant k\leqslant n-1. \tag{1.11}$$

存在常数 c_1 使得

$$|R_i^{k+\frac12}|\leqslant c_1(\tau^2+h^2),\quad 1\leqslant i\leqslant m-1,\quad 0\leqslant k\leqslant n-1. \tag{1.12}$$

注意到初边值条件 (1.2)–(1.3), 有

$$U_i^0=\varphi(x_i),\quad 1\leqslant i\leqslant m-1, \tag{1.13}$$

$$U_0^k=0,\quad U_m^k=0,\quad 0\leqslant k\leqslant n. \tag{1.14}$$

在 (1.11) 中略去小量项 $R_i^{k+\frac{1}{2}}$, 用 u_i^k 代替 U_i^k, 对问题 (1.1)–(1.3) 建立如下差分格式

$$\delta_t u_i^{k+\frac{1}{2}} + \frac{1}{3}(u_{i-1}^{k+\frac{1}{2}} + u_i^{k+\frac{1}{2}} + u_{i+1}^{k+\frac{1}{2}})\Delta_x u_i^{k+\frac{1}{2}} = \nu\delta_x^2 u_i^{k+\frac{1}{2}},$$
$$1 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{1.15}$$

$$u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{1.16}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{1.17}$$

差分格式 (1.15)–(1.17) 是一个二层非线性差分格式.

1.2.3　差分格式解的守恒性和有界性

差分格式 (1.15) 中的非线性项可作如下变形

$$\begin{aligned}
&\frac{1}{3}(u_{i-1}^{k+\frac{1}{2}} + u_i^{k+\frac{1}{2}} + u_{i+1}^{k+\frac{1}{2}})\Delta_x u_i^{k+\frac{1}{2}} \\
=&\frac{1}{3}[u_i^{k+\frac{1}{2}}\Delta_x u_i^{k+\frac{1}{2}} + (u_{i+1}^{k+\frac{1}{2}} + u_{i-1}^{k+\frac{1}{2}})\Delta_x u_i^{k+\frac{1}{2}}] \\
=&\frac{1}{3}\big[u_i^{k+\frac{1}{2}}\Delta_x u_i^{k+\frac{1}{2}} + \Delta_x(u_i^{k+\frac{1}{2}} u_i^{k+\frac{1}{2}})\big].
\end{aligned}$$

设 $v, w \in \mathcal{U}_h$, 定义

$$\psi(v,w)_i = \frac{1}{3}[v_i\Delta_x w_i + \Delta_x(vw)_i], \quad 1 \leqslant i \leqslant m-1.$$

则

$$\begin{aligned}
&\frac{1}{3}(u_{i-1}^{k+\frac{1}{2}} + u_i^{k+\frac{1}{2}} + u_{i+1}^{k+\frac{1}{2}})\Delta_x u_i^{k+\frac{1}{2}} = \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i, \\
&\frac{1}{3}(U_{i-1}^{k+\frac{1}{2}} + U_i^{k+\frac{1}{2}} + U_{i+1}^{k+\frac{1}{2}})\Delta_x U_i^{k+\frac{1}{2}} = \psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}})_i.
\end{aligned}$$

于是 (1.15) 可以写为

$$\delta_t u_i^{k+\frac{1}{2}} + \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i = \nu\delta_x^2 u_i^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n-1.$$

将 uu_x 写为 $\frac{1}{3}[uu_x + (u^2)_x]$, 可将 $\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}})_i$ 看成为后者在 $(x_i, t_{k+\frac{1}{2}})$ 处的离散化.

算子 ψ 具有如下结论.

引理 1.3　设 $v \in \mathcal{U}_h, w \in \mathring{\mathcal{U}}_h$, 则有

$$(\psi(v,w), w) = 0.$$

证明

$$
\begin{aligned}
&(\psi(v,w),w)\\
=&\frac{1}{3}\big(v\Delta_x w+\Delta_x(vw),w\big)\\
=&\frac{1}{3}\Big[\big(v\Delta_x w,w\big)+\big(\Delta_x(vw),w\big)\Big)\\
=&\frac{1}{3}\Big[\big(\Delta_x w,vw\big)+\big(\Delta_x(vw),w\big)\Big]\\
=&0.
\end{aligned}
$$

□

定理 1.2 设 $\{u_i^k\,|\,0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$ 是差分格式 (1.15)–(1.17) 的解. 令

$$E^k=\|u^k\|^2+2\nu\tau\sum_{l=0}^{k-1}|u^{l+\frac{1}{2}}|_1^2,\quad 0\leqslant k\leqslant n,$$

则有

$$E^k=E^0,\quad 1\leqslant k\leqslant n. \tag{1.18}$$

证明 注意到差分格式 (1.15) 可写成

$$\delta_t u_i^{k+\frac{1}{2}}+\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}})_i-\nu\delta_x^2u_i^{k+\frac{1}{2}}=0,\quad 1\leqslant i\leqslant m-1,\ 0\leqslant k\leqslant n-1.$$

用 $u^{k+\frac{1}{2}}$ 与上式作内积, 可得

$$(\delta_t u^{k+\frac{1}{2}},u^{k+\frac{1}{2}})+(\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}}),u^{k+\frac{1}{2}})-\nu(\delta_x^2u^{k+\frac{1}{2}},u^{k+\frac{1}{2}})=0.$$

注意到 $u^{k+\frac{1}{2}}\in\mathring{\mathcal{U}}_h$, 有

$$
\begin{aligned}
&(\delta_t u^{k+\frac{1}{2}},u^{k+\frac{1}{2}})=\frac{1}{2\tau}(\|u^{k+1}\|^2-\|u^k\|^2),\\
&(\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}}),u^{k+\frac{1}{2}})=0,\\
&-(\delta_x^2u^{k+\frac{1}{2}},u^{k+\frac{1}{2}})=|u^{k+\frac{1}{2}}|_1^2.
\end{aligned}
$$

因而

$$\frac{1}{2\tau}(\|u^{k+1}\|^2-\|u^k\|^2)+\nu|u^{k+\frac{1}{2}}|_1^2=0,\quad 0\leqslant k\leqslant n-1.$$

将上式中的 k 换为 l, 并对 l 从 0 到 $k-1$ 求和, 得

$$\frac{1}{2\tau}(\|u^k\|^2-\|u^0\|^2)+\nu\sum_{l=0}^{k-1}|u^{l+\frac{1}{2}}|_1^2=0,\quad 1\leqslant k\leqslant n.$$

将上式变形即得 (1.18). □

由定理 1.2 易知

$$\|u^k\|\leqslant\|u^0\|,\quad 1\leqslant k\leqslant n.$$

1.2.4 差分格式解的存在性和唯一性

我们借助于下列 Browder 定理证明差分格式解的存在性.

定理 1.3 (Browder 定理 [9, 10]) 设 $(H,(\cdot,\cdot))$ 是一个有限维内积空间, $\|\cdot\|$ 是导出范数算子, $\Pi: H\rightarrow H$ 是连续的. 进一步假设存在常数 $\alpha>0$, 对于任意的 $z\in H$, $\|z\|=\alpha$, 有 $\mathrm{Re}(\Pi(z),z)\geqslant 0$, 则存在 $z^*\in H$ 使得 $\Pi(z^*)=0$, 且 $\|z^*\|\leqslant\alpha$.

定理 1.4 差分格式 (1.15)–(1.17) 存在解.

证明 由 (1.16)–(1.17) 知第 0 层值 u^0 已给定. 设已求得第 k 层的解 u^k. 令

$$w_i=u_i^{k+\frac{1}{2}},\quad 0\leqslant i\leqslant m,$$

则有

$$u_i^{k+1}=2w_i-u_i^k,\quad 0\leqslant i\leqslant m.$$

由 (1.15), (1.17) 可得关于 $w=(w_0,w_1,\cdots,w_m)$ 的方程组

$$\frac{2}{\tau}(w_i-u_i^k)+\psi(w,w)_i-\nu\delta_x^2 w_i=0,\quad 1\leqslant i\leqslant m-1, \tag{1.19}$$

$$w_0=0,\quad w_m=0. \tag{1.20}$$

对于任意的 $u,v\in\mathring{\mathcal{U}}_h$, 定义内积

$$(u,v)=h\sum_{i=1}^{m-1}u_iv_i,$$

则 $\mathring{\mathcal{U}}_h$ 为一个内积空间, $\|u\|=\sqrt{(u,u)}$ 为导出范数.

定义 $\Pi:\mathring{\mathcal{U}}_h\rightarrow\mathring{\mathcal{U}}_h$

$$\Pi(w)_i=\frac{2}{\tau}(w_i-u_i^k)+\psi(w,w)_i-\nu\delta_x^2 w_i,\quad 1\leqslant i\leqslant m-1.$$

则 $\Pi(w)$ 为 $\mathring{\mathcal{U}}_h$ 上的连续函数, 应用引理 1.3, 可得

$$\begin{aligned}(\Pi(w),w)&=\frac{2}{\tau}[(w,w)-(u^k,w)]+(\psi(w,w),w)-\nu(\delta_x^2 w,w)\\&=\frac{2}{\tau}[\|w\|^2-(u^k,w)]+\nu|w|_1^2\\&\geqslant\frac{2}{\tau}(\|w\|^2-\|u^k\|\cdot\|w\|)\\&\geqslant\frac{2}{\tau}(\|w\|-\|u^k\|)\cdot\|w\|.\end{aligned}$$

因而当 $\|w\|=\|u^k\|$ 时, $(\Pi(w),w)\geqslant 0$. 由定理 1.3 知存在 $w^*\in\mathring{\mathcal{U}}_h$ 使得 $\Pi(w^*)=0$, 且 $\|w^*\|\leqslant\|u^k\|$, 即 (1.19),(1.20) 存在解 w^*. □

定理 1.5 记 $c_2=\|u^0\|$. 因当 $\tau<\dfrac{4\nu^3}{c_2^4}$ 时, 差分格式 (1.15)–(1.17) 的解是唯一的.

证明 由定理 1.4 的证明可知只要证明 (1.19)–(1.20) 的解是唯一的.

设 (1.19)–(1.20) 有两个解 $X,Y\in\mathring{\mathcal{U}}_h$, 即

$$\frac{2}{\tau}(X_i-u_i^k)+\psi(X,X)_i-\nu\delta_x^2X_i=0,\quad 1\leqslant i\leqslant m-1,\tag{1.21}$$

$$X_0=0,\quad X_m=0;\tag{1.22}$$

$$\frac{2}{\tau}(Y_i-u_i^k)+\psi(Y,Y)_i-\nu\delta_x^2Y_i=0,\quad 1\leqslant i\leqslant m-1,\tag{1.23}$$

$$Y_0=0,\quad Y_m=0.\tag{1.24}$$

令

$$z=X-Y.$$

将 (1.21)–(1.22) 与 (1.23)–(1.24) 相减, 得

$$\frac{2}{\tau}z_i+\psi(X,X)_i-\psi(Y,Y)_i-\nu\delta_x^2z_i=0,\quad 1\leqslant i\leqslant m-1,\tag{1.25}$$

$$z_0=0,\quad z_m=0.\tag{1.26}$$

由定理 1.2 知存在常数 c_2 使得

$$\|X\|\leqslant c_2,\quad \|Y\|\leqslant c_2.$$

用 z 与 (1.25) 作内积, 得

$$\frac{2}{\tau}\|z\|^2+(\psi(X,X)-\psi(Y,Y),z)+\nu|z|_1^2=0.\tag{1.27}$$

由

$$\psi(X,X)-\psi(Y,Y)=\psi(X,X)-\psi(X-z,X-z)=\psi(z,X)+\psi(X,z)-\psi(z,z)$$

及引理 1.3 得

$$(\psi(X,X)-\psi(Y,Y),z)=(\psi(z,X),z).$$

因而

$$\begin{aligned}&-(\psi(X,z)-\psi(Y,Y),z)\\=&-\frac{1}{3}h\sum_{i=1}^{m-1}[z_i\Delta_xX_i+\Delta_x(zX)_i]z_i\\=&\frac{1}{3}h\sum_{i=1}^{m-1}[X_i\Delta_x(z_i^2)+(zX)_i\Delta_xz_i]\end{aligned}$$

$$\begin{aligned}&\leqslant\frac{1}{3}(2\|z\|_\infty\cdot\|X\|\cdot|z|_1+\|z\|_\infty\cdot\|X\|\cdot|z|_1)\\&=\|X\|\cdot\|z\|_\infty\cdot|z|_1\\&\leqslant c_2\|z\|_\infty\cdot|z|_1.\end{aligned}$$

由 (1.27) 得

$$\frac{2}{\tau}\|z\|^2+\nu|z|_1^2\leqslant c_2\|z\|_\infty\cdot|z|_1.$$

由引理 1.1(c) 知, 对任意 $\varepsilon>0$, 有

$$\|z\|_\infty\leqslant\varepsilon|z|_1+\frac{1}{2\varepsilon}\|z\|.$$

因而

$$\begin{aligned}\frac{2}{\tau}\|z\|^2+\nu|z|_1^2&\leqslant c_2\left(\varepsilon|z|_1+\frac{1}{2\varepsilon}\|z\|\right)|z|_1\\&=c_2\varepsilon|z|_1^2+\frac{c_2}{2\varepsilon}\|z\|\cdot|z|_1\\&\leqslant c_2\varepsilon|z|_1^2+c_2\varepsilon|z|_1^2+\frac{1}{4c_2\varepsilon}\left(\frac{c_2}{2\varepsilon}\right)^2\|z\|^2\\&=2c_2\varepsilon|z|_1^2+\frac{c_2}{16\varepsilon^3}\|z\|^2.\end{aligned}$$

取 $\varepsilon=\dfrac{\nu}{2c_2}$, 则有

$$\frac{2}{\tau}\|z\|^2\leqslant\frac{c_2^4}{2\nu^3}\|z\|^2.$$

当 $\tau<\dfrac{4\nu^3}{c_2^4}$ 时 $\|z\|=0$, 即 (1.19)–(1.20) 的解是唯一的. □

1.2.5 差分格式解的收敛性

我们先给出重要的 Gronwall 不等式.

定理 1.6 (a) 设 $\{F^k\}_{k=0}^\infty$ 是一个非负序列, c 和 g 是两个非负常数, 且满足

$$F^{k+1}\leqslant(1+c\tau)F^k+\tau g,\quad k=0,1,2,\cdots,$$

则有

$$F^k\leqslant e^{ck\tau}\left(F^0+\frac{g}{c}\right),\quad k=1,2,3,\cdots.$$

(b) 设 $\{F^k\}_{k=0}^\infty$ 和 $\{g^k\}_{k=0}^\infty$ 是两个非负序列, c 为非负常数, 且满足

$$F^{k+1}\leqslant(1+c\tau)F^k+\tau g^k,\quad k=0,1,2,\cdots,$$

则有

$$F^k \leqslant e^{ck\tau}\left(F^0 + \tau\sum_{l=0}^{k-1} g^l\right), \quad k = 0, 1, 2, \cdots.$$

(c) 设 $\{F^k\}_{k=0}^{\infty}$ 是非负序列, c 和 g 是两个非负常数, 且满足

$$F^k \leqslant c\tau\sum_{k=0}^{k-1} F^l + g, \quad k = 0, 1, 2, \cdots,$$

则有

$$F^k \leqslant e^{ck\tau} g, \quad k = 0, 1, 2, \cdots.$$

(d) 设 $\{F^k\}_{k=0}^{\infty}$ 是非负序列, $\{g^k\}_{k=0}^{\infty}$ 是非负单调递增 (不必严格单调), 且满足

$$F^k \leqslant c\tau\sum_{l=0}^{k-1} F^l + g^k, \quad k = 0, 1, 2, \cdots,$$

则有

$$F^k \leqslant e^{ck\tau} g^k, \quad k = 0, 1, 2, \cdots.$$

证明 (a)

$$\begin{aligned}
F^{k+1} &\leqslant (1+c\tau)F^k + \tau g \\
&\leqslant (1+c\tau)[(1+c\tau)F^{k-1} + \tau g] + \tau g \\
&= (1+c\tau)^2 F^{k-1} + [(1+c\tau)+1]\tau g \\
&\leqslant (1+c\tau)^2[(1+c\tau)F^{k-2} + \tau g] + [(1+c\tau)+1]\tau g \\
&= (1+c\tau)^3 F^{k-2} + [(1+c\tau)^2 + (1+c\tau) + 1]\tau g \\
&\leqslant \cdots \\
&\leqslant (1+c\tau)^k F^1 + [(1+c\tau)^{k-1} + (1+c\tau)^{k-2} + \cdots + 1]\tau g \\
&\leqslant (1+c\tau)^k[(1+c\tau)F^0 + \tau g] + [(1+c\tau)^{k-1} + (1+c\tau)^{k-2} + \cdots + 1]\tau g \\
&= (1+c\tau)^{k+1} F^0 + [(1+c\tau)^k + (1+c\tau)^{k-1} + \cdots + 1]\tau g \\
&= (1+c\tau)^{k+1} F^0 + \frac{(1+c\tau)^{k+1} - 1}{c\tau} \cdot \tau g \\
&\leqslant e^{c(k+1)\tau}\left(F^0 + \frac{g}{c}\right), \quad k = 0, 1, \cdots.
\end{aligned}$$

(b)

$$
\begin{aligned}
F^{k+1} &\leqslant (1+c\tau)F^k + \tau g^k \\
&\leqslant (1+c\tau)[(1+c\tau)F^{k-1} + \tau g^{k-1}] + \tau g^k \\
&= (1+c\tau)^2 F^{k-1} + (1+c\tau)\tau g^{k-1} + \tau g^k \\
&\leqslant (1+c\tau)^2[(1+c\tau)F^{k-2} + \tau g^{k-2}] + (1+c\tau)\tau g^{k-1} + \tau g^k \\
&= (1+c\tau)^3 F^{k-2} + (1+c\tau)^2\tau g^{k-2} + (1+c\tau)\tau g^{k-1} + \tau g^k \\
&\leqslant (1+c\tau)^3[(1+c\tau)F^{k-3} + \tau g^{k-3}] + (1+c\tau)^2\tau g^{k-2} + (1+c\tau)\tau g^{k-1} + \tau g^k \\
&= (1+c\tau)^4 F^{k-3} + (1+c\tau)^3\tau g^{k-3} + (1+c\tau)^2\tau g^{k-2} + (1+c\tau)\tau g^{k-1} + \tau g^k \\
&\leqslant (1+c\tau)^{k+1} F^0 + \tau \sum_{l=0}^{k} (1+c\tau)^{k-l} g^l \\
&\leqslant (1+c\tau)^{k+1} \Big(F^0 + \tau \sum_{l=0}^{k} g^l\Big), \quad k = 0, 1, 2, \cdots.
\end{aligned}
$$

(c) 易知

$$F^0 \leqslant g.$$

令

$$G^k = c\tau \sum_{l=0}^{k-1} F^l + g, \quad k = 0, 1, 2, \cdots.$$

则有

$$
\begin{aligned}
&G^0 = g, \\
&F^k \leqslant G^k, \quad k = 0, 1, 2, \cdots, \\
&G^k = G^{k-1} + c\tau F^{k-1} \leqslant G^{k-1} + c\tau G^{k-1} = (1+c\tau)G^{k-1}, \quad k = 1, 2, 3, \cdots,
\end{aligned}
$$

递推可得

$$G^k \leqslant (1+c\tau)^k G^0 \leqslant e^{ck\tau} g, \quad k = 0, 1, 2, \cdots,$$

因而

$$F^k \leqslant G^k \leqslant e^{ck\tau} g, \quad k = 0, 1, 2, \cdots.$$

(d) 易知

$$F^0 \leqslant g^0.$$

令

$$G^k = c\tau \sum_{l=0}^{k-1} F^l + g^k, \quad k = 0, 1, 2, \cdots,$$

则

$$\begin{aligned}
&G^0 = g^0, \\
&F^k \leqslant G^k, \quad k = 0, 1, 2, \cdots, \\
&G^k = c\tau \sum_{l=0}^{k-2} F^l + g^{k-1} + c\tau F^{k-1} + (g^k - g^{k-1}) \\
&\quad = G^{k-1} + c\tau F^{k-1} + (g^k - g^{k-1}) \\
&\quad \leqslant (1 + c\tau) G^{k-1} + (g^k - g^{k-1}), \quad k = 1, 2, \cdots.
\end{aligned}$$

应用 (b) 之结果, 得

$$F^k \leqslant G^k \leqslant e^{ck\tau} \left[G^0 + \sum_{l=1}^{k} (g^l - g^{l-1}) \right] = e^{ck\tau} g^k, \quad k = 0, 1, 2, \cdots. \qquad \square$$

记

$$c_3 = \max_{0 \leqslant x \leqslant L, 0 \leqslant t \leqslant T} |u_x(x, t)|. \tag{1.28}$$

定理 1.7 设 $\{U_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为问题 (1.1)–(1.3) 的解, $\{u_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (1.15)–(1.17) 的解, 记

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant m, \ 0 \leqslant k \leqslant n.$$

则存在常数 c_4 使得

$$\|e^k\| \leqslant c_4(\tau^2 + h^2), \quad 0 \leqslant k \leqslant n. \tag{1.29}$$

证明 将 (1.11), (1.13), (1.14) 和 (1.15)–(1.17) 相减, 得到误差方程组

$$\delta_t e_i^{k+\frac{1}{2}} + \psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}})_i - \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i = \nu \delta_x^2 e_i^{k+\frac{1}{2}} + R_i^{k+\frac{1}{2}},$$
$$1 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{1.30}$$

$$e_i^0 = 0, \quad 1 \leqslant i \leqslant m-1, \tag{1.31}$$

$$e_0^k = 0, \quad e_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{1.32}$$

用 $e^{k+\frac{1}{2}}$ 与 (1.30) 作内积, 得

$$\begin{aligned}
&(\delta_t e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}) + (\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}) - \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}), e^{k+\frac{1}{2}}) + \nu |e^{k+\frac{1}{2}}|_1^2 \\
&= (R^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}), \quad 0 \leqslant k \leqslant n-1.
\end{aligned} \tag{1.33}$$

易知

$$(\delta_t e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}) = \frac{1}{2\tau}(\|e^{k+1}\|^2 - \|e^k\|^2). \tag{1.34}$$

注意到

$$
\begin{aligned}
&\psi(U^{k+\frac{1}{2}},U^{k+\frac{1}{2}})-\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}})\\
=&\psi(U^{k+\frac{1}{2}},U^{k+\frac{1}{2}})-\psi(U^{k+\frac{1}{2}}-e^{k+\frac{1}{2}},U^{k+\frac{1}{2}}-e^{k+\frac{1}{2}})\\
=&\psi(e^{k+\frac{1}{2}},U^{k+\frac{1}{2}})+\psi(U^{k+\frac{1}{2}},e^{k+\frac{1}{2}})-\psi(e^{k+\frac{1}{2}},e^{k+\frac{1}{2}}),
\end{aligned}
$$

再应用引理 1.3, 可得

$$
\begin{aligned}
&-(\psi(U^{k+\frac{1}{2}},U^{k+\frac{1}{2}})-\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}}),e^{k+\frac{1}{2}})\\
=&-(\psi(e^{k+\frac{1}{2}},U^{k+\frac{1}{2}}),e^{k+\frac{1}{2}})\\
=&-\frac{1}{3}h\sum_{i=1}^{m-1}[e_i^{k+\frac{1}{2}}\Delta_x U_i^{k+\frac{1}{2}}+\Delta_x(e^{k+\frac{1}{2}}U^{k+\frac{1}{2}})_i]e_i^{k+\frac{1}{2}}\\
=&-\frac{1}{3}h\sum_{i=1}^{m-1}(e_i^{k+\frac{1}{2}})^2\Delta_x U_i^{k+\frac{1}{2}}+\frac{1}{3}h\sum_{i=1}^{m-1}e_i^{k+\frac{1}{2}}U_i^{k+\frac{1}{2}}\Delta_x e_i^{k+\frac{1}{2}}\\
=&-\frac{1}{3}h\sum_{i=1}^{m-1}(e_i^{k+\frac{1}{2}})^2\Delta_x U_i^{k+\frac{1}{2}}-\frac{1}{6}h\sum_{i=0}^{m-1}\frac{U_{i+1}^{k+\frac{1}{2}}-U_i^{k+\frac{1}{2}}}{h}e_i^{k+\frac{1}{2}}e_{i+1}^{k+\frac{1}{2}}\\
\leqslant&\frac{1}{2}c_3\|e^{k+\frac{1}{2}}\|^2.
\end{aligned}
\tag{1.35}
$$

将 (1.34) 和 (1.35) 代入 (1.33), 可得

$$
\begin{aligned}
&\frac{1}{2\tau}(\|e^{k+1}\|^2-\|e^k\|^2)\\
\leqslant&\frac{1}{2}c_3\|e^{k+\frac{1}{2}}\|^2+\|R^{k+\frac{1}{2}}\|\cdot\|e^{k+\frac{1}{2}}\|\\
\leqslant&\frac{1}{2}c_3\left(\frac{\|e^k\|+\|e^{k+1}\|}{2}\right)^2+\|R^{k+\frac{1}{2}}\|\cdot\frac{\|e^k\|+\|e^{k+1}\|}{2},\quad 0\leqslant k\leqslant n-1.
\end{aligned}
$$

上式两边约去 $\dfrac{\|e^{k+1}\|+\|e^k\|}{2}$, 得到

$$
\frac{1}{\tau}(\|e^{k+1}\|-\|e^k\|)\leqslant\frac{c_3}{4}(\|e^k\|+\|e^{k+1}\|)+\|R^{k+\frac{1}{2}}\|,\quad 0\leqslant k\leqslant n-1,
$$

即

$$
\left(1-\frac{c_3}{4}\tau\right)\|e^{k+1}\|\leqslant\left(1+\frac{c_3\tau}{4}\right)\|e^k\|+\tau\|R^{k+\frac{1}{2}}\|,\quad 0\leqslant k\leqslant n-1.
$$

当 $\dfrac{c_3}{4}\tau\leqslant\dfrac{1}{3}$ 时,

$$
\begin{aligned}
\|e^{k+1}\|&\leqslant\left(1+\frac{3c_3\tau}{4}\right)\|e^k\|+\frac{3}{2}\tau\|R^{k+\frac{1}{2}}\|\\
&\leqslant\left(1+\frac{3c_3}{4}\tau\right)\|e^k\|+\frac{3}{2}c_1\sqrt{L}\tau(\tau^2+h^2),\quad 0\leqslant k\leqslant n-1.
\end{aligned}
$$

由 Gronwall 不等式 (定理 1.6), 并注意到 $\|e^0\| = 0$, 得

$$\|e^{k+1}\| \leqslant e^{\frac{3c_3}{4}k\tau} \cdot \frac{2c_1\sqrt{L}}{c_3}(\tau^2 + h^2) = c_4(\tau^2 + h^2), \quad 0 \leqslant k \leqslant n-1,$$

其中 $c_4 = e^{\frac{3c_3}{4}T} \cdot \dfrac{2c_1\sqrt{L}}{c_3}$. □

1.3 三层线性化差分格式

1.3.1 差分格式的建立

在点 (x_i, t_0) 处考虑方程 (1.1), 并注意到 (1.2), 有

$$u_t(x_i, 0) = \nu\varphi''(x_i) - \varphi(x_i)\varphi'(x_i).$$

记

$$\hat{u}_i = \varphi(x_i) + \frac{\tau}{2}[\nu\varphi''(x_i) - \varphi(x_i)\varphi'(x_i)], \quad 0 \leqslant i \leqslant m.$$

在点 $(x_i, t_{\frac{1}{2}})$ 处考虑方程 (1.1), 并应用 Taylor 展开式, 有

$$\delta_t U_i^{\frac{1}{2}} + \psi(\hat{u}, U^{\frac{1}{2}})_i = \nu\delta_x^2 U_i^{\frac{1}{2}} + R_i^0, \quad 1 \leqslant i \leqslant m-1, \tag{1.36}$$

且存在常数 c_5 使得

$$|R_i^0| \leqslant c_5(\tau^2 + h^2), \quad 1 \leqslant i \leqslant m-1. \tag{1.37}$$

在点 (x_i, t_k) 处考虑方程 (1.1), 并应用 Taylor 展开式, 得到

$$\Delta_t U_i^k + \psi(U^k, U^{\bar{k}})_i = \nu\delta_x^2 U_i^{\bar{k}} + R_i^k, \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{1.38}$$

且存在常数 c_6 使得

$$|R_i^k| \leqslant c_6(\tau^2 + h^2), \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1. \tag{1.39}$$

注意到初边值条件 (1.2) 和 (1.3), 有

$$U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{1.40}$$

$$U_0^k = 0, \quad U_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{1.41}$$

在 (1.36), (1.38) 中略去小量项, 对问题 (1.1)–(1.3) 建立如下差分格式

$$\delta_t u_i^{\frac{1}{2}} + \psi(\hat{u}, u^{\frac{1}{2}})_i = \nu\delta_x^2 u_i^{\frac{1}{2}}, \quad 1 \leqslant i \leqslant m-1, \tag{1.42}$$

$$\Delta_t u_i^k + \psi(u^k, u^{\bar{k}})_i = \nu\delta_x^2 u_i^{\bar{k}}, \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{1.43}$$

$$u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{1.44}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{1.45}$$

1.3.2　差分格式解的存在性和唯一性

定理 1.8　差分格式 (1.42)–(1.45) 的解是存在唯一的.

证明　由 (1.44) 和 (1.45) 知第 0 层的值 u^0 已给定. 由 (1.42) 和 (1.45) 可得关于第 1 层值 u^1 的线性方程组. 考虑其齐次方程组

$$\frac{1}{\tau}u_i^1+\frac{1}{2}\psi(\hat{u},u^1)_i=\frac{1}{2}\nu\delta_x^2u_i^1,\quad 1\leqslant i\leqslant m-1, \tag{1.46}$$
$$u_0^1=0,\quad u_m^1=0. \tag{1.47}$$

用 u^1 与 (1.46) 作内积, 得

$$\frac{1}{\tau}\|u^1\|^2+\frac{1}{2}(\psi(\hat{u},u^1),u^1)=\frac{1}{2}\nu(\delta_x^2u^1,u^1).$$

由 $(\psi(\hat{u},u^1),u^1)=0$ 和 $(\delta_x^2u^1,u^1)=-|u^1|_1^2$ 得

$$\frac{1}{\tau}\|u^1\|^2+\frac{1}{2}\nu|u^1|_1^2=0.$$

因而 $\|u^1\|=0$. 方程组 (1.46)–(1.47) 只有零解. (1.42), (1.45) 唯一确定 u^1.

现设第 $k-1$ 层值 u^{k-1} 和第 k 层值 u^k 已确定, 则由 (1.43) 和 (1.45) 可得关于 u^{k+1} 的线性方程组. 考虑其齐次方程组

$$\frac{1}{2\tau}u_i^{k+1}+\frac{1}{2}\psi(u^k,u^{k+1})_i=\frac{1}{2}\nu\delta_x^2u_i^{k+1},\quad 1\leqslant i\leqslant m-1, \tag{1.48}$$
$$u_0^{k+1}=0,\quad u_m^{k+1}=0. \tag{1.49}$$

用 u^{k+1} 与 (1.48) 作内积, 得

$$\frac{1}{2\tau}\|u^{k+1}\|^2+\frac{1}{2}(\psi(u^k,u^{k+1}),u^{k+1})=\frac{1}{2}\nu(\delta_x^2u^{k+1},u^{k+1}).$$

由 $(\psi(u^k,u^{k+1}),u^{k+1})=0$ 及 $(\delta_x^2u^{k+1},u^{k+1})=-|u^{k+1}|_1^2$ 得

$$\frac{1}{2\tau}\|u^{k+1}\|^2+\frac{1}{2}\nu|u^{k+1}|_1^2=0.$$

因而 $\|u^{k+1}\|=0$. 方程组 (1.48)–(1.49) 只有零解. 因而 (1.43) 和 (1.45) 唯一确定 u^{k+1}.

由归纳原理, 定理证毕. □

1.3.3　差分格式解的守恒性和有界性

定理 1.9　设 $\{u_i^k\,|\,0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$ 为 (1.42)–(1.45) 的解. 则有

$$\frac{1}{2}(\|u^1\|^2+\|u^0\|^2)+\nu\tau|u^{\frac{1}{2}}|_1^2=\|u^0\|^2, \tag{1.50}$$
$$E(u^{k+1},u^k)=E(u^1,u^0),\quad k=1,2,3,\cdots,n-1, \tag{1.51}$$

其中

$$E(u^{k+1},u^k)=\frac{1}{2}(\|u^{k+1}\|^2+\|u^k\|^2)+2\nu\tau\sum_{l=1}^{k}|u^{\bar{l}}|_1^2,\quad k=0,1,\cdots,n-1.$$

证明 (I) 用 $u^{\frac{1}{2}}$ 与 (1.42) 作内积, 得

$$(\delta_t u^{\frac{1}{2}},u^{\frac{1}{2}})+(\psi(\hat{u},u^{\frac{1}{2}}),u^{\frac{1}{2}})=\nu(\delta_x^2 u^{\frac{1}{2}},u^{\frac{1}{2}}).$$

由

$$\begin{aligned}&(\delta_t u^{\frac{1}{2}},u^{\frac{1}{2}})=\frac{1}{2\tau}(\|u^1\|^2-\|u^0\|^2),\\&(\psi(\hat{u},u^{\frac{1}{2}}),u^{\frac{1}{2}})=0,\\&(\delta_x^2 u^{\frac{1}{2}},u^{\frac{1}{2}})=-|u^{\frac{1}{2}}|_1^2,\end{aligned}$$

得

$$\frac{1}{2\tau}(\|u^1\|^2-\|u^0\|^2)+\nu|u^{\frac{1}{2}}|_1^2=0, \tag{1.52}$$

即

$$\frac{1}{2}(\|u^1\|^2+\|u^0\|^2)+\nu\tau|u^{\frac{1}{2}}|_1^2=\|u^0\|^2.$$

(II) 用 $u^{\bar{k}}$ 与 (1.43) 作内积, 得到

$$(\Delta_t u^k,u^{\bar{k}})+(\psi(u^k,u^{\bar{k}}),u^{\bar{k}})=\nu(\delta_x^2 u^{\bar{k}},u^{\bar{k}}),\quad 1\leqslant k\leqslant n-1.$$

易得

$$\frac{1}{4\tau}(\|u^{k+1}\|^2-\|u^{k-1}\|^2)+\nu|u^{\bar{k}}|_1^2=0,\quad 1\leqslant k\leqslant n-1, \tag{1.53}$$

或

$$\frac{1}{2\tau}\left(\frac{\|u^{k+1}\|^2+\|u^k\|^2}{2}-\frac{\|u^k\|^2+\|u^{k-1}\|^2}{2}\right)+\nu|u^{\bar{k}}|_1^2=0,\quad 1\leqslant k\leqslant n-1.$$

上式可进一步写为

$$\frac{1}{2\tau}[E(u^{k+1},u^k)-E(u^k,u^{k-1})]=0,\quad 1\leqslant k\leqslant n-1.$$

因而

$$E(u^{k+1},u^k)=E(u^1,u^0),\quad 1\leqslant k\leqslant n-1.$$ □

注 1.1 (1.50) 和 (1.51) 可以统一写为

$$\frac{1}{2}(\|u^{k+1}\|^2+\|u^k\|^2)+\nu\tau|u^{\frac{1}{2}}|_1^2+2\nu\tau\sum_{l=1}^{k}|u^{\bar{l}}|_1^2=\|u^0\|^2,\quad k=0,1,\cdots,n-1.$$

注 1.2 由 (1.52) 和 (1.53) 可得

$$\|u^k\|\leqslant\|u^0\|,\quad 1\leqslant k\leqslant n.$$

1.3.4　差分格式解的收敛性

定理 1.10　设 $\{U_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为问题 (1.1)–(1.3) 的解, $\{u_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (1.42)–(1.45) 的解. 记

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n.$$

则存在常数 c_7, 当 $\tau^2 + h^2 \leqslant \dfrac{1}{c_7}$ 时成立

$$|e^k|_1 \leqslant c_7(\tau^2 + h^2), \quad 0 \leqslant k \leqslant n, \tag{1.54}$$

$$\|e^k\|_\infty \leqslant \frac{\sqrt{L}}{2} c_7(\tau^2 + h^2), \quad 0 \leqslant k \leqslant n. \tag{1.55}$$

证明　将 (1.36), (1.38), (1.40)–(1.41) 与 (1.42)–(1.45) 相减, 得误差方程组

$$\delta_t e_i^{\frac{1}{2}} + \psi(\hat{u}, e^{\frac{1}{2}})_i = \nu \delta_x^2 e_i^{\frac{1}{2}} + R_i^0, \quad 1 \leqslant i \leqslant m-1, \tag{1.56}$$

$$\Delta_t e_i^k + \psi(U^k, U^{\bar{k}})_i - \psi(u^k, u^{\bar{k}})_i = \nu \delta_x^2 e_i^{\bar{k}} + R_i^k,$$
$$1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{1.57}$$

$$e_i^0 = 0, \quad 1 \leqslant i \leqslant m-1, \tag{1.58}$$

$$e_0^k = 0, \quad e_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{1.59}$$

我们将用数学归纳法证明所要求的结果.

由 (1.58)–(1.59) 得

$$|e^0|_1 = 0, \quad \|e^0\|_\infty = 0. \tag{1.60}$$

故 (1.54) 及 (1.55) 对 $k = 0$ 成立.

(I) 用 $\delta_t e^{\frac{1}{2}}$ 与 (1.56) 作内积, 得

$$\|\delta_t e^{\frac{1}{2}}\|^2 + (\psi(\hat{u}, e^{\frac{1}{2}}), \delta_t e^{\frac{1}{2}}) = \nu(\delta_x^2 e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) + (R^0, \delta_t e^{\frac{1}{2}}),$$

注意到

$$e_i^0 = 0, \quad 0 \leqslant i \leqslant m,$$

有

$$\frac{1}{\tau^2}\|e^1\|^2 + \frac{1}{2\tau}(\psi(\hat{u}, e^1), e^1) = -\frac{\nu}{2\tau}|e^1|_1^2 + \frac{1}{\tau}(R^0, e^1).$$

再注意到

$$(\psi(\hat{u}, e^1), e^1) = 0,$$

有

$$\frac{1}{\tau^2}\|e^1\|^2 + \frac{\nu}{2\tau}|e|_1^2 = \frac{1}{\tau}(R^0, e^1) \leqslant \frac{1}{\tau^2}\|e^1\|^2 + \frac{1}{4}\|R^0\|^2.$$

再由 (1.37), 得到

$$|e^1|_1^2 \leqslant \frac{2\tau}{\nu} \cdot \frac{1}{4}\|R^0\|^2 \leqslant \frac{\tau}{2\nu} L c_5^2 (\tau^2 + h^2)^2.$$

当 $\tau \leqslant 2\nu$ 时

$$|e^1|_1^2 \leqslant L c_5^2 (\tau^2 + h^2)^2,$$

或

$$|e^1|_1 \leqslant \sqrt{L} c_5 (\tau^2 + h^2). \tag{1.61}$$

(II) 将 $\Delta_t e^k$ 与 (1.57) 作内积, 得

$$\|\Delta_t e^k\|^2 + (\psi(U^k, U^{\bar{k}}) - \psi(u^k, u^{\bar{k}}), \Delta_t e^k) = \nu(\delta_x^2 e^{\bar{k}}, \Delta_t e^k) + (R^k, \Delta_t e^k),$$
$$1 \leqslant k \leqslant n-1, \tag{1.62}$$

或

$$\begin{aligned}&\|\Delta_t e^k\|^2 + \frac{\nu}{4\tau}(|e^{k+1}|_1^2 - |e^{k-1}|_1^2)\\ =& -(\psi(U^k, U^{\bar{k}}) - \psi(u^k, u^{\bar{k}}), \Delta_t e^k) + (R^k, \Delta_t e^k), \quad 1 \leqslant k \leqslant n-1.\end{aligned}$$

易知

$$|U^k|_1 \leqslant \sqrt{L} c_3, \quad \|U^k\|_\infty \leqslant \frac{\sqrt{L}}{2}|U^k|_1 \leqslant \frac{L}{2} c_3, \quad 0 \leqslant k \leqslant n. \tag{1.63}$$

设 (1.54) 对 $k = 1, 2, \cdots, l$ 成立. 则当 $c_7(\tau^2 + h^2) \leqslant 1$, 有

$$|u^k|_1 \leqslant |U^k|_1 + |e^k|_1 \leqslant \sqrt{L} c_3 + 1, \quad 1 \leqslant k \leqslant l, \tag{1.64}$$

$$\|u^k\|_\infty \leqslant \frac{\sqrt{L}}{2}|u^k|_1 \leqslant \frac{\sqrt{L}}{2}(\sqrt{L} c_3 + 1), \quad 1 \leqslant k \leqslant l. \tag{1.65}$$

注意到

$$\begin{aligned}&\psi(U^k, U^{\bar{k}})_i - \psi(u^k, u^{\bar{k}})_i\\ =& \psi(e^k, U^{\bar{k}})_i + \psi(u^k, e^{\bar{k}})_i\\ =& \frac{1}{3}[e_i^k \Delta_x U_i^{\bar{k}} + \Delta_x(e^k U^{\bar{k}})_i] + \frac{1}{3}[u_i^k \Delta_x e_i^{\bar{k}} + \Delta_x(u^k e^{\bar{k}})_i]\\ =& \frac{1}{3}\left[e_i^k \Delta_x U_i^{\bar{k}} + \frac{1}{2}(\delta_x e_{i+\frac{1}{2}}^k) U_{i+1}^{\bar{k}} + e_i^k \Delta_x U_i^{\bar{k}} + \frac{1}{2}(\delta_x e_{i-\frac{1}{2}}^k) U_{i-1}^{\bar{k}}\right]\\ &+ \frac{1}{3}\left[u_i^k \Delta_x e_i^{\bar{k}} + \frac{1}{2}(\delta_x u_{i+\frac{1}{2}}^k) e_{i+1}^{\bar{k}} + u_i^k \Delta_x e_i^{\bar{k}} + \frac{1}{2}(\delta_x u_{i-\frac{1}{2}}^k) e_{i-1}^{\bar{k}}\right],\end{aligned}$$

以及 (1.63)–(1.65), 有

$$\begin{aligned}
&-(\psi(U^k,U^{\bar{k}})-\psi(u^k,u^k),\Delta_t e^k)\\
\leqslant&\frac{1}{3}(\|e^k\|_\infty|U^{\bar{k}}|_1+\|U^{\bar{k}}\|_\infty|e^k|_1+\|e^k\|_\infty|U^{\bar{k}}|_1)\|\Delta_t e^k\|\\
&+\frac{1}{3}(\|u^k\|_\infty|e^{\bar{k}}|_1+\|e^{\bar{k}}\|_\infty|u^k|_1+\|u^k\|_\infty|e^{\bar{k}}|_1)\|\Delta_t e^k\|\\
\leqslant&\frac{1}{3}\left(2\sqrt{L}c_3\|e^k\|_\infty+\frac{L}{2}c_3|e^k|_1\right)\|\Delta_t e^k\|\\
&+\frac{1}{3}\left(2\cdot\frac{\sqrt{L}}{2}(\sqrt{L}c_3+1)|e^{\bar{k}}|_1+(\sqrt{L}c_3+1)\|e^{\bar{k}}\|_\infty\right)\|\Delta_t e^k\|\\
\leqslant&\frac{1}{3}\left(2\sqrt{L}c_3\frac{\sqrt{L}}{2}|e^k|_1+\frac{L}{2}c_3|e^k|_1\right)\|\Delta_t e^k\|\\
&+\frac{1}{3}\left[\sqrt{L}(\sqrt{L}c_3+1)|e^{\bar{k}}|_1+(\sqrt{L}c_3+1)\frac{\sqrt{L}}{2}|e^{\bar{k}}|_1\right]\|\Delta_t e^k\|\\
=&\frac{1}{2}Lc_3|e^k|_1\cdot\|\Delta_t e^k\|+\frac{1}{2}\sqrt{L}(\sqrt{L}c_3+1)|e^{\bar{k}}|_1\cdot\|\Delta_t e^k\|\\
\leqslant&\frac{1}{4}\|\Delta_t e^k\|^2+\frac{L^2c_3^2}{4}|e^k|_1^2+\frac{1}{4}\|\Delta_t e^k\|^2+\frac{L(\sqrt{L}c_3+1)^2}{4}|e^{\bar{k}}|_1^2,\quad 1\leqslant k\leqslant l.
\end{aligned}$$

此外, 有

$$(R^k,\Delta_t e^k)\leqslant\frac{1}{2}\|\Delta_t e^k\|^2+\frac{1}{2}\|R^k\|^2.$$

将以上两式代入 (1.62), 得

$$\begin{aligned}
&\frac{\nu}{4\tau}(|e^{k+1}|_1^2-|e^{k-1}|_1^2)\\
\leqslant&\frac{L^2c_3^2}{4}|e^k|_1^2+\frac{L(\sqrt{L}c_3+1)^2}{4}|e^{\bar{k}}|_1^2+\frac{1}{2}\|R^k\|^2\\
\leqslant&\frac{L^2c_3^2}{4}|e^k|_1^2+\frac{L(\sqrt{L}c_3+1)^2}{4}\cdot\frac{|e^{k+1}|_1^2+|e^{k-1}|_1^2}{2}+\frac{1}{2}Lc_6^2(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant l.
\end{aligned}$$

两边乘以 $\frac{4\tau}{\nu}$, 并移项, 得

$$\begin{aligned}
|e^{k+1}|_1^2\leqslant&|e^{k-1}|_1^2+\frac{L^2c_3^2}{\nu}\tau|e^k|_1^2+\frac{1}{2\nu}L(\sqrt{L}c_3+1)^2\tau(|e^{k+1}|_1^2+|e^{k-1}|_1^2)\\
&+\frac{2}{\nu}Lc_6^2\tau(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant l,
\end{aligned}$$

即

$$\left[1-\frac{L(\sqrt{L}c_3+1)^2}{2\nu}\tau\right]|e^{k+1}|_1^2$$
$$\leqslant\frac{L^2c_3^2}{\nu}\tau|e^k|_1^2+\left[1+\frac{L(\sqrt{L}c_3+1)^2}{2\nu}\tau\right]|e^{k-1}|_1^2+\frac{2}{\nu}Lc_6^2\tau(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant l.$$

当 $\dfrac{L(\sqrt{L}c_3+1)^2}{2\nu}\tau\leqslant\dfrac{1}{3}$ 时,

$$|e^{k+1}|_1^2\leqslant\frac{3L^2c_3^2}{2\nu}\tau|e^k|_1^2+\left[1+\frac{3L(\sqrt{L}c_3+1)^2}{2\nu}\tau\right]|e^{k-1}|_1^2+\frac{3}{\nu}Lc_6^2\tau(\tau^2+h^2)^2,$$
$$1\leqslant k\leqslant l.$$

易知

$$\max\{|e^k|_1^2,|e^{k+1}|_1^2\}$$
$$\leqslant\left[1+\frac{3L^2c_3^2+3L(\sqrt{L}c_3+1)^2}{2\nu}\tau\right]\max\{|e^{k-1}|_1^2,|e^k|_1^2\}+\frac{3}{\nu}Lc_6^2\tau(\tau^2+h^2)^2,\ 1\leqslant k\leqslant l.$$

由 Gronwall 不等式, 得

$$\max\{|e^l|_1^2,|e^{l+1}|_1^2\}$$
$$\leqslant\exp\left\{\frac{3L^3c_3^2+3L(\sqrt{L}c_3+1)^2}{2\nu}T\right\}$$
$$\cdot\left(\max\{|e^0|_1^2,|e^1|_1^2\}+\frac{2Lc_6^2}{L^2c_3^2+L(\sqrt{L}c_3+1)^2}(\tau^2+h^2)^2\right).$$

注意到 (1.60), (1.61) 可得

$$|e^{l+1}|_1^2\leqslant\exp\left\{\frac{3L^2c_3^2+3L(\sqrt{L}c_3+1)^2}{2\nu}T\right\}$$
$$\cdot\left(Lc_5^2+\frac{2c_6^2}{Lc_3^2+(\sqrt{L}c_3+1)^2}\right)(\tau^2+h^2)^2$$
$$\equiv c_7^2(\tau^2+h^2)^2,$$

其中

$$c_7=\exp\left\{\frac{3L^2c_3^2+3L(\sqrt{L}c_3+1)^2}{4\nu}T\right\}\cdot\left(Lc_5^2+\frac{2c_6^2}{Lc_3^2+(\sqrt{L}c_3+1)^2}\right)^{\frac{1}{2}}.$$

即 (1.54) 对 $k=l+1$ 成立. 由归纳原理知 (1.54) 对 $k=0,1,2,\cdots,n$ 成立.

由 (1.54) 及引理 1.1(b) 得到 (1.55). □

1.4　Hopf-Cole 变换与高阶差分格式

1.4.1　Hopf-Cole 变换

令

$$u(x,t) = -2\nu \frac{w_x(x,t)}{w(x,t)}, \tag{1.66}$$

则有

$$\begin{aligned} u_t &= -2\nu \left(\frac{w_x}{w}\right)_t = -2\nu \frac{w_{xt}w - w_x w_t}{w^2} = -2\nu \left(\frac{w_t}{w}\right)_x, \\ u_x &= -2\nu \left(\frac{w_x}{w}\right)_x, \\ u_{xx} &= -2\nu \left(\frac{w_x}{w}\right)_{xx}. \end{aligned}$$

将上式代入 (1.1), 得到

$$-2\nu \left(\frac{w_t}{w}\right)_x + \left(-2\nu \frac{w_x}{w}\right)\left(-2\nu \left(\frac{w_x}{w}\right)_x\right) = \nu \left[-2\nu \left(\frac{w_x}{w}\right)_{xx}\right],$$

即

$$\left(\frac{w_t}{w}\right)_x - \nu \left[\left(\frac{w_x}{w}\right)^2\right]_x = \nu \left(\frac{w_x}{w}\right)_{xx},$$

或

$$\left[\frac{w_t}{w} - \nu \left(\frac{w_x}{w}\right)^2 - \nu \left(\frac{w_x}{w}\right)_x\right]_x = 0.$$

上式又可以写成

$$\left(\frac{w_t - \nu w_{xx}}{w}\right)_x = 0.$$

因而

$$\frac{w_t - \nu w_{xx}}{w} = q(t).$$

可以将上式写成

$$w_t - q(t)w = \nu w_{xx}.$$

上式两边同乘以 $e^{-\int_0^t q(s)\mathrm{d}s}$, 则可得到

$$\left[w e^{-\int_0^t q(s)\mathrm{d}s}\right]_t = \nu \left[w e^{-\int_0^t q(s)\mathrm{d}s}\right]_{xx}.$$

令

$$\widetilde{w}(x,t) = w(x,t) e^{-\int_0^t q(s)\mathrm{d}s},$$

则

$$-2\nu\frac{\widetilde{w}_x}{\widetilde{w}}=-2\nu\frac{w_x}{w}=u(x,t).$$

即对于任意 $q(t)$, 不影响 $u(x,t)$. 因而取 $q(t)=0$. 于是得到如下等价问题

$$w_t-\nu w_{xx}=0,\quad 0<x<L,\ 0<t\leqslant T, \tag{1.67}$$

$$w(x,0)=\widetilde{\varphi}(x),\quad 0\leqslant x\leqslant L, \tag{1.68}$$

$$w_x(0,t)=0,\quad w_x(L,t)=0,\quad 0\leqslant t\leqslant T, \tag{1.69}$$

其中

$$\widetilde{\varphi}(x)=e^{-\frac{1}{2\nu}\int_0^x\varphi(s)\mathrm{d}s}. \tag{1.70}$$

称 (1.66) 为 **Hopf-Cole 变换**.

1.4.2 差分格式的建立

下面给出几个带积分余项的数值微分公式.

引理 1.4 ([20]) 记 $\alpha(s)=(1-s)^2[5-(1-s)^2]$.

(a) 设 $g(x)\in C^6[x_0,x_1]$, 有

$$\begin{aligned}&\frac{5}{6}g''(x_0)+\frac{1}{6}g''(x_1)-\frac{2}{h}\left[\frac{g(x_1)-g(x_0)}{h}-g'(x_0)\right]\\=&-\frac{h}{6}g'''(x_0)+\frac{h^3}{90}g^{(5)}(x_0)+\frac{h^4}{180}\int_0^1 g^{(6)}(x_0+sh)\alpha(s)ds\\=&-\frac{h}{6}g'''(x_0)+\frac{h^3}{90}g^{(5)}(x_0)+\frac{h^4}{240}g^{(6)}(x_0+\theta_0 h),\quad \theta_0\in(0,1).\end{aligned}$$

(b) 设 $g(x)\in C^6[x_{m-1},x_m]$, 有

$$\begin{aligned}&\frac{1}{6}g''(x_{m-1})+\frac{5}{6}g''(x_m)-\frac{2}{h}\left[g'(x_m)-\frac{g(x_m)-g(x_{m-1})}{h}\right]\\=&\frac{h}{6}g'''(x_m)-\frac{h^3}{90}g^{(5)}(x_m)+\frac{h^4}{180}\int_0^1 g^{(6)}(x_m-sh)\alpha(s)ds\\=&\frac{h}{6}g'''(x_m)-\frac{h^3}{90}g^{(5)}(x_m)+\frac{h^4}{240}g^{(6)}(x_m-\theta_m h),\quad \theta_m\in(0,1).\end{aligned}$$

(c) 设 $f(x)\in C^6[x_{i-1},x_{i+1}]$, 有

$$\begin{aligned}&\frac{1}{12}[g''(x_{i-1})+10g''(x_i)+g''(x_{i+1})]-\frac{1}{h^2}[g(x_{i+1})-2g(x_i)+g(x_{i-1})]\\=&\frac{h^4}{360}\int_0^1[g^{(6)}(x_i+sh)+g^{(6)}(x_i-sh)]\alpha(s)ds\\=&\frac{h^4}{240}g^{(6)}(x_i+\theta_i h),\quad \theta_i\in(-1,1).\end{aligned}$$

设 $v \in \mathcal{U}_h$. 定义平均值算子 $\mathcal{A}$:

$$\mathcal{A}v_i = \begin{cases} \dfrac{5}{6}v_0 + \dfrac{1}{6}v_1, & i = 0, \\ \dfrac{1}{12}(v_{i-1} + 10v_i + v_{i+1}), & 1 \leqslant i \leqslant m-1, \\ \dfrac{5}{6}v_m + \dfrac{1}{6}v_{m-1}, & i = m. \end{cases}$$

设 (1.67)–(1.70) 存在解 $w(x,t) \in C^{6,4}([0,L] \times [0,T])$. 定义网格函数

$$U_i^k = u(x_i, t_k), \quad w_i^k = W(x_i, t_k), \quad 0 \leqslant i \leqslant m, \quad 0 \leqslant k \leqslant n.$$

由 (1.67) 和 (1.69) 可得

$$w_{xxx}(0,t) = 0, \quad w_{xxx}(L,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{1.71}$$

$$w_{xxxxx}(0,t) = 0, \quad w_{xxxxx}(L,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{1.72}$$

在点 $(x_i, t_{k+\frac{1}{2}})$ 处考虑方程 (1.67), 有

$$w_t(x_i, t_{k+\frac{1}{2}}) - \nu w_{xx}(x_i, t_{k+\frac{1}{2}}) = 0, \quad 0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n-1.$$

应用引理 1.2, 可得

$$\begin{aligned} &w_t(x_i, t_{k+\frac{1}{2}}) - \frac{\nu}{2}[w_{xx}(x_i, t_k) + w_{xx}(x_i, t_{k+1})] = (R_{xt}w)_i^{k+\frac{1}{2}}, \\ &0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n-1, \end{aligned}$$

其中

$$(R_{xt}w)_i^{k+\frac{1}{2}} = -\frac{\nu\tau^2}{8}\frac{\partial^4 w}{\partial x^2 \partial t^2}(x_i, t_k + \eta_i^k \tau), \quad \eta_i^k \in (0,1),\ 0 \leqslant i \leqslant m.$$

用算子 $\mathcal{A}$ 作用上述等式的两边, 得

$$\begin{aligned} \mathcal{A}w_t(x_i, t_{k+\frac{1}{2}}) - \frac{\nu}{2}[\mathcal{A}w_{xx}(x_i, t_k) + \mathcal{A}w_{xx}(x_i, t_{k+1})] = \mathcal{A}(R_{xt}w)_i^{k+\frac{1}{2}}, \\ 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \end{aligned}$$

应用引理 1.4 并注意到 (1.71)–(1.72), 得到

$$\mathcal{A}\delta_t W_0^{k+\frac{1}{2}} - \nu\left(\frac{2}{h}\delta_x W_{\frac{1}{2}}^{k+\frac{1}{2}}\right) = R_0^{k+\frac{1}{2}}, \quad 0 \leqslant k \leqslant n-1, \tag{1.73}$$

$$\mathcal{A}\delta_t W_i^{k+\frac{1}{2}} - \nu\,\delta_x^2 W_i^{k+\frac{1}{2}} = R_i^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{1.74}$$

$$\mathcal{A}\delta_t W_m^{k+\frac{1}{2}} - \nu\left(-\frac{2}{h}\delta_x W_{m-\frac{1}{2}}^{k+\frac{1}{2}}\right) = R_m^{k+\frac{1}{2}}, \quad 0 \leqslant k \leqslant n-1, \tag{1.75}$$

存在常数 c_8 使得

$$|R_i^{k+\frac{1}{2}}| \leqslant c_8(\tau^2 + h^4), \quad 0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n-1. \tag{1.76}$$

注意到初值条件

$$W_i^0 = \widetilde{\varphi}(x_i), \quad 0 \leqslant i \leqslant m. \tag{1.77}$$

在 (1.73)–(1.75) 中略去小量项, 对 (1.67)–(1.69) 建立如下差分格式

$$\mathcal{A}\delta_t w_0^{k+\frac{1}{2}} - \nu\left(\frac{2}{h}\delta_x w_{\frac{1}{2}}^{k+\frac{1}{2}}\right) = 0, \quad 0 \leqslant k \leqslant n-1, \tag{1.78}$$

$$\mathcal{A}\delta_t w_i^{k+\frac{1}{2}} - \nu\delta_x^2 w_i^{k+\frac{1}{2}} = 0, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{1.79}$$

$$\mathcal{A}\delta_t w_m^{k+\frac{1}{2}} - \nu\left(-\frac{2}{h}\delta_x w_{m-\frac{1}{2}}^{k+\frac{1}{2}}\right) = 0, \quad 0 \leqslant k \leqslant n-1, \tag{1.80}$$

$$w_i^0 = \widetilde{\varphi}(x_i), \quad 0 \leqslant i \leqslant m. \tag{1.81}$$

1.4.3 差分格式解的存在性和唯一性

引理 1.5 设 $v = (v_0, v_1, \cdots, v_m) \in \mathcal{U}_h$, 则有

$$(\mathcal{A}v, v) = \|v\|^2 - \frac{h^2}{12}|v|_1^2,$$

$$\frac{2}{3}\|v\|^2 \leqslant (\mathcal{A}v, v) \leqslant \|v\|^2.$$

证明 由

$$(\mathcal{A}v)_i = \begin{cases} v_0 + \dfrac{h}{6}\delta_x v_{\frac{1}{2}}, & i = 0, \\ v_i + \dfrac{h^2}{12}\delta_x^2 v_i, & 1 \leqslant i \leqslant m-1, \\ v_m - \dfrac{h}{6}\delta_x v_{m-\frac{1}{2}}, & i = m. \end{cases}$$

得到

$$\begin{aligned}(\mathcal{A}v, v) =& h\left[\frac{1}{2}(\mathcal{A}v_0)v_0 + \sum_{i=1}^{m-1}(\mathcal{A}v_i)v_i + \frac{1}{2}(\mathcal{A}v_m)v_m\right] \\ =& h\left[\frac{1}{2}\left(v_0 + \frac{h}{6}\delta_x v_{\frac{1}{2}}\right)v_0 + \sum_{i=1}^{m-1}\left(v_i + \frac{h^2}{12}\delta_x^2 v_i\right)v_i + \frac{1}{2}\left(v_m - \frac{h}{6}\delta_x v_{m-\frac{1}{2}}\right)v_m\right] \\ =& h\left(\frac{1}{2}v_0^2 + \sum_{i=1}^{m-1}v_i^2 + \frac{1}{2}v_m^2\right)\end{aligned}$$

$$+\frac{h^2}{12}\left[(\delta_x v_{\frac{1}{2}})v_0+h\sum_{i=1}^{m-1}(\delta_x^2 v_i)v_i-(\delta_x v_{m-\frac{1}{2}})v_m\right]$$
$$=\|v\|^2-\frac{h^2}{12}\cdot h\sum_{i=0}^{m-1}(\delta_x v_{i+\frac{1}{2}})^2$$
$$=\|v\|^2-\frac{h^2}{12}|v|_1^2.$$

易知

$$(\mathcal{A}v,v)\leqslant\|v\|^2.$$

由

$$|v|_1^2\leqslant\frac{4}{h^2}\|v\|^2,$$

易得

$$(\mathcal{A}v,v)\geqslant\|v\|^2-\frac{h^2}{12}\cdot\frac{4}{h^2}\|v\|=\frac{2}{3}\|v\|^2.$$ □

定义 $\mathcal{U}_h$ 上的范数

$$\|v\|_{\mathcal{A}}=\sqrt{(\mathcal{A}v,v)}.$$

由引理 1.5 知 $\|v\|_{\mathcal{A}}$ 和 $\|v\|$ 等价.

定理 1.11　差分格式 (1.78)–(1.81) 的解是存在唯一的.

证明　第 0 层的值 w^0 是由 (1.81) 给定. 设已得到第 k 层的值 w^k, 则由 (1.78)–(1.80) 可得关于第 $k+1$ 层值 w^{k+1} 的线性方程组. 考虑其齐次方程组

$$\frac{1}{\tau}\mathcal{A}w_0^{k+1}-\nu\frac{1}{h}\delta_x w_{\frac{1}{2}}^{k+1}=0, \tag{1.82}$$
$$\frac{1}{\tau}\mathcal{A}w_i^{k+1}-\nu\frac{1}{2}\delta_x^2 w_i^{k+1}=0,\quad 1\leqslant i\leqslant m-1, \tag{1.83}$$
$$\frac{1}{\tau}\mathcal{A}w_m^{k+\frac{1}{2}}-\nu\Big(-\frac{1}{h}\delta_x w_{m-\frac{1}{2}}^{k+1}\Big)=0. \tag{1.84}$$

用 $\frac{1}{2}hw_0^{k+1}$ 与 (1.82) 相乘, 用 hw_i^{k+1} 与 (1.83) 相乘, 用 $\frac{1}{2}hw_m^{k+\frac{1}{2}}$ 与 (1.84) 相乘, 并将所得结果相加, 得

$$\frac{1}{\tau}(\mathcal{A}w^{k+1},w^{k+1})-\frac{1}{2}\nu\left[w_0^{k+\frac{1}{2}}\delta_x w_{\frac{1}{2}}^{k+1}+h\sum_{i=1}^{m-1}w_i^{k+1}\delta_x^2 w_i^{k+1}-w_m^{k+1}\delta_x w_{m-\frac{1}{2}}^{k+1}\right]=0.$$

由上式易得

$$\frac{1}{\tau}\|w^{k+1}\|_{\mathcal{A}}^2+\frac{1}{2}\nu|w^{k+1}|_1^2=0.$$

因而

$$w^{k+1}=0,$$

即 (1.82)–(1.84) 只有零解. 于是 (1.78)–(1.80) 唯一确定 w^{k+1}. □

1.4.4 差分格式解的收敛性

定理 1.12 设 $\{W_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 是问题 (1.67)–(1.69) 的解, $\{w_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 是差分格式 (1.78)–(1.81) 的解. 令

$$e_i^k = W_i^k - w_i^k, \quad 0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n.$$

则存在常数 c_9, c_{10} 使得

$$\|e^k\| \leqslant c_9(\tau^2 + h^4), \quad 0 \leqslant k \leqslant n, \tag{1.85}$$

$$|e^k|_1 \leqslant c_{10}(\tau^2 + h^4), \quad 0 \leqslant k \leqslant n, \tag{1.86}$$

$$\|e^k\|_\infty \leqslant \frac{\sqrt{L}}{2} c_{10}(\tau^2 + h^4), \quad 0 \leqslant k \leqslant n. \tag{1.87}$$

证明 将 (1.73)–(1.75), (1.77) 和 (1.78)–(1.81) 相减, 得到误差方程组

$$\mathcal{A}\delta_t e_0^{k+\frac{1}{2}} - \nu \cdot \frac{2}{h}\delta_x e_{\frac{1}{2}}^{k+\frac{1}{2}} = R_0^{k+\frac{1}{2}}, \quad 0 \leqslant k \leqslant n-1, \tag{1.88}$$

$$\mathcal{A}\delta_t e_i^{k+\frac{1}{2}} - \nu\delta_x^2 e_i^{k+\frac{1}{2}} = R_i^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{1.89}$$

$$\mathcal{A}\delta_t e_m^{k+\frac{1}{2}} - \nu\Big(-\frac{2}{h}\delta_x e_{m-\frac{1}{2}}^{k+\frac{1}{2}}\Big) = R_m^{k+\frac{1}{2}}, \quad 0 \leqslant k \leqslant n-1, \tag{1.90}$$

$$e_i^0 = 0, \quad 0 \leqslant i \leqslant m. \tag{1.91}$$

(I) 用 $\frac{1}{2}he_0^{k+\frac{1}{2}}$ 与 (1.88) 相乘, 用 $he_i^{k+\frac{1}{2}}$ 与 (1.89) 相乘, 用 $\frac{1}{2}he_m^{k+\frac{1}{2}}$ 与 (1.90) 相乘, 并将结果相加, 得到

$$\frac{1}{2\tau}(\|e^{k+1}\|_{\mathcal{A}}^2 - \|e^k\|_{\mathcal{A}}^2) - \nu\Big(e_0^{k+\frac{1}{2}}\delta_x e_{\frac{1}{2}}^{k+\frac{1}{2}} + h\sum_{i=1}^{m-1} e_i^{k+\frac{1}{2}}\delta_x^2 e_i^{k+\frac{1}{2}} - e_m^{k+\frac{1}{2}}\delta_x e_{m-\frac{1}{2}}^{k+\frac{1}{2}}\Big)$$
$$= (R^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}), \quad 0 \leqslant k \leqslant n-1,$$

即

$$\frac{1}{2\tau}(\|e^{k+1}\|_{\mathcal{A}}^2 - \|e^k\|_{\mathcal{A}}^2) + \nu|e^{k+\frac{1}{2}}|_1^2 = (R^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}), \quad 0 \leqslant k \leqslant n-1.$$

对上式右端用 Cauchy-Schwarz 不等式, 并应用引理 1.5, 得

$$\begin{aligned}
&\frac{1}{2\tau}(\|e^{k+1}\|_{\mathcal{A}}^2 - \|e^k\|_{\mathcal{A}}^2)\\
\leqslant&\|R^{k+\frac{1}{2}}\| \cdot \|e^{k+\frac{1}{2}}\|\\
\leqslant&\sqrt{\frac{3}{2}}\|R^{k+\frac{1}{2}}\| \cdot \|e^{k+\frac{1}{2}}\|_{\mathcal{A}}\\
\leqslant&\sqrt{\frac{3}{2}}\|R^{k+\frac{1}{2}}\| \cdot \frac{\|e^{k+1}\|_{\mathcal{A}} + \|e^k\|_{\mathcal{A}}}{2},
\end{aligned}$$

两边约去 $\frac{1}{2}(\|e^{k+1}\|_{\mathcal{A}}+\|e^k\|_{\mathcal{A}})$, 得到

$$\frac{1}{\tau}(\|e^{k+1}\|_{\mathcal{A}}-\|e^k\|_{\mathcal{A}})\leqslant\sqrt{\frac{3}{2}}\|R^{k+\frac{1}{2}}\|,\quad 0\leqslant k\leqslant n-1.$$

因而

$$\|e^{k+1}\|_{\mathcal{A}}\leqslant\|e^0\|_{\mathcal{A}}+\sqrt{\frac{3}{2}}\tau\sum_{l=0}^{k}\|R^{l+\frac{1}{2}}\|,\quad 0\leqslant k\leqslant n-1.$$

由 (1.76) 和 (1.91), 得到

$$\|e^{k+1}\|_A\leqslant\sqrt{\frac{3}{2}}(k+1)\tau\sqrt{L}c_8(\tau^2+h^4)\leqslant\sqrt{\frac{3}{2}}T\sqrt{L}c_8(\tau^2+h^4),\quad 0\leqslant k\leqslant n-1.$$

再次应用引理 1.5, 得

$$\|e^k\|\leqslant\sqrt{\frac{3}{2}}\|e^k\|_A\leqslant\frac{3}{2}T\sqrt{L}c_8(\tau^2+h^2),\quad 1\leqslant k\leqslant n.$$

(II) 用 $\frac{1}{2}h\delta_te_0^{k+\frac{1}{2}}$ 与 (1.88) 相乘, 用 $h\delta_te_i^{k+\frac{1}{2}}$ 与 (1.89) 相乘, 用 $\frac{1}{2}h\delta_te_m^{k+\frac{1}{2}}$ 与 (1.90) 相乘, 并将结果相加, 得

$$\begin{aligned}&\|\delta_te^{k+\frac{1}{2}}\|_{\mathcal{A}}^2-\nu\Big[(\delta_xe_{\frac{1}{2}}^{k+\frac{1}{2}})\delta_te_0^{k+\frac{1}{2}}+h\sum_{i=1}^{m-1}(\delta_x^2e_i^{k+\frac{1}{2}})\delta_te_i^{k+\frac{1}{2}}-(\delta_xe_{m-\frac{1}{2}}^{k+\frac{1}{2}})\delta_te_m^{k+\frac{1}{2}}\Big]\\&=(R^{k+\frac{1}{2}},\delta_te^{k+\frac{1}{2}}),\quad 0\leqslant k\leqslant n-1,\end{aligned}$$

即

$$\begin{aligned}&\|\delta_te^{k+\frac{1}{2}}\|_{\mathcal{A}}^2+\nu\cdot\frac{1}{2\tau}(|e^{k+1}|_1^2-|e^k|_1^2)\\&=(R^{k+\frac{1}{2}},\delta_te^{k+\frac{1}{2}})\\&\leqslant\|R^{k+\frac{1}{2}}\|\cdot\|\delta_te^{k+\frac{1}{2}}\|\\&\leqslant\frac{2}{3}\|\delta_te^{k+\frac{1}{2}}\|^2+\frac{3}{8}\|R^{k+\frac{1}{2}}\|^2\\&\leqslant\|\delta_te^{k+\frac{1}{2}}\|_{\mathcal{A}}^2+\frac{3}{8}\|R^{k+\frac{1}{2}}\|^2,\quad 0\leqslant k\leqslant n-1.\end{aligned}$$

因而

$$\frac{1}{2\tau}(|e^{k+1}|_1^2-|e^k|_1^2)\leqslant\frac{3}{8\nu}\|R^{k+\frac{1}{2}}\|^2,\quad 0\leqslant k\leqslant n-1.$$

递推可得

$$|e^{k+1}|_1^2\leqslant|e^0|_1^2+\frac{3}{4\nu}\tau\sum_{l=0}^{k}\|R^{l+\frac{1}{2}}\|^2\leqslant\frac{3}{4\nu}(k+1)\tau Lc_8^2(\tau^2+h^4)^2,\quad 0\leqslant k\leqslant n-1.$$

于是

$$|e^k|_1 \leqslant \frac{1}{2}\sqrt{\frac{3}{\nu}TL}\,c_8(\tau^2+h^4), \quad 1 \leqslant k \leqslant n.$$

(III) 由 (1.86) 和引理 1.1(b) 得

$$\|e^k\|_\infty \leqslant \frac{\sqrt{L}}{2}|e^k|_1 \leqslant \frac{\sqrt{L}}{2}c_{10}(\tau^2+h^4), \quad 1 \leqslant k \leqslant n. \qquad \square$$

1.4.5 原问题解的计算

设 $g(x) \in C^5[x_0, x_m]$, 则存在常数 $\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_6$ 使得

$$g'(x_1) = \hat{c}_1\delta_x g_{\frac{1}{2}} + \hat{c}_2\delta_x g_{\frac{3}{2}} + \hat{c}_3\delta_x g_{\frac{5}{2}} + \hat{c}_4\delta_x g_{\frac{7}{2}} + O(h^4), \tag{1.92}$$

$$g'(x_i) = \hat{c}_5\delta_x g_{i-\frac{3}{2}} + \hat{c}_6\delta_x g_{i-\frac{1}{2}} + \hat{c}_6\delta_x g_{i+\frac{1}{2}} + \hat{c}_5\delta_x g_{i+\frac{3}{2}} + O(h^4),$$
$$2 \leqslant i \leqslant m-2, \tag{1.93}$$

$$g'(x_{m-1}) = \hat{c}_1\delta_x g_{m-\frac{1}{2}} + \hat{c}_2\delta_x g_{m-\frac{3}{2}} + \hat{c}_3\delta_x g_{m-\frac{5}{2}} + \hat{c}_4\delta_x g_{m-\frac{7}{2}} + O(h^4). \tag{1.94}$$

由变换 (1.66) 有

$$u(x_i, t_k) = -2\nu\frac{w_x(x_i, t_k)}{w(x_i, t_k)}.$$

利用 (1.92)–(1.94) 可得

$$U_1^k = -\frac{2\nu}{W_1^k}\left(\hat{c}_1\delta_x W_{\frac{1}{2}}^k + \hat{c}_2 W_{\frac{3}{2}}^k + \hat{c}_3\delta_x W_{\frac{5}{2}}^k + \hat{c}_4\delta_x W_{\frac{7}{2}}^k\right) + \hat{R}_1^k, \quad 1 \leqslant k \leqslant n, \tag{1.95}$$

$$U_i^k = -\frac{2\nu}{W_i^k}\left(\hat{c}_5\delta_x W_{i-\frac{3}{2}}^k + \hat{c}_6\delta_x W_{i-\frac{1}{2}}^k + \hat{c}_6\delta_x \hat{W}_{i+\frac{1}{2}}^k + \hat{c}_5\delta_x W_{i+\frac{3}{2}}^k\right) + \hat{R}_i^k,$$
$$2 \leqslant i \leqslant m-2,\ 1 \leqslant k \leqslant n. \tag{1.96}$$

$$U_{m-1}^k = -\frac{2\nu}{W_{m-1}^k}\left(\hat{c}_4\delta_x W_{m-\frac{7}{2}}^k + \hat{c}_3\delta_x W_{m-\frac{5}{2}}^k + \hat{c}_2\delta_x W_{m-\frac{3}{2}}^k + \hat{c}_1\delta_x W_{m-\frac{1}{2}}^k\right) + \hat{R}_{m-1}^k,$$
$$1 \leqslant k \leqslant n, \tag{1.97}$$

存在常数 c_{11} 使得

$$|\hat{R}_i^k| \leqslant c_{11}h^4, \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n. \tag{1.98}$$

在 (1.95)–(1.97) 中略去小量项, 得到如下计算格式

$$u_1^k = -\frac{2\nu}{w_1^k}\left(\hat{c}_1\delta_x w_{\frac{1}{2}}^k + \hat{c}_2\delta_x w_{\frac{3}{2}}^k + \hat{c}_3\delta_x w_{\frac{5}{2}}^k + \hat{c}_4\delta_x w_{\frac{7}{2}}^k\right), \quad 1 \leqslant k \leqslant n, \tag{1.99}$$

$$u_i^k = -\frac{2\nu}{w_i^k}\left(\hat{c}_5\delta_x w_{i-\frac{3}{2}}^k + \hat{c}_6\delta_x w_{i-\frac{1}{2}}^k + \hat{c}_6\delta_x w_{i+\frac{1}{2}}^k + \hat{c}_5\delta_x w_{i+\frac{3}{2}}^k\right),$$
$$2 \leqslant i \leqslant m-2,\ 1 \leqslant k \leqslant n, \tag{1.100}$$

$$u_{m-1}^k = -\frac{2\nu}{w_{m-1}^k}\left(\hat{c}_4\delta_x w_{m-\frac{7}{2}}^k + \hat{c}_3\delta_x w_{m-\frac{5}{2}}^k + \hat{c}_2\delta_x w_{m-\frac{3}{2}}^k + \hat{c}_1\delta_x w_{m-\frac{1}{2}}^k\right),$$
$$1 \leqslant k \leqslant n. \tag{1.101}$$

利用定理 1.12 的结果可以证明

$$\sqrt{h\sum_{i=1}^{m-1}(U_i^k - u_i^k)^2} \leqslant c_{12}(\tau^2 + h^4), \quad 1 \leqslant k \leqslant n.$$

1.5　小结与延拓

本章讨论了 Burgers 方程的差分方法. 首先证明了问题 (1.1)–(1.3) 的解满足能量守恒性. 接着在 1.2 节和 1.3 节分别介绍了二层非线性差分格式和三层线性化差分格式. 证明了差分格式解的存在性、唯一性、有界性和收敛性. 三层线性化差分格式的有关结果主要取材于 [30].

对于问题 (1.1)–(1.3) 可建立如下二层线性化差分格式

$$\delta_t u_i^{k+\frac{1}{2}} + \frac{1}{2}(u_i^k \Delta_x u_i^{k+1} + u_i^{k+1}\Delta_x u_i^k) = \nu\delta_x^2 u_i^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{1.102}$$

$$u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{1.103}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{1.104}$$

可以证明差分格式 (1.102)–(1.104) 是唯一可解的, 在无穷范数下关于时间步长和空间步长均是二阶收敛的.

文 [39] 研究了二维 Burgers 方程的二阶差分方法.

应用 Hopf-Cole 变换可将 Burgers 方程的初边值问题 (1.1)–(1.3) 变为线性的热传导方程的初边值问题 (1.67)–(1.69). 对 (1.67)–(1.69), 我们建立了紧致差分格式 (1.78)–(1.81). 证明了 (1.78)–(1.81) 解的存在性和唯一性以及解在无穷范数下关于时间步长 2 阶、空间步长 4 阶的收敛性. 如果在 (1.78)–(1.81) 中用单位算子 $\mathcal{I}$ 代替平均值算子 $\mathcal{A}$, 得到如下格式

$$\begin{aligned}
&\delta_t w_0^{k+\frac{1}{2}} - \nu\Big(\frac{2}{h}\delta_x w_{\frac{1}{2}}^{k+\frac{1}{2}}\Big) = 0, \quad 0 \leqslant k \leqslant n-1,\\
&\delta_t w_i^{k+\frac{1}{2}} - \nu\,\delta_x^2 w_i^{k+\frac{1}{2}} = 0, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1,\\
&\delta_t w_m^{k+\frac{1}{2}} - \Big(-\frac{2}{h}\delta_x w_{m-\frac{1}{2}}^{k+\frac{1}{2}}\Big) = 0, \quad 0 \leqslant k \leqslant n-1,\\
&w_i^0 = \widetilde{\varphi}(x_i), \quad 0 \leqslant i \leqslant m.
\end{aligned}$$

该差分格式是唯一可解的, 在无穷范数下关于时间步长和空间步长均是二阶收敛的.

我们借助于 Browder 定理证明了非线性方程组 (1.19)–(1.20) 解的存在性. 与 Browder 定理相伴的还有一个 Leray-Schauder 定理 [43].

设 H 是一个有限维内积空间, $\|\cdot\|$ 是导出范数. 考虑 $H \to H$ 的算子 $T_\lambda(w)$, 其中 $\lambda \in [0,1]$ 为参数. 如果 $T_\lambda(w)$ 满足如下条件:

(a) $T_\lambda(w)$ 是 H 上的连续算子;

(b) $T_0(w)=0$ 有唯一解;

(c) $T_\lambda(w)=0$ 的一切可能解有一致的界,

则对任意 $\lambda \in [0,1], T_\lambda(w)=0$ 存在解. 特别地, $T_1(w)=0$ 存在解.

现在用上述结论来证明定理 1.4, 即证明 (1.19)–(1.20) 存在解.

令 $H=\mathring{\mathcal{U}}_h$. 对任意的 $w \in \mathring{\mathcal{U}}_h$, 定义

$$
\begin{aligned}
&T_\lambda(w)_i = \frac{2}{\tau}(w_i - u_i^k) + \lambda\psi(w,w)_i - \nu\delta_x^2 w_i, \quad 1 \leqslant i \leqslant m-1,\\
&T_\lambda(w)_0 = 0,\\
&T_\lambda(w)_m = 0.
\end{aligned}
$$

易知 (a) $T_\lambda(w)$ 是连续的; (b) $T_0(w)_i=0,\ i=1,2,\cdots,m-1$ 是一个严格对角占优的三对角线性方程组, 故有唯一解. 现在来检验 (c). 设 w 是 $T_\lambda(w)=0$ 可能的解. 用 w 和 $T_\lambda(w)=0$ 作内积, 得

$$
\frac{2}{\tau}\big((w,w)-(u^k,w)\big)+\lambda\big(\psi(w,w),w\big)-\nu\big(\delta_x^2 w,w\big)=0.
$$

利用

$$
(\psi(w,w),w)=0, \qquad -(\delta_x^2 w,w)=|w|_1^2,
$$

得

$$
\frac{2}{\tau}\big(\|w\|^2-(u^k,w)\big)+\nu|w|_1^2=0.
$$

于是

$$
\|w\|^2 \leqslant (u^k,w) \leqslant \|u^k\|\cdot\|w\|.
$$

易知

$$
\|w\| \leqslant \|u^k\|.
$$

条件 (c) 满足. 由 Leray-Schauder 定理. (1.19)–(1.20) 存在解.

第2章 正则长波方程的差分方法

2.1 引 言

正则长波 (Regularized Long Wave) 方程是非线性长波方程的一种表述形式. 在进行非线性扩散波的研究时, 正则长波方程因其能描述大量的物理现象, 如浅水波和离子波而占有重要的地位.

本章考虑正则长波方程初边值问题

$$u_t - \mu u_{xxt} + \gamma u u_x + u_x = 0, \quad 0 < x < L, \quad 0 < t \leqslant T, \tag{2.1}$$

$$u(x,0) = \varphi(x), \quad 0 < x < L, \tag{2.2}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{2.3}$$

其中 μ, γ 为正常数, $\varphi(0) = \varphi(L) = 0$.

在介绍差分方法之前, 我们先用能量方法给出问题 (2.1)–(2.3) 解的先验估计式.

定理 2.1 设 $u(x,t)$ 为问题 (2.1)–(2.3) 的解, 记

$$E(t) = \int_0^L u^2(x,t)\mathrm{d}x + \mu \int_0^L u_x^2(x,t)\mathrm{d}x, \tag{2.4}$$

则有

$$E(t) \equiv E(0), \quad 0 < t \leqslant T. \tag{2.5}$$

证明 用 u 乘以方程 (2.1) 的两边, 对 x 从 0 到 L 积分, 得

$$\int_0^L u u_t \mathrm{d}x - \mu \int_0^L u u_{xxt} \mathrm{d}x + \gamma \int_0^L u^2 u_x dx + \int_0^L u u_x \mathrm{d}x = 0.$$

利用分部求积公式以及边界条件 (2.3), 可得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2 \mathrm{d}x + \frac{1}{2}\mu \frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u_x^2 \mathrm{d}x = 0,$$

即

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = 0, \quad 0 < t \leqslant T.$$

因而

$$E(t) \equiv E(0), \quad 0 < t \leqslant T. \qquad \square$$

称 (2.5) 为能量守恒律.

2.2 二层非线性差分格式

以下设问题 (2.1)–(2.3) 的解 $u(x,t) \in C_{x,t}^{4,3}([0,L]\times[0,T])$.

与第 2 章一样, 引进记号

$$\psi(v,w)_i = \frac{1}{3}[v_i\Delta_x w_i + \Delta_x(vw)_i], \quad 1 \leqslant i \leqslant m-1.$$

2.2.1 差分格式的建立

在结点 $(x_i, t_{k+\frac{1}{2}})$ 处考虑方程 (2.1), 得到

$$u_t(x_i,t_{k+\frac{1}{2}}) - \mu u_{xxt}(x_i,t_{k+\frac{1}{2}}) + \gamma u(x_i,t_{k+\frac{1}{2}})u_x(x_i,t_{k+\frac{1}{2}}) + u_x(x_i,t_{k+\frac{1}{2}}) = 0,$$
$$1 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{2.6}$$

应用数值微分公式, 可得

$$\delta_t U_i^{k+\frac{1}{2}} - \mu\delta_t\delta_x^2 U_i^{k+\frac{1}{2}} + \gamma\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}})_i + \Delta_x U_i^{k+\frac{1}{2}} = R_i^{k+\frac{1}{2}},$$
$$1 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{2.7}$$

存在正常数 c_1 使得

$$|R_i^{k+\frac{1}{2}}| \leqslant c_1(\tau^2+h^2), \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{2.8}$$

在 (2.7) 中略去小量项 $R_i^{k+\frac{1}{2}}$, 并注意到初边值条件

$$U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{2.9}$$

$$U_0^k = 0, \quad U_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{2.10}$$

对 (2.1)–(2.3) 建立如下差分格式

$$\delta_t u_i^{k+\frac{1}{2}} - \mu\delta_t\delta_x^2 u_i^{k+\frac{1}{2}} + \gamma\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i + \Delta_x u_i^{k+\frac{1}{2}} = 0,$$
$$1 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{2.11}$$

$$u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{2.12}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{2.13}$$

2.2.2 差分格式解的存在性

差分格式 (2.11)–(2.13) 是一个二层非线性差分格式. 当 k 层的值 u^k 已确定时, 可将其看成关于平均值 $\{u_i^{k+\frac{1}{2}} \,|\, 0 \leqslant i \leqslant m\}$ 的非线性方程组. 当求得 $\{u_i^{k+\frac{1}{2}} \,|\, 0 \leqslant i \leqslant m\}$ 时,

$$u_i^{k+1} = 2u_i^{k+\frac{1}{2}} - u_i^k, \quad 0 \leqslant i \leqslant m.$$

定理 2.2 差分格式 (2.11)–(2.13) 存在解.

证明 记

$$w_i = u_i^{k+\frac{1}{2}}, \quad 0 \leqslant i \leqslant m.$$

则有

$$\frac{2}{\tau}(w_i - u_i^k) - \frac{2\mu}{\tau}\delta_x^2(w_i - u_i^k) + \gamma\psi(w,w)_i + \Delta_x w_i = 0, \quad 1 \leqslant i \leqslant m-1, \tag{2.14}$$

$$w_0 = 0, \quad w_m = 0. \tag{2.15}$$

令

$$\Pi(w)_i = \frac{2}{\tau}(w_i - u_i^k) - \frac{2\mu}{\tau}\delta_x^2(w_i - u_i^k) + \gamma\psi(w,w)_i + \Delta_x w_i, \quad 1 \leqslant i \leqslant m-1,$$

则

$$\begin{aligned}(\Pi(w), w) &= \frac{2}{\tau}[(w,w) - (u^k, w)] - \frac{2\mu}{\tau}(\delta_x^2(w - u^k), w) + \gamma(\psi(w,w), w) + (\Delta_x w, w)\\ &= \frac{2}{\tau}[\|w\|^2 - (u^k, w)] + \frac{2\mu}{\tau}[|w|_1^2 - (\delta_x u^k, \delta_x w)]\\ &\geqslant \frac{2}{\tau}\left[\|w\|^2 - \frac{1}{2}(\|u^k\|^2 + \|w\|^2)\right] + \frac{2\mu}{\tau}\left[|w|_1^2 - \left(|w|_1^2 + \frac{1}{4}|u^k|_1^2\right)\right]\\ &= \frac{1}{\tau}\left[\|w\|^2 - \left(\|u^k\|^2 + \frac{\mu}{2}|u^k|_1^2\right)\right].\end{aligned}$$

当 $\|w\| = \|u^k\|^2 + \frac{\mu}{2}|u^k|_1^2$ 时, $(\Pi(w), w) \geqslant 0$. 由 Browder 定理 (定理 1.3) 知 (2.14)–(2.15) 存在解. □

2.2.3 差分格式解的守恒性和有界性

定理 2.3 设 $\{u_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (2.11)–(2.13) 的解. 记

$$E^k = \|u^k\|^2 + \mu|u^k|_1^2,$$

则有

$$E^k \equiv E^0, \quad 1 \leqslant k \leqslant n.$$

证明 用 $hu_i^{k+\frac{1}{2}}$ 乘以 (2.11) 的两边, 并对 i 从 1 到 $m-1$ 求和, 得到

$$\begin{aligned}&h\sum_{i=1}^{m-1} u_i^{k+\frac{1}{2}}\delta_t u_i^{k+\frac{1}{2}} - \mu h\sum_{i=1}^{m-1} u_i^{k+\frac{1}{2}}\delta_x^2\delta_t u_i^{k+\frac{1}{2}}\\ &+\gamma h\sum_{i=1}^{m-1} u_i^{k+\frac{1}{2}}\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i + h\sum_{i=1}^{m-1} u_i^{k+\frac{1}{2}}\Delta_x u_i^{k+\frac{1}{2}} = 0.\end{aligned} \tag{2.16}$$

由 (2.13) 可得

$$u_0^{k+\frac{1}{2}} = 0, \quad u_m^{k+\frac{1}{2}} = 0.$$

现在分析 (2.16) 中的每一项.

$$h\sum_{i=1}^{m-1} u_i^{k+\frac{1}{2}} \delta_t u_i^{k+\frac{1}{2}} = \frac{1}{2\tau} h \sum_{i=1}^{m-1} [(u_i^{k+1})^2 - (u_i^k)^2] = \frac{1}{2\tau}(\|u^{k+1}\|^2 - \|u^k\|^2),$$

$$\begin{aligned}-h\sum_{i=1}^{m-1} u_i^{k+\frac{1}{2}} \delta_x^2 \delta_t u_i^{k+\frac{1}{2}} &= h\sum_{i=0}^{m-1} (\delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}})(\delta_x \delta_t u_{i+\frac{1}{2}}^{k+\frac{1}{2}}) \\ &= \frac{1}{2\tau} h \sum_{i=0}^{m-1} [(\delta_x u_{i+\frac{1}{2}}^{k+1})^2 - (\delta_x u_{i+\frac{1}{2}}^k)^2] = \frac{1}{2\tau}(|u^{k+1}|_1^2 - |u^k|_1^2),\end{aligned}$$

$$h\sum_{i=1}^{m-1} u_i^{k+\frac{1}{2}} \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i = (\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}), u^{k+\frac{1}{2}}) = 0,$$

$$h\sum_{i=1}^{m-1} u_i^{k+\frac{1}{2}} \Delta_x u_i^{k+\frac{1}{2}} = 0.$$

将以上四式代入 (2.16), 得

$$\frac{1}{2\tau}(\|u^{k+1}\|^2 - \|u^k\|^2) + \mu \cdot \frac{1}{2\tau}(|u^{k+1}|_1^2 - |u^k|_1^2) = 0, \quad 0 \leqslant k \leqslant n-1,$$

即

$$\frac{1}{2\tau}(E^{k+1} - E^k) = 0, \quad 0 \leqslant k \leqslant n-1.$$

于是

$$E^k \equiv E^0, \quad 1 \leqslant k \leqslant n.$$

□

2.2.4 差分格式解的唯一性

定理 2.4 存在 $\tau_0 > 0$, 当 $\tau < \tau_0$ 时, 差分格式 (2.11)–(2.13) 的解是唯一的.

证明 由定理 2.2 的证明可知只要证明 (2.14)–(2.15) 的解是唯一的.

设差分格式 (2.14)–(2.15) 还有另一解 $\{v_i\}$, 即 $\{v_i\}$ 满足

$$\frac{2}{\tau}(v_i - u_i^k) - \mu \cdot \frac{2}{\tau}\delta_x^2(v_i - u_i^k) + \gamma\psi(v,v)_i + \Delta_x v_i = 0, \quad 1 \leqslant i \leqslant m-1, \tag{2.17}$$

$$v_0 = 0, \quad v_m = 0. \tag{2.18}$$

令

$$z_i = w_i - v_i, \quad 0 \leqslant i \leqslant m.$$

将 (2.14)–(2.15) 和 (2.17)–(2.18) 相减, 得

$$\frac{2}{\tau}z_i-\frac{2\mu}{\tau}\delta_x^2z_i+\gamma\big[\psi(w,w)_i-\psi(v,v)_i\big]+\Delta_xz_i=0,\quad 1\leqslant i\leqslant m-1, \tag{2.19}$$

$$z_0=0,\quad z_m=0. \tag{2.20}$$

记

$$c_2=\sqrt{\|u^0\|^2+\mu|u^0|_1^2}.$$

由定理 2.3 有

$$\|w\|^2+\mu|w|_1^2\leqslant c_2^2. \tag{2.21}$$

注意到

$$\begin{aligned}&\psi(w,w)_i-\psi(v,v)_i\\=&\psi(w,w)_i-\psi(w-z,w-z)_i\\=&\psi(z,w)_i+\psi(w,z)_i-\psi(z,z)_i,\end{aligned}$$

用 z 与 (2.19) 作内积, 得到

$$\frac{2}{\tau}(z,z)-\mu\cdot\frac{2}{\tau}(\delta_x^2z,z)+\gamma[(\psi(z,w),z)+(\psi(w,z),z)-(\psi(z,z),z)]+(\Delta_xz,z)=0.$$

注意到

$$-(\delta_x^2z,z)=|z|_1^2,\qquad(\psi(w,z),z)=0,\qquad(\psi(z,z),z)=0,\qquad(\Delta_xz,z)=0,$$

得

$$\frac{2}{\tau}(\|z\|^2+\mu|z|_1^2)=-\gamma(\psi(z,w),z)\leqslant\gamma\cdot\frac{1}{3}\|z\|_\infty(2\|z\|\cdot|w|_1+|z|_1\|w\|),$$

由 (2.21), 得

$$\frac{2}{\tau}(\|z\|^2+\mu|z|_1^2)\leqslant\frac{\gamma}{3}\|z\|_\infty\left(2\|z\|\frac{c_2}{\sqrt{\mu}}+|z|_1c_2\right).$$

由

$$\|z\|_\infty\leqslant\frac{\sqrt{L}}{2}|z|_1,\qquad\|z\|\leqslant\frac{L}{\sqrt{6}}|z|_1,$$

得

$$\frac{2}{\tau}\mu|z|_1^2\leqslant\frac{\gamma}{3}\cdot\frac{\sqrt{L}}{2}|z|_1\left(\frac{2c_2}{\sqrt{\mu}}\cdot\frac{L}{\sqrt{6}}|z|_1+c_2|z|_1\right).$$

当 $\frac{2}{\tau}\mu > \frac{\gamma}{3}\cdot\frac{\sqrt{L}}{2}\left(\frac{2L}{\sqrt{6\mu}}+1\right)c_2$, 即 $\tau < \frac{2\mu}{\frac{\gamma}{3}\cdot\frac{\sqrt{L}}{2}\left(\frac{2L}{\sqrt{6\mu}}+1\right)c_2}$ 时,

$$|z|_1^2 = 0.$$

从而

$$z = 0. \qquad \square$$

2.2.5 差分格式解的收敛性

定理 2.5 设 $\{U_i^k\,|\,0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为问题 (2.1)–(2.3) 的解, $\{u_i^k\,|\,0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (2.11)–(2.13) 的解. 记

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant m, \quad 0 \leqslant k \leqslant n.$$

则存在常数 c_3 使得

$$\|e^k\|_\infty \leqslant c_3(\tau^2 + h^2), \quad 0 \leqslant k \leqslant n. \tag{2.22}$$

证明 将 (2.7), (2.9), (2.10) 分别和 (2.11)—(2.13) 相减, 得误差方程组

$$\delta_t e_i^{k+\frac{1}{2}} - \mu\delta_t\delta_x^2 e_i^{k+\frac{1}{2}} + \gamma[\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}})_i - \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i]$$
$$+\Delta_x e_i^{k+\frac{1}{2}} = R_i^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{2.23}$$

$$e_i^0 = 0, \quad 1 \leqslant i \leqslant m-1, \tag{2.24}$$

$$e_0^k = 0, \quad e_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{2.25}$$

用 $e^{k+\frac{1}{2}}$ 与 (2.23) 作内积, 得

$$\left(\delta_t e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}\right) - \mu\left(\delta_t\delta_x^2 e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}\right) + \gamma\left(\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}) - \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}), e^{k+\frac{1}{2}}\right)$$
$$+(\Delta_x e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}) = (R^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}), \quad 0 \leqslant k \leqslant n-1. \tag{2.26}$$

注意到

$$\begin{aligned}&\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}})_i - \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i \\ =&\,\psi(U^{k+\frac{1}{2}}, e^{k+\frac{1}{2}})_i + \psi(e^{k+\frac{1}{2}}, U^{k+\frac{1}{2}})_i - \psi(e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}})_i,\end{aligned}$$

可得

$$\begin{aligned}&\left(\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}) - \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}), e^{k+\frac{1}{2}}\right) \\ =&\left(\psi(U^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}), e^{k+\frac{1}{2}}\right) + \left(\psi(e^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}), e^{k+\frac{1}{2}}\right) - \left(\psi(e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}), e^{k+\frac{1}{2}}\right) \\ =&\left(\psi(e^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}), e^{k+\frac{1}{2}}\right).\end{aligned}$$

注意到

$$\left(\delta_t e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}\right) = \frac{1}{2\tau}(\|e^{k+1}\|^2 - \|e^k\|^2),$$
$$-(\delta_t\delta_x^2 e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}) = \frac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2),$$
$$(\Delta_x e^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}) = 0,$$

可得

$$\begin{aligned}
&\frac{1}{2\tau}[(\|e^{k+1}\|^2 + \mu|e^{k+1}|_1^2) - (\|e^k\|^2 + \mu|e^k|_1^2)] \\
=& -\gamma(\psi(e^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}), e^{k+\frac{1}{2}}) + (R^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}) \\
\leqslant& \frac{\gamma}{3}[2\|\Delta_x U^{k+\frac{1}{2}}\|_\infty \|e^{k+\frac{1}{2}}\|^2 + \|U^{k+\frac{1}{2}}\|_\infty \|e^{k+\frac{1}{2}}\| \cdot |e^{k+\frac{1}{2}}|_1] + \frac{1}{2}\|e^{k+\frac{1}{2}}\|^2 + \frac{1}{2}\|R^{k+\frac{1}{2}}\|^2 \\
\leqslant& c_4(\|e^{k+\frac{1}{2}}\|^2 + \mu|e^{k+\frac{1}{2}}|_1^2) + c_4(\tau^2 + h^2)^2 \\
\leqslant& \frac{c_4}{2}(\|e^{k+1}\|^2 + \mu|e^{k+1}|_1^2 + \|e^k\|^2 + \mu|e^k|_1^2) + c_4(\tau^2 + h^2)^2, \quad 0 \leqslant k \leqslant n-1.
\end{aligned}$$

变形, 得到

$$\begin{aligned}
&(1 - c_4\tau)(\|e^{k+1}\|^2 + \mu|e^{k+1}|_1^2) \\
\leqslant& (1 + c_4\tau)(\|e^k\|^2 + \mu|e^k|_1^2) + 2c_4\tau(\tau^2 + h^2)^2, \quad 0 \leqslant k \leqslant n-1.
\end{aligned}$$

当 $c_4\tau \leqslant \dfrac{1}{3}$ 时,

$$\|e^{k+1}\|^2 + \mu|e^{k+1}|_1^2 \leqslant (1 + 3c_4\tau)(\|e^k\|^2 + \mu|e^k|_1^2) + 3c_4\tau(\tau^2 + h^2)^2, \quad 0 \leqslant k \leqslant n-1.$$

由 Gronwall 不等式得到

$$\|e^k\|^2 + \mu|e^k|_1^2 \leqslant e^{3c_4T} \cdot (\tau^2 + h^2)^2, \quad 0 \leqslant k \leqslant n. \qquad \square$$

2.3　三层线性化差分格式

2.3.1　差分格式的建立

在 $(x_i, t_{\frac{1}{2}})$ 处考虑方程 (2.1), 有

$$u_t\left(x_i, t_{\frac{1}{2}}\right) - \mu u_{xxt}(x_i, t_{\frac{1}{2}}) + \gamma u(x_i, t_{\frac{1}{2}}) u_x(x_i, t_{\frac{1}{2}}) + u_x(x_i, t_{\frac{1}{2}}) = 0, \quad 1 \leqslant i \leqslant m-1.$$

应用 Taylor 展开式和数值微分公式, 可得

$$\delta_t U_i^{\frac{1}{2}} - \mu\delta_t\delta_x^2 U_i^{\frac{1}{2}} + \gamma\psi(U^0, U^{\frac{1}{2}})_i + \Delta_x U_i^{\frac{1}{2}} = P_i^0, \quad 1 \leqslant i \leqslant m-1, \tag{2.27}$$

且存在常数 c_5 使得

$$|P_i^0| \leqslant c_5(\tau + h^2), \quad 1 \leqslant i \leqslant m-1. \tag{2.28}$$

在 (x_i, t_k) 考虑方程 (2.1) 有

$$u_t(x_i, t_k) - \mu u_{xxt}(x_i, t_k) + \gamma u(x_i, t_k) u_x(x_i, t_k) + u_x(x_i, t_k) = 0.$$

应用微分公式有

$$\Delta_t U_i^k - \mu \Delta_t \delta_x^2 U_i^k + \gamma\psi(U^k, U^{\bar{k}})_i + \Delta_x U_i^{\bar{k}} = P_i^k, \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{2.29}$$

存在常数 c_6 使得

$$|P_i^k| \leqslant c_6(\tau^2 + h^2), \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{2.30}$$

注意到初边值条件

$$U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{2.31}$$

$$U_0^k = 0, \quad U_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{2.32}$$

对 (2.1)-(2.3) 建立如下三层线性化差分格式

$$\delta_t u_i^{\frac{1}{2}} - \mu \delta_t \delta_x^2 u_i^{\frac{1}{2}} + \gamma\psi(u^0, u^{\frac{1}{2}})_i + \Delta_x u_i^{\frac{1}{2}} = 0, \quad 1 \leqslant i \leqslant m-1, \tag{2.33}$$

$$\Delta_t u_i^k - \mu \Delta_t \delta_x^2 u_i^k + \gamma\psi(u^k, u^{\bar{k}})_i + \Delta_x u_i^{\bar{k}} = 0, \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{2.34}$$

$$u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{2.35}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{2.36}$$

2.3.2 差分格式解的守恒性和有界性

定理 2.6 设 $\{u_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (2.33)-(2.36) 的解. 记

$$E^k = \|u^k\|^2 + \mu|u^k|_1^2, \quad 0 \leqslant k \leqslant n,$$

则有

$$E^k \equiv E^0, \quad 0 \leqslant k \leqslant n-1.$$

证明　用 $u^{\frac{1}{2}}$ 与 (2.33) 作内积, 可得

$$\frac{1}{2\tau}(\|u^1\|^2-\|u^0\|^2)+\frac{\mu}{2\tau}(|u^1|_1^2-|u^0|_1^2)=0,$$

即

$$E^1=E^0. \tag{2.37}$$

用 $u^{\bar{k}}$ 与 (2.34) 作内积, 可得

$$\frac{1}{4\tau}(\|u^{k+1}\|^2-\|u^{k-1}\|^2)+\frac{\mu}{4\tau}(|u^{k+1}|_1^2-|u^{k-1}|_1^2)=0,\quad 1\leqslant k\leqslant n-1.$$

变形, 得到

$$\frac{1}{4\tau}(E^{k+1}-E^{k-1})=0,\quad 1\leqslant k\leqslant n-1. \tag{2.38}$$

综合 (2.37) 和 (2.38), 得

$$E^k\equiv E^0,\quad 1\leqslant k\leqslant n.$$ □

2.3.3　差分格式解的存在性和唯一性

定理 2.7　差分格式 (2.33)–(2.36) 的解是存在唯一的.

证明　由 (2.35), (2.36) 式第 0 层的值 u^0 已知. 由 (2.33), (2.36) 可得第 1 层的值 u^1 的线性方程组. 考虑其齐次方程组

$$\frac{1}{\tau}u_i^1-\frac{\mu}{\tau}\delta_x^2u_i^1+\frac{1}{2}\gamma\psi(u^0,u^1)_i+\frac{1}{2}\Delta_xu_i^1=0,\quad 1\leqslant i\leqslant m-1, \tag{2.39}$$

$$u_0^1=0,\quad u_m^1=0. \tag{2.40}$$

由 u^1 与 (2.39) 作内积, 可得

$$\frac{1}{\tau}\|u^1\|^2+\mu\frac{1}{\tau}|u^1|_1^2=0.$$

因而

$$\|u^1\|=0,$$

即得关于 u^1 的线性方程组解是唯一的.

设 u^{k-1} 和 u^k 已知, 则由差分格式 (2.34) 和 (2.36) 可得关于 u^{k+1} 的线性方程组. 考虑其齐次方程组

$$\frac{1}{2\tau}u_i^{k+1}-\frac{\mu}{2\tau}\delta_x^2u_i^{k+1}+\frac{1}{2}\gamma\psi(u^k,u^{k+1})_i+\frac{1}{2}\Delta_xu_i^{k+1}=0,\ 1\leqslant i\leqslant m-1, \tag{2.41}$$

$$u_0^{k+1}=0,\quad u_m^{k+1}=0. \tag{2.42}$$

用 u^{k+1} 与 (2.41) 作内积, 得

$$\frac{1}{2\tau}\|u^{k+1}\|^2+\mu\cdot\frac{1}{2\tau}|u^{k+1}|_1^2=0.$$

因而

$$\|u^{k+1}\| = 0,$$

即关于 u^{k+1} 的线性方程组是唯一可解的. □

2.3.4 差分格式解的收敛性

定理 2.8 设 $\{U_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为问题 (2.1)–(2.3) 的解, $\{u_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (2.33)–(2.36) 的解. 记

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant m,\ 0 \leqslant k \leqslant n,$$

则存在常数 c_7 使得

$$\|e^k\|_\infty \leqslant c_7(\tau^2 + h^2), \quad 0 \leqslant k \leqslant n. \tag{2.43}$$

证明 将 (2.27), (2.29), (2.31)–(2.32) 与 (2.33)–(2.36) 相减, 得误差方程

$$\delta_t e_i^{\frac{1}{2}} - \mu\delta_t\delta_x^2 e_i^{\frac{1}{2}} + \gamma[\psi(U^0, U^{\frac{1}{2}})_i - \psi(u^0, u^{\frac{1}{2}})_i] + \Delta_x e_i^{\frac{1}{2}} = P_i^0,$$
$$1 \leqslant i \leqslant m-1, \tag{2.44}$$

$$\Delta_t e_i^k - \mu\Delta_t\delta_x^2 e_i^k + \gamma[\psi(U^k, U^{\bar{k}})_i - \psi(u^k, u^{\bar{k}})_i] + \Delta_x e_i^{\bar{k}} = P_i^k,$$
$$1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{2.45}$$

$$e_i^0 = 0, \quad 1 \leqslant i \leqslant m-1, \tag{2.46}$$

$$e_0^k = 0, \quad e_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{2.47}$$

易知

$$\|e^0\|^2 = 0, \quad |e^0|_1 = 0. \tag{2.48}$$

用 $e^{\frac{1}{2}}$ 与 (2.44) 作内积, 可得

$$\begin{aligned}
&\frac{1}{2\tau}(\|e^1\|^2 - \|e^0\|^2) + \frac{\mu}{2\tau}(|e^1|_1^2 - |e^0|_1^2) \\
=&-\gamma(\psi(U^0, U^{\frac{1}{2}}) - \psi(u^0, u^{\frac{1}{2}}), e^{\frac{1}{2}}) + (P^0, e^{\frac{1}{2}}) \\
=&-\gamma(\psi(U^0, e^{\frac{1}{2}}), e^{\frac{1}{2}}) + (P^0, e^{\frac{1}{2}}) \\
=&(P^0, e^{\frac{1}{2}}) \\
\leqslant&\|P^0\|\|e^{\frac{1}{2}}\|,
\end{aligned}$$

即

$$\begin{aligned}
\|e^1\|^2 + \mu|e^1|_1^2 &\leqslant 2\tau\|P^0\| \cdot \|e^{\frac{1}{2}}\| \\
&= \tau\|P^0\| \cdot \|e^1\| \\
&\leqslant \tau^2\|P^0\|^2 + \frac{1}{4}\|e^1\|^2.
\end{aligned}$$

存在常数 c_8 使得

$$\|e^1\|^2+\mu|e^1|_1^2\leqslant c_8\tau^2(\tau+h^2)^2\leqslant c_8(\tau^2+h^2)^2. \tag{2.49}$$

用 $e^{\bar{k}}$ 与 (2.45) 作内积, 得

$$\begin{aligned}&\frac{1}{4\tau}(\|e^{k+1}\|^2-\|e^{k-1}\|^2)+\frac{\mu}{4\tau}(|e^{k+1}|_1^2-|e^{k-1}|_1^2)\\=&-\gamma(\psi(U^k,U^{\bar{k}})-\psi(u^k,u^{\bar{k}}),e^{\bar{k}})+(P^k,e^{\bar{k}})\\=&-\gamma[(\psi(U^k,e^{\bar{k}}),e^{\bar{k}})+(\psi(e^k,U^{\bar{k}}),e^{\bar{k}})-(\psi(e^k,e^{\bar{k}}),e^{\bar{k}})]+(P^k,e^k)\\=&-\gamma(\psi(e^k,U^{\bar{k}}),e^{\bar{k}})+(P^k,e^{\bar{k}})\\\leqslant&\frac{1}{3}\gamma(2\|\Delta_xU^{\bar{k}}\|_\infty\|e^k\|\cdot\|e^{\bar{k}}\|+\|U^{\bar{k}}\|_\infty|e^k|_1\cdot\|e^{\bar{k}}\|)+\frac{1}{2}\|P^k\|\cdot\|e^{\bar{k}}\|,\ 1\leqslant k\leqslant n-1.\end{aligned}$$

存在常数 c_9 使得

$$\begin{aligned}&\frac{1}{4\tau}(\|e^{k+1}\|^2-\|e^{k-1}\|^2)+\frac{\mu}{4\tau}(|e^{k+1}|_1^2-|e^{k-1}|_1^2)\\\leqslant&c_9(\|e^k\|^2+\|e^{\bar{k}}\|^2+\mu|e^k|_1^2)+c_9\|P^k\|^2,\quad 1\leqslant k\leqslant n-1.\end{aligned}$$

注意到 (2.30) 可得

$$\begin{aligned}&\frac{1}{2\tau}\left(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2}-\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}\right)\\&+\frac{\mu}{2\tau}\left(\frac{|e^{k+1}|_1^2+|e^k|_1^2}{2}-\frac{|e^k|_1^2+|e^{k-1}|_1^2}{2}\right)\\\leqslant&c_9\left[\left(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2}+\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}\right)\right.\\&\left.+\mu\left(\frac{|e^{k+1}|_1^2+|e^k|_1^2}{2}+\frac{|e^k|_1^2+|e^{k-1}|_1^2}{2}\right)\right]\\&+c_9c_6^2(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n-1.\end{aligned}$$

记

$$F^k=\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2}+\mu\cdot\frac{|e^{k+1}|_1^2+|e^k|_1^2}{2},$$

则可将上式写为

$$\frac{1}{2\tau}(F^k-F^{k-1})\leqslant c_9(F^k+F^{k-1})+c_9c_6^2(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n-1.$$

变形得

$$(1-2c_9\tau)F^k\leqslant(1+2c_9\tau)F^{k-1}+2c_9c_6^2\tau(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n-1.$$

当 $2c_9\tau \leqslant \dfrac{1}{3}$ 时,

$$F^k \leqslant (1+6c_9\tau)F^{k-1} + 3c_9c_6^2\tau(\tau^2+h^2)^2, \quad 1 \leqslant k \leqslant n-1.$$

由 Gronwall 不等式并利用 (2.48), (2.49), 得到

$$F^k \leqslant e^{6c_9T}\left[F^0 + \frac{1}{2}c_6^2(\tau^2+h^2)^2\right] \leqslant \frac{1}{2}e^{6c_9T}\left(c_8^2+c_6^2\right)(\tau^2+h^2)^2, \quad 0 \leqslant k \leqslant n. \quad \square$$

2.4 小结与延拓

本章对正则长波方程的初边值问题介绍了二层非线性差分格式和三层线性化差分格式.

对于二层非线性差分格式, 用不动点定理证明了差分格式解的存在性, 用能量方法证明了差分格式解满足守恒性, 给出了解的先验估计式, 证明了差分格式解的唯一性, 证明了差分格式在无穷模下的无条件收敛性, 收敛阶关于空间步长和时间步长均是二阶收敛的.

对于三层线性化差分格式, 证明了差分格式的解满足守恒性和有界性, 证明了差分格式的解是存在唯一的, 差分格式解在无穷模下是无条件收敛的, 收敛阶关于空间步长和时间步长均为 2.

2.2 节、2.3 节的基本素材分别取自 [3], [5].

我们也可以构造如下三层外推型线性化差分格式

$$\delta_t u_i^{k+\frac{1}{2}} - \mu\delta_t\delta_x^2 u_i^{k+\frac{1}{2}} + \gamma\psi\left(\frac{3}{2}u^k - \frac{1}{2}u^{k-1}, u^{k+\frac{1}{2}}\right)_i + \Delta_x u_i^{k+\frac{1}{2}} = 0,$$
$$1 \leqslant i \leqslant m-1, \quad 1 \leqslant k \leqslant n-1. \tag{2.50}$$

可以证明差分格式 (2.33), (2.50), (2.35)–(2.36) 解的守恒性、有界性、收敛性.

第 3 章 Korteweg-de Vries 方程的差分方法

3.1 引 言

KdV 浅水波方程是非线性色散方程的典型代表. 因其具有无穷多个守恒律, 在固体、液体、气体以及等离子体等学科领域中得到了广泛应用. KdV 方程是 1895 年由荷兰数学家 Diederik Korteweg 和 Gustav de Vries 在研究浅水波中小振幅长波运动时共同发现的一种单向运动浅水波偏微分方程. Boussinesq 于 1877 年首先引入了 KdV 方程.

本章研究 KdV 方程初边值问题

$$u_t + \gamma u u_x + u_{xxx} = 0, \quad 0 < x < L,\ 0 < t \leqslant T, \tag{3.1}$$

$$u(x,0) = \varphi(x), \quad 0 < x < L, \tag{3.2}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0, \quad 0 < t \leqslant T \tag{3.3}$$

的差分方法, 其中 γ 为常数, $\varphi(0) = \varphi(L) = \varphi'(L) = 0$.

在介绍差分方法之前, 我们先用能量方法给出问题 (3.1)–(3.3) 解的先验估计式.

定理 3.1 设 $u(x,t)$ 为问题 (3.1)–(3.3) 的解, 记

$$E(t) = \int_0^L u^2(x,t)\mathrm{d}x + \int_0^t u_x^2(0,s)\mathrm{d}s,$$

则有

$$E(t) \equiv E(0), \quad 0 < t \leqslant T. \tag{3.4}$$

证明 用 $u(x,t)$ 乘以 (3.1) 的两边, 对 x 从 0 到 L 积分, 得

$$\int_0^L u u_t \mathrm{d}x + \gamma \int_0^L u^2 u_x \mathrm{d}x + \int_0^L u u_{xxx} \mathrm{d}x = 0,$$

即

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2 \mathrm{d}x + \frac{1}{3}\gamma \int_0^L (u^3)_x \mathrm{d}x + \int_0^L [(u u_{xx})_x - u_x u_{xx}]\mathrm{d}x = 0.$$

易得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2 \mathrm{d}x + \frac{\gamma}{3} u^3\Big|_{x=0}^L + u u_{xx}\Big|_{x=0}^L - \frac{1}{2}u_x^2\Big|_{x=0}^L = 0.$$

应用 (3.3) 得到

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2\mathrm{d}x+\frac{1}{2}u_x^2(0,t)=0,$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L u^2(x,t)\mathrm{d}x+\int_0^t u_x^2(0,s)\mathrm{d}s\right]=0.$$

因而

$$E(t)\equiv E(0),\quad 0<t\leqslant T.$$ □

称 (3.4) 为能量守恒律.

3.2 空间一阶二层非线性差分格式

以下设问题 (3.1)–(3.3) 存在解 $u(x,t)\in C_{x,t}^{4,3}([0,L]\times[0,T])$.

3.2.1 差分格式的建立

与第 2 章一样, 记

$$\psi(v,w)_i=\frac{1}{3}\left[v_i\Delta_x w_i+\Delta_x(vw)_i\right],\quad 1\leqslant i\leqslant m-1.$$

在方程 (3.1) 中令 $x=L$, 并注意到 $u(L,t)=0$, 得

$$u_{xxx}(L,t)=0.$$

将方程 (3.1) 的两边关于 x 求导, 得

$$u_{xt}+\gamma\left(u_x^2+uu_{xx}\right)+u_{xxxx}=0.$$

令 $x=L$, 并注意到 $u(L,t)=u_x(L,t)=0$, 可得

$$u_{xxxx}(L,t)=0.$$

在点 (x_i,t) 处考虑方程 (3.1), 有

$$u_t(x_i,t)+\gamma u(x_i,t)u_x(x_i,t)+u_{xxx}(x_i,t)=0,\quad 1\leqslant i\leqslant m-1. \tag{3.5}$$

由 Taylor 展开式可得

$$u_{xxx}(x_i,t)=\frac{1}{h^3}[u(x_{i+2},t)-3u(x_{i+1},t)+3u(x_i,t)-u(x_{i-1},t)]+O(h),\\ 1\leqslant i\leqslant m-2. \tag{3.6}$$

由 $u(x_m,t)=0, u_x(x_m,t)=0$ 和 Taylor 展开式可得

$$u_{xxx}(x_m,t)=\frac{3}{2}\cdot\frac{1}{h^3}\big[(u(x_{m-1},t)-u(x_{m-2},t))-3(u(x_m,t)-u(x_{m-1},t))\big]\\+\frac{3}{4}hu_{xxxx}(x_m,t)+O(h^2). \tag{3.7}$$

再注意到 $u_{xxx}(x_m,t)=0$, $u_{xxxx}(x_m,t)=0$, 对任意常数 α 有

$$u_{xxx}(x_{m-1},t)=u_{xxx}(x_m,t)-hu_{xxxx}(x_m,t)+O(h^2)\\=\frac{\alpha}{h^3}\big[(u(x_{m-1},t)-u(x_{m-2},t))-3(u(x_m,t)-u(x_{m-1},t))\big]+O(h^2). \tag{3.8}$$

以下取 $\alpha=1$.

注 3.1　以 $x_{i-1},x_i,x_{i+1},x_{i+2}$ 为插值节点作 $u(x,\cdot)$ 的 3 次 Lagrange 插值多项式 $L_i(x)$, 则

$$L_i'''(x)|_{x=x_i}=\frac{1}{h^3}[u(x_{i+2},t)-3u(x_{i+1},t)+3u(x_i,t)-u(x_{i-1},t)].$$

注 3.2　以

$$H_{m-1}(x_{m-2})=u(x_{m-2},t),\quad H_{m-1}(x_{m-1})=u(x_{m-1},t),\quad H_{m-1}(x_m)=u(x_m,t),$$

$$H'_{m-1}(x_m)=u_x(x_m,t)=0$$

为插值条件, 作 $u(x,\cdot)$ 的 3 次 Hermite 插值多项式 $H_{m-1}(x)$, 则

$$H'''_{m-1}(x)|_{x=x_{m-1}}=\frac{3}{2}\cdot\frac{1}{h^3}\big[(u(x_{m-1},t)-u(x_{m-2},t))-3(u(x_m,t)-u(x_{m-1},t))\big].$$

定义网格函数

$$U_i^k=u(x_i,t_k),\quad 0\leqslant i\leqslant m,\ 0\leqslant k\leqslant n. \tag{3.9}$$

在 (3.5) 中令 $t=t_k$ 和 $t=t_{k+1}$, 将两式作平均, 并利用 (3.6) 和 (3.8), 得

$$\delta_t U_i^{k+\frac12}+\gamma\psi(U^{k+\frac12},U^{k+\frac12})_i+\delta_x^2(\delta_x U_{i+\frac12}^{k+\frac12})=P_i^{k+\frac12},\\ 1\leqslant i\leqslant m-2,\quad 0\leqslant k\leqslant n-1, \tag{3.10}$$

$$\delta_t U_{m-1}^{k+\frac12}+\gamma\psi(U^{k+\frac12},U^{k+\frac12})_{m-1}+\frac{1}{h^2}(\delta_x U_{m-\frac32}^{k+\frac12}-3\delta_x U_{m-\frac12}^{k+\frac12})=P_{m-1}^{k+\frac12},\\ 0\leqslant k\leqslant n-1, \tag{3.11}$$

存在常数 c_1 使得

$$|P_i^{k+\frac12}|\leqslant\begin{cases}c_1(\tau^2+h), & 1\leqslant i\leqslant m-2,\ 0\leqslant k\leqslant n-1,\\ c_1(\tau^2+h^2), & i=m-1,\ 0\leqslant k\leqslant n-1.\end{cases} \tag{3.12}$$

由初边值条件 (3.2)–(3.3), 有

$$U_i^0=\varphi(x_i),\quad 1\leqslant i\leqslant m-1, \tag{3.13}$$

$$U_0^k=0,\quad U_m^k=0,\quad 0\leqslant k\leqslant n, \tag{3.14}$$

在 (3.10)–(3.11) 中略去小量项, 对 (3.1)–(3.3) 建立如下差分格式

$$\delta_t u_i^{k+\frac12}+\gamma\psi(u^{k+\frac12},u^{k+\frac12})_i+\delta_x^2(\delta_x u_{i+\frac12}^{k+\frac12})=0,$$
$$1\leqslant i\leqslant m-2,\ 0\leqslant k\leqslant n-1, \tag{3.15}$$

$$\delta_t u_{m-1}^{k+\frac12}+\gamma\psi(u^{k+\frac12},u^{k+\frac12})_{m-1}+\frac{1}{h^2}\big(\delta_x u_{m-\frac32}^{k+\frac12}-3\delta_x u_{m-\frac12}^{k+\frac12}\big)=0,$$
$$0\leqslant k\leqslant n-1, \tag{3.16}$$

$$u_i^0=\varphi(x_i),\quad 1\leqslant i\leqslant m-1, \tag{3.17}$$

$$u_0^k=0,\quad u_m^k=0,\quad 0\leqslant k\leqslant n. \tag{3.18}$$

定义

$$v_m^{k+\frac12}=0,\quad v_i^{k+\frac12}=\delta_x u_{i+\frac12}^{k+\frac12},\quad i=m-1,m-2,\cdots,0,\quad 0\leqslant k\leqslant n-1,$$

则可将 (3.15)–(3.18) 写成如下方程组

$$\begin{aligned}
&\delta_t u_i^{k+\frac12}+\gamma\psi(u_i^{k+\frac12},u_i^{k+\frac12})+\delta_x^2 v_i^{k+\frac12}=0,\\
&\qquad\qquad 1\leqslant i\leqslant m-2,0\leqslant k\leqslant n-1,\\
&\delta_t u_{m-1}^{k+\frac12}+\gamma\psi(u_{m-1}^{k+\frac12},u_{m-1}^{k+\frac12})+\frac{1}{h^2}\big(v_{m-2}^{k+\frac12}-3v_{m-1}^{k+\frac12}\big)=0,\quad 0\leqslant k\leqslant n-1,\\
&u_0^k=0,\quad u_m^k=0,\quad 0\leqslant k\leqslant n,\\
&u_i^0=\varphi(x_i),\quad 1\leqslant i\leqslant m-1,\\
&v_m^{k+\frac12}=0,\quad v_i^{k+\frac12}=\delta_x u_{i+\frac12}^{k+\frac12},\quad i=m-1,m-2,\cdots,0,\ 0\leqslant k\leqslant n-1.
\end{aligned}$$

3.2.2 差分格式解的存在性

引理 3.1 设 $w\in\mathring{\mathcal{U}}_h$, 则有

$$\begin{aligned}
&h\sum_{i=1}^{m-2}(\delta_x^2\delta_x w_{i+\frac12})w_i+\frac{1}{h}\big(\delta_x w_{m-\frac32}-3\delta_x w_{m-\frac12}\big)w_{m-1}\\
=&\frac12 h|w|_2^2+\frac12(\delta_x w_{\frac12})^2+\frac32(\delta_x w_{m-\frac12})^2.
\end{aligned}$$

证明 定义

$$v_m=0,\quad v_i=\delta_x w_{i+\frac12},\quad i=m-1,m-2,\cdots,0,$$

则

$$\begin{aligned}
&h\sum_{i=1}^{m-2}(\delta_x^2\delta_x w_{i+\frac{1}{2}})w_i+\frac{1}{h}\big(\delta_x w_{m-\frac{3}{2}}-3\delta_x w_{m-\frac{1}{2}}\big)w_{m-1}\\
=&h\sum_{i=1}^{m-2}(\delta_x^2 v_i)w_i+\frac{1}{h}\big(v_{m-2}-3v_{m-1}\big)w_{m-1}\\
=&\sum_{i=1}^{m-2}(\delta_x v_{i+\frac{1}{2}}-\delta_x v_{i-\frac{1}{2}})w_i+\frac{1}{h}\big(v_{m-2}-3v_{m-1}\big)w_{m-1}\\
=&\sum_{i=1}^{m-2}(\delta_x v_{i+\frac{1}{2}})w_i-\sum_{i=0}^{m-3}(\delta_x v_{i+\frac{1}{2}})w_{i+1}+\frac{1}{h}\big(v_{m-2}-3v_{m-1}\big)w_{m-1}\\
=&\sum_{i=0}^{m-2}(\delta_x v_{i+\frac{1}{2}})(w_i-w_{i+1})+(\delta_x v_{m-\frac{3}{2}})w_{m-1}+\frac{1}{h}\big(v_{m-2}-3v_{m-1}\big)w_{m-1}\\
=&-h\sum_{i=0}^{m-2}(\delta_x v_{i+\frac{1}{2}})v_i+\frac{1}{h}(v_{m-1}-v_{m-2}+v_{m-2}-3v_{m-1})w_{m-1}\\
=&\frac{1}{2}\sum_{i=0}^{m-2}(v_{i+1}-v_i)^2+\frac{1}{2}v_0^2+\frac{3}{2}v_{m-1}^2\\
=&\frac{1}{2}\sum_{i=1}^{m-1}(v_i-v_{i-1})^2+\frac{1}{2}v_0^2+\frac{3}{2}v_{m-1}^2\\
=&\frac{1}{2}h^2\sum_{i=1}^{m-1}(\delta_x^2 w_i)^2+\frac{1}{2}(\delta_x w_{\frac{1}{2}})^2+\frac{3}{2}(\delta_x w_{m-\frac{1}{2}})^2\\
=&\frac{1}{2}h|w|_2^2+\frac{1}{2}(\delta_x w_{\frac{1}{2}})^2+\frac{3}{2}(\delta_x w_{m-\frac{1}{2}})^2.
\end{aligned}$$

□

定理 3.2　差分格式 (3.15)–(3.18) 的解是存在的.

证明　由 (3.17)–(3.18) 知第 0 层的解 u^0 唯一确定.

设第 k 层的解 u^k 已确定. 令

$$w_i=u_i^{k+\frac{1}{2}},\quad 0\leqslant i\leqslant m.$$

则可得关于 w 的方程组

$$\begin{aligned}
&\frac{2}{\tau}(w_i-u_i^k)+\gamma\psi(w,w)_i+\delta_x^2(\delta_x w_{i+\frac{1}{2}})=0,\quad 1\leqslant i\leqslant m-2,\\
&\frac{2}{\tau}(w_{m-1}-u_{m-1}^k)+\gamma\psi(w,w)_{m-1}+\frac{1}{h^2}\big(\delta_x w_{m-\frac{3}{2}}-3\delta_x w_{m-\frac{1}{2}}\big)=0,\\
&w_0=0,\quad w_m=0.
\end{aligned}$$

对 $w \in \mathring{\mathcal{U}}_h$, 定义

$$\Pi(w)_i = \begin{cases} \dfrac{2}{\tau}(w_i - u_i^k) + \gamma\psi(w,w)_i + \delta_x^2(\delta_x w_{i+\frac{1}{2}}), & 1 \leqslant i \leqslant m-2, \\ \dfrac{2}{\tau}(w_{m-1} - u_{m-1}^k) + \gamma\psi(w,w)_{m-1} \\ +\dfrac{1}{h^2}(\delta_x w_{m-\frac{3}{2}} - 3\delta_x w_{m-\frac{1}{2}}), & i = m-1. \end{cases}$$

计算可得

$$\begin{aligned}(\Pi(w), w) =& \frac{2}{\tau}\left[\|w\|^2 - (u^k, w)\right] + \gamma\big(\psi(w,w), w\big) \\ &+ h\sum_{i=1}^{m-2}(\delta_x^2\delta_x w_{i+\frac{1}{2}})w_i + \frac{1}{h}(\delta_x w_{m-\frac{3}{2}} - 3\delta_x w_{m-\frac{1}{2}})w_{m-1}.\end{aligned}$$

由引理 3.1 及 $(\psi(w,w),w) = 0$, 得

$$(\Pi(w), w) \geqslant \frac{2}{\tau}[\|w\|^2 - (u^k, w)] \geqslant \frac{2}{\tau}\|w\|(\|w\| - \|u^k\|).$$

当 $\|w\| = \|u^k\|$ 时 $(\Pi(w), w) \geqslant 0$.

由 Browder 定理 (定理 1.3) 存在 $w^* \in \mathring{\mathcal{U}}_h$ 且 $\|w^*\| \leqslant \|u^k\|$ 使得

$$\Pi(w^*) = 0.$$

□

3.2.3 差分格式解的守恒性和有界性

定理 3.3 设 $\{u_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为 (3.15)–(3.18) 的解, 则有

$$\|u^{k+1}\|^2 + \tau\sum_{l=0}^{k}\left[(\delta_x u_{\frac{1}{2}}^{l+\frac{1}{2}})^2 + 3(\delta_x u_{m-\frac{1}{2}}^{l+\frac{1}{2}})^2 + h|w^{l+\frac{1}{2}}|_2^2\right] = \|u^0\|^2, \quad 0 \leqslant k \leqslant n-1.$$

证明 用 $hu_i^{k+\frac{1}{2}}$ 与 (3.15) 相乘, 用 $hu_{m-1}^{k+\frac{1}{2}}$ 与 (3.16) 相乘, 并将所得结果相加, 得

$$\begin{aligned}&\frac{1}{2\tau}(\|u^{k+1}\|^2 - \|u^k\|^2) + \gamma(\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}), u^{k+\frac{1}{2}}) \\ &+h\sum_{i=1}^{m-2}(\delta_x^2\delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}})u_i^{k+\frac{1}{2}} + \frac{1}{h}(\delta_x u_{m-\frac{3}{2}}^{k+\frac{1}{2}} - 3\delta_x u_{m-\frac{1}{2}}^{k+\frac{1}{2}})u_{m-1}^{k+\frac{1}{2}} = 0. \end{aligned}\tag{3.19}$$

由引理 3.1 得

$$\begin{aligned}&h\sum_{i=1}^{m-2}(\delta_x^2\delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}})u_i^{k+\frac{1}{2}} + \frac{1}{h}(\delta_x u_{m-\frac{3}{2}}^{k+\frac{1}{2}} - 3\delta_x u_{m-\frac{1}{2}}^{k+\frac{1}{2}})u_{m-1}^{k+\frac{1}{2}} \\ =& \frac{1}{2}h|u^{k+\frac{1}{2}}|_2^2 + \frac{1}{2}(\delta_x u_{\frac{1}{2}}^{k+\frac{1}{2}})^2 + \frac{3}{2}(\delta_x u_{m-\frac{1}{2}}^{k+\frac{1}{2}})^2.\end{aligned}\tag{3.20}$$

将上式代入 (3.19) 并注意到 $(\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}}),u^{k+\frac{1}{2}})=0$, 得

$$\frac{1}{2\tau}(\|u^{k+1}\|^2-\|u^k\|^2)+\frac{1}{2}h|u^{k+\frac{1}{2}}|_2^2+\frac{1}{2}(\delta_x u_{\frac{1}{2}}^{k+\frac{1}{2}})^2+\frac{3}{2}(\delta_x u_{m-\frac{1}{2}}^{k+\frac{1}{2}})^2=0,$$

即

$$\frac{1}{2\tau}(\|u^{k+1}\|^2-\|u^k\|^2)+\frac{1}{2}h|u^{k+\frac{1}{2}}|_2^2+\frac{1}{2}(\delta_x u_{\frac{1}{2}}^{k+\frac{1}{2}})^2+\frac{3}{2}(\delta_x u_{m-\frac{1}{2}}^{k+\frac{1}{2}})^2\leqslant 0,\quad 0\leqslant k\leqslant n-1.$$

将上式中的 k 换为 l, 并对 l 从 0 到 k 求和, 得

$$\|u^{k+1}\|^2+\tau\sum_{l=0}^{k}\left[(\delta_x u_{\frac{1}{2}}^{l+\frac{1}{2}})^2+3(\delta_x u_{m-\frac{1}{2}}^{l+\frac{1}{2}})+h|u^{l+\frac{1}{2}}|_2^2\right]=\|u^0\|^2,\quad 0\leqslant k\leqslant n-1.\ \square$$

3.2.4 差分格式解的收敛性

定理 3.4 设 $\{U_i^k\,|\,0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$ 为 (3.1)–(3.3) 的解, $\{u_i^k\,|\,0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$ 为 (3.15)–(3.18) 的解. 记

$$e_i^k=U_i^k-u_i^k,\quad 0\leqslant i\leqslant m,\ 0\leqslant k\leqslant n,$$

则存在常数 c_2 使得

$$\|e^k\|\leqslant c_2(\tau^2+h),\quad 0\leqslant k\leqslant n. \tag{3.21}$$

证明 将 (3.10)–(3.11), (3.13)–(3.14) 和 (3.15)–(3.18) 相减, 得误差方程

$$\delta_t e_i^{k+\frac{1}{2}}+\gamma\left[\psi(U_i^{k+\frac{1}{2}},U_i^{k+\frac{1}{2}})-\psi(u_i^{k+\frac{1}{2}},u_i^{k+\frac{1}{2}})\right]+\delta_x^2\delta_x e_{i+\frac{1}{2}}^{k+\frac{1}{2}}=P_i^{k+\frac{1}{2}},$$
$$1\leqslant i\leqslant m-2,\ 0\leqslant k\leqslant n-1, \tag{3.22}$$

$$\delta_t e_{m-1}^{k+\frac{1}{2}}+\gamma\left[\psi(U_{m-1}^{k+\frac{1}{2}},U_{m-1}^{k+\frac{1}{2}})-\psi(u_{m-1}^{k+\frac{1}{2}},u_{m-1}^{k+\frac{1}{2}})\right]$$
$$+\frac{1}{h^2}\left(\delta_x e_{m-\frac{3}{2}}^{k+\frac{1}{2}}-3\delta_x e_{m-\frac{1}{2}}^{k+\frac{1}{2}}\right)=P_{m-1}^{k+\frac{1}{2}},\quad 0\leqslant k\leqslant n-1, \tag{3.23}$$

$$e_i^0=0,\quad 1\leqslant i\leqslant m-1, \tag{3.24}$$

$$e^k=0,\quad e_m^k=0,\quad 0\leqslant k\leqslant n. \tag{3.25}$$

用 $he_i^{k+\frac{1}{2}}$ 乘以 (3.22), 用 $he_{m-1}^{k+\frac{1}{2}}$ 乘以 (3.23), 将所得结果相加, 得

$$\frac{1}{2\tau}(\|e^{k+1}\|^2-\|e^k\|^2)+\gamma(\psi(U^{k+\frac{1}{2}},U^{k+\frac{1}{2}})-\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}}),e^{k+\frac{1}{2}})$$
$$+h\sum_{i=1}^{m-2}(\delta_x^2\delta_x e_{i+\frac{1}{2}}^{k+\frac{1}{2}})e_i^{k+\frac{1}{2}}+\frac{1}{h}(-3\delta_x e_{m-\frac{1}{2}}^{k+\frac{1}{2}}+\delta_x e_{m-\frac{3}{2}}^{k+\frac{1}{2}})e_{m-1}^{k+\frac{1}{2}}$$
$$=(P^{k+\frac{1}{2}},e^{k+\frac{1}{2}}),\quad 0\leqslant k\leqslant n-1. \tag{3.26}$$

由引理 3.1 可得

$$\begin{aligned}&h\sum_{i=1}^{m-2}(\delta_x^2\delta_x e_{i+\frac{1}{2}}^{k+\frac{1}{2}})e_i^{k+\frac{1}{2}}+\frac{1}{h}(\delta_x e_{m-\frac{3}{2}}^{k+\frac{1}{2}}-3\delta_x e_{m-\frac{1}{2}}^{k+\frac{1}{2}})e_{m-1}^{k+\frac{1}{2}}\\=&\frac{1}{2}h^2\sum_{i=1}^{m-1}(\delta_x^2 e_i^{k+\frac{1}{2}})^2+\frac{1}{2}(\delta_x e_{\frac{1}{2}}^{k+\frac{1}{2}})^2+\frac{3}{2}(\delta_x e_{m-\frac{1}{2}}^{k+\frac{1}{2}})^2.\end{aligned}\tag{3.27}$$

下面分析 (3.26) 左端第二项.

$$\begin{aligned}&\left(\psi(U^{k+\frac{1}{2}},U^{k+\frac{1}{2}})-\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}}),e^{k+\frac{1}{2}}\right)\\=&\left(\psi(U^{k+\frac{1}{2}},U^{k+\frac{1}{2}})-\psi(U^{k+\frac{1}{2}}-e^{k+\frac{1}{2}},U^{k+\frac{1}{2}}-e^{k+\frac{1}{2}}),e^{k+\frac{1}{2}}\right)\\=&\left(\psi(e^{k+\frac{1}{2}},U^{k+\frac{1}{2}})+\psi(U^{k+\frac{1}{2}},e^{k+\frac{1}{2}})-\psi(e^{k+\frac{1}{2}},e^{k+\frac{1}{2}}),e^{k+\frac{1}{2}}\right)\\=&\left(\psi(e^{k+\frac{1}{2}},U^{k+\frac{1}{2}}),e^{k+\frac{1}{2}}\right)\\=&\frac{1}{3}h\sum_{i=1}^{m-1}\left[e_i^{k+\frac{1}{2}}\Delta_x U_i^{k+\frac{1}{2}}+\Delta_x(eU)_i^{k+\frac{1}{2}}\right]e_i^{k+\frac{1}{2}}\\=&\frac{1}{3}\left[h\sum_{i=1}^{m-1}(\Delta_x U_i^{k+\frac{1}{2}})(e_i^{k+\frac{1}{2}})^2+\frac{1}{2}\sum_{i=1}^{m-1}(e_{i+1}^{k+\frac{1}{2}}U_{i+1}^{k+\frac{1}{2}}-e_{i-1}^{k+\frac{1}{2}}U_{i-1}^{k+\frac{1}{2}})e_i^{k+\frac{1}{2}}\right]\\=&\frac{1}{3}\left[h\sum_{i=1}^{m-1}(\Delta_x U_i^{k+\frac{1}{2}})(e_i^{k+\frac{1}{2}})^2+\frac{1}{2}h\sum_{i=1}^{m-2}e_{i+1}^{k+\frac{1}{2}}e_i^{k+\frac{1}{2}}\delta_x U_{i+\frac{1}{2}}^{k+\frac{1}{2}}\right].\end{aligned}$$

记

$$c_3=\max_{0\leqslant x\leqslant l,0\leqslant t\leqslant T}|u_x(x,t)|.$$

则

$$\begin{aligned}&-\left(\psi(U^{k+\frac{1}{2}},U^{k+\frac{1}{2}})-\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}}),e^{k+\frac{1}{2}}\right)\\\leqslant&\frac{1}{3}c_3\left[h\sum_{i=1}^{m-1}(e_i^{k+\frac{1}{2}})^2+\frac{1}{2}h\sum_{i=1}^{m-2}|e_{i+1}^{k+\frac{1}{2}}e_i^{k+\frac{1}{2}}|\right]\\\leqslant&\frac{1}{2}c_3\|e^{k+\frac{1}{2}}\|^2.\end{aligned}\tag{3.28}$$

将 (3.27) 和 (3.28) 代入 (3.26) 得到

$$\frac{1}{2\tau}(\|e^{k+1}\|^2-\|e^k\|^2)\leqslant\frac{1}{2}\gamma c_3\|e^{k+\frac{1}{2}}\|^2+\|P^{k+\frac{1}{2}}\|\cdot\|e^{k+\frac{1}{2}}\|,\quad 0\leqslant k\leqslant n-1.$$

两边约去 $\frac{1}{2}(\|e^{k+1}\|+\|e^k\|)$, 得到

$$\frac{1}{\tau}(\|e^{k+1}\|-\|e^k\|)\leqslant\frac{1}{2}\gamma c_3\frac{\|e^{k+1}\|+\|e^k\|}{2}+\|P^{k+\frac{1}{2}}\|,\quad 0\leqslant k\leqslant n-1.$$

注意到 (3.12), 得

$$\left(1-\frac{\gamma c_3}{4}\tau\right)\|e^{k+1}\| \leqslant \left(1+\frac{\gamma c_3}{4}\tau\right)\|e^k\| + \tau\sqrt{L}c_1(\tau^2+h), \quad 0 \leqslant k \leqslant n-1.$$

当 $\frac{\gamma c_3}{4}\tau \leqslant \frac{1}{3}$ 时

$$\|e^{k+1}\| \leqslant \left(1+\frac{3\gamma c_3}{4}\tau\right)\|e^k\| + \frac{3}{2}\sqrt{L}c_1\tau(\tau^2+h), \quad 0 \leqslant k \leqslant n-1.$$

由 Gronwall 不等式得到

$$\|e^{k+1}\| \leqslant e^{\frac{3\gamma c_3}{4}T}\frac{2\sqrt{L}c_1}{\gamma c_3}(\tau^2+h), \quad 0 \leqslant k \leqslant n-1.$$ □

3.3 空间一阶三层线性化差分格式

3.3.1 差分格式的建立

在点 (x_i, t_0) 处考虑方程 (3.1) 并注意到初边值条件 (3.2), 有

$$\begin{aligned} u_t(x_i,t_0) &= -u(x_i,t_0)u_x(x_i,t_0) - u_{xxx}(x_i,t_0) \\ &= -\varphi(x_i)\varphi'(x_i) - \varphi'''(x_i), \quad 0 \leqslant i \leqslant m. \end{aligned}$$

记

$$\hat{u}_i = \varphi(x_i) + \frac{\tau}{2}[-\varphi(x_i)\varphi'(x_i) - \varphi'''(x_i)], \quad 0 \leqslant i \leqslant m.$$

在 (3.5) 中令 $t=t_0$ 和 $t=t_1$, 将两式作平均, 应用数值微分公式, 可得

$$\delta_t U_i^{\frac{1}{2}} + \gamma\psi(\hat{u}, U^{\frac{1}{2}})_i + \delta_x^2\delta_x U_{i+\frac{1}{2}}^{\frac{1}{2}} = \hat{P}_i^0, \quad 1 \leqslant i \leqslant m-2, \tag{3.29}$$

$$\delta_t U_{m-1}^{\frac{1}{2}} + \gamma\psi(\hat{u}, U^{\frac{1}{2}})_{m-1} + \frac{1}{h^2}\left(\delta_x U_{m-\frac{3}{2}}^{\frac{1}{2}} - 3\delta_x U_{m-\frac{1}{2}}^{\frac{1}{2}}\right) = \hat{P}_{m-1}^0. \tag{3.30}$$

在 (3.5) 中令 $t=t_{k-1}$ 和 $t=t_{k+1}$, 将两式作平均, 应用数值微分公式可得

$$\Delta_t U_i^k + \gamma\psi(U^k, U^{\bar{k}})_i + \delta_x^2\delta_x U_{i+\frac{1}{2}}^{\bar{k}} = \hat{P}_i^k, \quad 1 \leqslant i \leqslant m-2,\ 1 \leqslant k \leqslant n-1, \tag{3.31}$$

$$\Delta_t U_{m-1}^k + \gamma\psi(U^k, U^{\bar{k}})_{m-1} + \frac{1}{h^2}\left(\delta_x U_{m-\frac{3}{2}}^{\bar{k}} - 3\delta_x U_{m-\frac{1}{2}}^{\bar{k}}\right) = \hat{P}_{m-1}^k, \quad 1 \leqslant k \leqslant n-1. \tag{3.32}$$

存在常数 c_4 使得

$$|\hat{P}_i^k| \leqslant \begin{cases} c_4(\tau^2+h), & 1 \leqslant i \leqslant m-2, \quad 0 \leqslant k \leqslant n-1, \\ c_4(\tau^2+h^2), & i=m-1,\ 0 \leqslant k \leqslant n-1. \end{cases} \tag{3.33}$$

注意到初边值条件

$$U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{3.34}$$

$$U_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{3.35}$$

在 (3.29)–(3.32) 中略去小量项, 对 (3.1)–(3.3) 建立如下线性化差分格式

$$\delta_t u_i^{\frac{1}{2}} + \gamma\psi(\hat{u}, u^{\frac{1}{2}})_i + \delta_x^2\delta_x u_{i+\frac{1}{2}}^{\frac{1}{2}} = 0, \quad 1 \leqslant i \leqslant m-2, \tag{3.36}$$

$$\delta_t u_{m-1}^{\frac{1}{2}} + \gamma\psi(\hat{u}, u^{\frac{1}{2}})_{m-1} + \frac{1}{h^2}(\delta_x u_{m-\frac{3}{2}}^{\frac{1}{2}} - 3\delta_x u_{m-\frac{1}{2}}^{\frac{1}{2}}) = 0, \tag{3.37}$$

$$\Delta_t u_i^k + \gamma\psi(u^k, u^{\bar{k}})_i + \delta_x^2\delta_x u_{i+\frac{1}{2}}^{\bar{k}} = 0, \quad 1 \leqslant i \leqslant m-2,\ 1 \leqslant k \leqslant n-1, \tag{3.38}$$

$$\Delta_t u_{m-1}^k + \gamma\psi(u^k, u^{\bar{k}})_{m-1} + \frac{1}{h^2}(\delta_x u_{m-\frac{3}{2}}^{\bar{k}} - 3\delta_x u_{m-\frac{1}{2}}^{\bar{k}}) = 0, \quad 1 \leqslant k \leqslant n-1, \tag{3.39}$$

$$u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{3.40}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{3.41}$$

3.3.2 差分格式的可解性

定理 3.5 差分格式 (3.36)–(3.41) 是唯一可解的.

证明 由 (3.40)–(3.41) 知第 0 层值 u^0 已给定.

由 (3.36) 和 (3.41) 可得关于第 1 层值 u^1 的线性方程组. 考虑其齐次方程组

$$\frac{1}{\tau}u_i^1 + \frac{1}{2}\gamma\psi(\hat{u}, u^1)_i + \frac{1}{2}\delta_x^2\delta_x u_{i+\frac{1}{2}}^1 = 0, \quad 1 \leqslant i \leqslant m-2, \tag{3.42}$$

$$\frac{1}{\tau}u_{m-1}^1 + \frac{1}{2}\gamma\psi(\hat{u}, u^1)_{m-1} + \frac{1}{2}\cdot\frac{1}{h^2}(\delta_x u_{m-\frac{3}{2}}^1 - 3\delta_x u_{m-\frac{1}{2}}^1) = 0, \tag{3.43}$$

$$u_0^1 = 0, \quad u_m^1 = 0. \tag{3.44}$$

由 hu_i^1 乘以 (3.42), 用 hu_{m-1}^1 乘以 (3.43), 并将结果相加, 得

$$\frac{1}{\tau}\|u^1\|^2 + \frac{1}{2}\gamma(\psi(\hat{u}, u^1), u^1) + \frac{1}{2}\Big[h\sum_{i=1}^{m-2}(\delta_x^2\delta_x u_{i+\frac{1}{2}}^1)u_i^1 + \frac{1}{h}(\delta_x u_{m-\frac{3}{2}}^1 - 3\delta_x u_{m-\frac{1}{2}}^1)u_{m-1}^1\Big] = 0.$$

由 $(\psi(\hat{u}, u^1), u^1) = 0$ 及引理 3.1 得

$$\|u^1\| = 0.$$

因而 (3.36), (3.37) 及 (3.41) 唯一确定 u^1.

设第 $k-1$ 层值 u^{k-1} 和第 k 层值 u^k 已唯一确定. 则由 (3.38), (3.39) 及 (3.41) 可得关于第 $k+1$ 层值 u^{k+1} 的线性方程组. 考虑其齐次方程组

$$\frac{1}{2\tau}u_i^{k+1}+\frac{1}{2}\gamma\psi(u^k,u^{k+1})_i+\frac{1}{2}\delta_x^2\delta_x u_{i+\frac{1}{2}}^{k+1}=0,\quad 1\leqslant i\leqslant m-2, \tag{3.45}$$

$$\frac{1}{2\tau}u_{m-1}^{k+1}+\frac{1}{2}\gamma\psi(u^k,u^{k+1})_{m-1}+\frac{1}{2}\cdot\frac{1}{h^2}(\delta_x u_{m-\frac{3}{2}}^{k+1}-3\delta_x u_{m-\frac{1}{2}}^{k+1})=0, \tag{3.46}$$

$$u_0^{k+1}=0,\quad u_m^{k+1}=0. \tag{3.47}$$

用 $2hu_i^{k+1}$ 乘以 (3.45), 用 $2hu_{m-1}^{k+1}$ 乘以 (3.46), 并将结果相加, 得到

$$\begin{aligned}&\frac{1}{\tau}\|u^{k+1}\|^2+\gamma(\psi(u^k,u^{k+1}),u^{k+1})+h\sum_{i=2}^{m-1}(\delta_x^2\delta_x u_{i+\frac{1}{2}}^{k+1})u_i^{k+1}\\&+\frac{1}{h}\left(\delta_x u_{m-\frac{3}{2}}^{k+1}-3\delta_x u_{m-\frac{1}{2}}^{k+1}\right)u_{m-\frac{1}{2}}^{k+1}=0.\end{aligned}$$

由 $(\psi(u^k,u^{k+1}),u^{k+1})=0$ 以及引理 3.1, 得

$$\|u^{k+1}\|^2=0.$$

因而 (3.38)–(3.39) 及 (3.41) 唯一确定 u^{k+1}. □

3.3.3 差分格式解的守恒性和有界性

定理 3.6 设 $\{u_i^k\,|\,0\leqslant i\leqslant m, 0\leqslant k\leqslant n\}$ 为 (3.36)–(3.41) 的解, 则有

$$\begin{aligned}&\frac{1}{2}(\|u^{k+1}\|^2+\|u^k\|^2)+\frac{1}{2}\tau[(\delta_x u_{\frac{1}{2}}^{\frac{1}{2}})^2+3(\delta_x u_{m-\frac{1}{2}}^{\frac{1}{2}})^2+h\|u^{\frac{1}{2}}|_2^2]\\&+\tau\sum_{l=1}^{k}[(\delta_x u_{\frac{1}{2}}^{\bar{l}})^2+3(\delta_x u_{m-\frac{1}{2}}^{\bar{l}})^2+h|u^{\bar{l}}|_2^2]=\|u^0\|^2,\quad 0\leqslant k\leqslant n-1.\end{aligned} \tag{3.48}$$

证明 用 $hu_i^{\frac{1}{2}}$ 乘以 (3.36), 用 $hu_{m-1}^{\frac{1}{2}}$ 乘以 (3.37), 将结果相加, 得到

$$\begin{aligned}&\frac{1}{2\tau}(\|u^1\|^2-\|u^0\|^2)+\gamma(\psi(\hat{u},u^{\frac{1}{2}}),u^{\frac{1}{2}})\\&+h\sum_{i=1}^{m-2}(\delta_x^2\delta_x u_{i+\frac{1}{2}}^{\frac{1}{2}})u_i^{\frac{1}{2}}+\frac{1}{h}(\delta_x u_{m-\frac{3}{2}}^{\frac{1}{2}}-3\delta_x u_{m-\frac{1}{2}}^{\frac{1}{2}})u_{m-1}^{\frac{1}{2}}=0.\end{aligned}$$

由 $(\psi(\hat{u},u^{\frac{1}{2}}),u^{\frac{1}{2}})=0$ 及引理 3.1 得

$$\frac{1}{2\tau}(\|u^1\|^2-\|u^0\|^2)+\frac{1}{2}h^2\sum_{i=1}^{m-1}(\delta_x^2 u_i^{\frac{1}{2}})^2+\frac{1}{2}(\delta_x u_{\frac{1}{2}}^{\frac{1}{2}})^2+\frac{3}{2}(\delta_x u_{m-\frac{1}{2}}^{\frac{1}{2}})^2=0. \tag{3.49}$$

用 $hu_i^{\bar{k}}$ 乘以 (3.38), 用 $hu_{m-1}^{\bar{k}}$ 乘以 (3.39), 将结果相加, 得到

$$\frac{1}{4\tau}(\|u^{k+1}\|^2 - \|u^{k-1}\|^2) + \gamma(\psi(u^k, u^{\bar{k}}), u^{\bar{k}}) + h\sum_{i=1}^{m-2}(\delta_x^2\delta_x u_{i+\frac{1}{2}}^{\bar{k}})u_i^{\bar{k}}$$
$$+\frac{1}{2}(\delta_x u_{m-\frac{3}{2}}^{\bar{k}} - 3\delta_x u_{m-\frac{1}{2}}^{\bar{k}})u_{m-1}^{\bar{k}} = 0, \quad 1 \leqslant k \leqslant n-1.$$

由 $\big(\psi(u^k, u^{\bar{k}}), u^{\bar{k}}\big) = 0$ 及引理 3.1, 得

$$\frac{1}{4\tau}(\|u^{k+1}\|^2 - \|u^{k-1}\|^2) + \frac{1}{2}h^2\sum_{i=1}^{m-1}(\delta_x^2 u_i^{\bar{k}})^2 + \frac{1}{2}(\delta_x u_{\frac{1}{2}}^{\bar{k}})^2 + \frac{3}{2}(\delta_x u_{m-\frac{1}{2}}^{\bar{k}})^2 = 0,$$
$$1 \leqslant k \leqslant n-1.$$

由 (3.49) 和 (3.50) 可得

$$\frac{1}{2}(\|u^{k+1}\|^2 + \|u^k\|^2) + \frac{1}{2}\tau\big[(\delta_x u_{\frac{1}{2}}^{\frac{1}{2}})^2 + 3(\delta_x u_{m-\frac{1}{2}}^{\frac{1}{2}})^2 + h\|u^{\frac{1}{2}}|_2^2\big]$$
$$+\tau\sum_{l=1}^{k}\big[(\delta_x u_{\frac{1}{2}}^{\bar{l}})^2 + 3(\delta_x u_{m-\frac{1}{2}}^{\bar{l}})^2 + h\|u^{\bar{l}}|_2^2\big] = \|u^0\|^2, \quad 0 \leqslant k \leqslant n-1.$$

□

3.3.4 差分格式解的收敛性

记

$$c_0 = \max_{0\leqslant x\leqslant L, 0\leqslant t\leqslant L}|u(x,t)|, \quad \lambda = \frac{2|\gamma|c_0}{3}\frac{\tau}{h}.$$

定理 3.7 设 $\{U_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 是定解问题 (3.1)–(3.3) 的解, $\{u_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 是差分格式 (3.36)–(3.41) 的解. 记

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant m, \quad 0 \leqslant k \leqslant n.$$

设

$$\lambda < 1, \tag{3.50}$$

则存在一个常数 c_5, 有下式成立

$$\|e^k\| \leqslant c_5(\tau^2 + h), \quad 0 \leqslant k \leqslant n. \tag{3.51}$$

证明 令

$$c_3 = \max_{0\leqslant x\leqslant L, 0\leqslant t\leqslant T}|u_x(x,t)|, \quad c_6 = \max_{0\leqslant x\leqslant L, 0\leqslant t\leqslant T}|u_t(x,t)|.$$

将式 (3.29)-(3.32), (3.34)-(3.35) 与 (3.36)-(3.41) 相减, 得到误差方程

$$\delta_t e_i^{\frac{1}{2}} + \gamma\psi(\hat{u}, e^{\frac{1}{2}})_i + \delta_x^2\left(\delta_x e_{i+\frac{1}{2}}^{\frac{1}{2}}\right) = \hat{P}_i^0, \quad 1 \leqslant i \leqslant m-2, \tag{3.52}$$

$$\delta_t e_{m-1}^{\frac{1}{2}} + \gamma\psi(\hat{u}, e^{\frac{1}{2}})_{m-1} + \frac{1}{h^2}(\delta_x e_{m-\frac{3}{2}}^{\frac{1}{2}} - 3\delta_x e_{m-\frac{1}{2}}^{\frac{1}{2}}) = \hat{P}_{m-1}^0, \tag{3.53}$$

$$\Delta_t e_i^k + \gamma\left[\psi(U^k, U^{\bar{k}})_i - \psi(u^k, u^{\bar{k}})_i\right] + \delta_x^2\left(\delta_x e_{i+\frac{1}{2}}^{\bar{k}}\right) = \hat{P}_i^k,$$
$$1 \leqslant i \leqslant m-2, 1 \leqslant k \leqslant n-1, \tag{3.54}$$

$$\Delta_t e_i^k + \gamma\left[\psi(U^k, U^{\bar{k}})_{m-1} - \psi(u^k, u^{\bar{k}})_{m-1}\right] + \frac{1}{h^2}\left(\delta_x e_{m-\frac{3}{2}}^{\bar{k}} - 3\delta_x e_{m-\frac{1}{2}}^{\bar{k}}\right) = \hat{P}_{m-1}^k,$$
$$1 \leqslant k \leqslant n-1, \tag{3.55}$$

$$e_i^0 = 0, \quad 1 \leqslant i \leqslant m-1, \tag{3.56}$$

$$e_0^k = 0, \quad e_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{3.57}$$

将 (3.52) 乘以 $he_i^{\frac{1}{2}}$, 将 (3.53) 乘以 $he_{m-1}^{\frac{1}{2}}$, 并将结果相加, 得

$$(\delta_t e^{\frac{1}{2}}, e^{\frac{1}{2}}) + \gamma(\psi(\hat{u}, e^{\frac{1}{2}}), e^{\frac{1}{2}}) + \left[h\sum_{i=1}^{m-2}(\delta_x^2\delta_x e_{i+\frac{1}{2}}^{\frac{1}{2}})e_i^{\frac{1}{2}} + \frac{1}{h}(\delta_x e_{m-\frac{3}{2}}^{\frac{1}{2}} - \delta_x e_{m-\frac{1}{2}}^{\frac{1}{2}})e_{m-1}^{\frac{1}{2}}\right] = (\hat{P}^0, e^{\frac{1}{2}}). \tag{3.58}$$

应用引理 1.3 和引理 3.1, 得

$$(\psi(\hat{u}, e^{\frac{1}{2}})) = 0, \tag{3.59}$$

$$h\sum_{i=1}^{m-2}(\delta_x^2\delta_x e_{i+\frac{1}{2}}^{\frac{1}{2}})e_i^{\frac{1}{2}} + \frac{1}{h}(\delta_x e_{m-\frac{3}{2}}^{\frac{1}{2}} - \delta_x e_{m-\frac{1}{2}}^{\frac{1}{2}})e_{m-1}^{\frac{1}{2}} \geqslant 0. \tag{3.60}$$

将 (3.59) 式和 (3.60) 式代入 (3.58) 式, 可得

$$(\delta_t e^{\frac{1}{2}}, e^{\frac{1}{2}}) \leqslant (\hat{P}^0, e^{\frac{1}{2}}).$$

注意到初边值条件 (3.56) 和 (3.57), 得

$$\frac{1}{2\tau}\|e^1\|^2 \leqslant \frac{1}{2}(\hat{P}^0, e^1) \leqslant \frac{1}{2}\|\hat{P}^0\| \cdot \|e^1\|.$$

易知

$$\|e^1\| \leqslant \tau\|\hat{P}^0\| \leqslant c_4\sqrt{L}\tau(\tau^2 + h). \tag{3.61}$$

将 (3.54) 乘以 $he_i^{\bar{k}}$, 将 (3.55) 乘以 $he_{m-1}^{\bar{k}}$, 并将结果相加, 得到

$$(\Delta_t e^k, e^{\bar{k}}) + \gamma(\psi(U^k, U^{\bar{k}}) - \psi(u^k, u^{\bar{k}}), e^{\bar{k}}) + \left[h\sum_{i=1}^{m-2}(\delta_x^2\delta_x e_{i+\frac{1}{2}}^{\bar{k}})e_i^{\bar{k}} + \frac{1}{h}(\delta_x e_{m-\frac{3}{2}}^{\bar{k}} - 3\delta_x e_{m-\frac{1}{2}}^{\bar{k}})e_{m-1}^{\bar{k}}\right] = (\hat{P}^k, e^{\bar{k}}), \quad 1 \leqslant k \leqslant n-1. \tag{3.62}$$

易知

$$\begin{aligned}(\Delta_t e^k, e^{\bar{k}}) &= \frac{1}{4\tau}\|e^{k+1}\|^2 - \|e^k\|^2)\\ &= \frac{1}{2\tau}\left(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2} - \frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}\right).\end{aligned} \tag{3.63}$$

应用引理 3.1, 有

$$h\sum_{i=1}^{m-2}(\delta_x^2\delta_x e_{i+\frac{1}{2}}^{\bar{k}})e_i^{\bar{k}} + \frac{1}{h}(\delta_x e_{m-\frac{3}{2}}^{\bar{k}} - 3\delta_x e_{m-\frac{1}{2}}^{\bar{k}})e_{m-1}^{\bar{k}} \geqslant 0. \tag{3.64}$$

应用引理 1.3, 可得

$$\begin{aligned}&(\psi(U^k,U^{\bar{k}}) - \psi(u^k,u^{\bar{k}}), e^{\bar{k}})\\ =&(\psi(U^k,e^{\bar{k}}) + \psi(e^k,U^{\bar{k}}) - \psi(e^k,e^{\bar{k}}), e^{\bar{k}})\\ =&(\psi(e^k,U^{\bar{k}}), e^{\bar{k}})\\ =&\frac{1}{3}h\sum_{i=1}^{m-1}\left[e_i^k\Delta_x U_i^{\bar{k}} + \Delta_x(e^kU^{\bar{k}})_i\right]e_i^{\bar{k}}\\ =&\frac{1}{3}h\sum_{i=1}^{m-1}(\Delta_x U_i^{\bar{k}})e_i^k e_i^{\bar{k}} - \frac{1}{3}h\sum_{i=1}^{m-1}U_i^{\bar{k}}e_i^k\Delta_x e_i^{\bar{k}}\\ =&\frac{1}{3}h\sum_{i=1}^{m-1}(\Delta_x U_i^{\bar{k}})e_i^k e_i^{\bar{k}} + \frac{1}{6}\sum_{i=1}^{m-2}(U_{i+1}^{\bar{k}}e_{i+1}^k e_i^{\bar{k}} - U_i^{\bar{k}}e_i^k e_{i+1}^{\bar{k}})\\ =&\frac{1}{3}h\sum_{i=1}^{m-1}(\Delta_x U_i^{\bar{k}})e_i^k e_i^{\bar{k}} + \frac{1}{6}\sum_{i=1}^{m-2}(U_{i+1}^{\bar{k}} - U_i^{\bar{k}})e_{i+1}^k e_i^{\bar{k}} + \frac{1}{6}\sum_{i=1}^{m-2}U_i^{\bar{k}}(e_i^{\bar{k}}e_{i+1}^k - e_i^k e_{i+1}^{\bar{k}})\\ =&\frac{1}{3}h\sum_{i=1}^{m-1}(\Delta_x U_i^{\bar{k}})e_i^k e_i^{\bar{k}} + \frac{1}{6}h\sum_{i=1}^{m-2}(\delta_x U_{i+\frac{1}{2}}^{\bar{k}})e_{i+1}^k e_i^{\bar{k}}\\ &+\frac{1}{12}\sum_{i=1}^{m-2}U_i^{\bar{k}}\left[(e_i^{k+1}e_{i+1}^k - e_i^k e_{i+1}^{k+1}) - (e_i^k e_{i+1}^{k-1} - e_i^{k-1}e_{i+1}^k)\right]\\ =&\frac{1}{12}\left[\sum_{i=1}^{m-2}U_i^{k+\frac{1}{2}}(e_i^{k+1}e_{i+1}^k - e_i^k e_{i+1}^{k+1}) - \sum_{i=1}^{m-2}U_i^{k-\frac{1}{2}}(e_i^k e_{i+1}^{k-1} - e_i^{k-1}e_{i+1}^k)\right]\\ &+\frac{1}{3}h\sum_{i=1}^{m-1}(\Delta_x U_i^{\bar{k}})e_i^k e_i^{\bar{k}} + \frac{1}{6}h\sum_{i=1}^{m-2}(\delta_x U_{i+\frac{1}{2}}^{\bar{k}})e_{i+1}^k e_i^{\bar{k}}\\ &+\frac{1}{12}\sum_{i=1}^{m-2}(U_i^{\bar{k}} - U_i^{k+\frac{1}{2}})(e_i^{k+1}e_{i+1}^k - e_i^k e_{i+1}^{k+1})\\ &+\frac{1}{12}\sum_{i=1}^{m-1}(U_i^{k-\frac{1}{2}} - U_i^{\bar{k}})(e_i^k e_{i+1}^{k-1} - e_i^{k-1}e_{i+1}^k).\end{aligned} \tag{3.65}$$

记

$$G^{k-\frac{1}{2}} = \frac{\gamma}{6}\tau \sum_{i=1}^{m-2} U_i^{k-\frac{1}{2}} (e_i^k e_{i+1}^{k-1} - e_i^{k-1} e_{i+1}^k).$$

将 (3.63)–(3.65) 式代入 (3.62), 得

$$\begin{aligned}
&\frac{1}{2\tau}\left\{[\frac{\|e^{k+1}\|^2 + \|e^k\|^2}{2} + G^{k+\frac{1}{2}}] - [\frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2} + G^{k-\frac{1}{2}}]\right\} \\
\leqslant & -\gamma\left\{\frac{1}{3}h\sum_{i=1}^{m-1}(\Delta_x U_i^{\bar{k}})e_i^k e_i^{\bar{k}} + \frac{1}{6}h\sum_{i=1}^{m-2}(\delta_x U_{i+\frac{1}{2}}^{\bar{k}})e_{i+1}^k e_i^{\bar{k}}\right. \\
&+\frac{1}{12}\sum_{i=1}^{m-2}(U_i^{k-1} - U_i^k)(e_i^{k+1}e_{i+1}^k - e_i^k e_{i+1}^{k+1}) \\
&\left.+\frac{1}{12}\sum_{i=1}^{m-2}(U_i^k - U_i^{k+1})(e_i^k e_{i+1}^{k-1} - e_i^{k-1}e_{i+1}^k)\right\} + (\hat{P}^k, e^{\bar{k}}) \\
\leqslant & |\gamma|\left[\frac{1}{2}c_3\|e^k\|\|e^{\bar{k}}\| + \frac{1}{6}c_6\frac{\tau}{h}\|e^k\|\cdot\|e^{k+1}\| + \frac{1}{6}c_6\frac{\tau}{h}\|e^k\|\cdot\|e^{k-1}\|\right] \\
&+\|\hat{P}^k\|\|e^{\bar{k}}\| \\
\leqslant & \frac{1}{2}|\gamma|c_3\|e^k\|\|e^{\bar{k}}\| + \frac{1}{6}|\gamma|c_6\frac{\tau}{h}\left(\frac{\|e^k\|^2 + \|e^{k+1}\|^2}{2} + \frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2}\right) \\
&+\frac{1}{2}\left(\|\hat{P}^k\|^2 + \|e^{\bar{k}}\|^2\right), \quad 1 \leqslant k \leqslant n-1.
\end{aligned} \tag{3.66}$$

我们容易得到

$$|G^{k-\frac{1}{2}}| \leqslant \frac{\gamma}{6}\frac{c_0\tau}{h}h\sum_{i=1}^{m-2}\left(|e_i^k e_{i+1}^{k-1}| + |e_i^{k-1}e_{i+1}^k|\right) \leqslant \frac{2c_0\gamma}{6}\frac{\tau}{h}\frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2}$$

令

$$E^k = \frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2} + G^{k-\frac{1}{2}},$$

有

$$(1-\lambda)\frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2} \leqslant E^k \leqslant (1+\lambda)\frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2}.$$

应用 (3.66) 式, 存在一个常数 c_7

$$\frac{1}{2\tau}(E^{k+1} - E^k) \leqslant \frac{1}{2}c_7(E^{k+1} + E^k) + \frac{1}{2}\|\hat{P}^k\|^2, \quad 1 \leqslant k \leqslant n-1.$$

两边乘以 2τ, 并移项得到

$$(1 - c_7\tau)E^{k+1} \leqslant (1 + c_7\tau)E^k + \tau\|\hat{P}^k\|^2, \quad 1 \leqslant k \leqslant n-1.$$

当 $c_7\tau \leqslant \frac{1}{3}$ 有

$$\begin{aligned} E^{k+1} &\leqslant (1+3c_7\tau)E^k + \frac{3}{2}\tau\|\hat{P}^k\|^2 \\ &\leqslant (1+3c_7\tau)E^k + \frac{3}{2}\tau c_4^2 L(\tau^2+h)^2, \quad 1 \leqslant k \leqslant n-1. \end{aligned}$$

再由 Gronwall 不等式, 可得

$$E^k \leqslant e^{3c_7 k\tau}\left[E^1 + \frac{c_4^2 L}{2c_7}(\tau^2+h)^2\right], \quad 1 \leqslant k \leqslant n-1.$$

又注意到 (3.61) 式, 有

$$E^k \leqslant e^{3c_7 T}\left(c_4^2 L + \frac{c_4^2 L}{2c_7}(\tau^2+h)^2\right), \quad 1 \leqslant k \leqslant n.$$

因而

$$\|e^k\| \leqslant e^{\frac{3}{2}c_7 T}\sqrt{\left(1+\frac{1}{2c_7}\right)L}\, c_4(\tau^2+h), \quad 1 \leqslant k \leqslant n. \qquad \square$$

注 3.3 从数值计算的结果可以看出条件 (3.50) 不是必要的. 作者想去掉这一条件, 但证明的技巧还不够.

3.4 空间二阶二层非线性差分格式

令

$$v = u_x,$$

则 (3.1)–(3.3) 等价于

$$u_t + \gamma u u_x + v_{xx} = 0, \quad 0 < x < L, \ 0 < t \leqslant T, \tag{3.67}$$

$$v = u_x, \quad 0 < x < L, \quad 0 < t \leqslant T, \tag{3.68}$$

$$u(x,0) = \varphi(x), \quad 0 < x < L, \tag{3.69}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad v(L,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{3.70}$$

3.4.1 差分格式的建立

定义网格函数 U, V 如下

$$U_i^k = u(x_i, t_k), \quad V_i^k = v(x_i, t_k), \quad 0 \leqslant i \leqslant m, \ 0 \leqslant k \leqslant n.$$

在点 $(x_i, t_{k+\frac{1}{2}})$ 和 $(x_{i+\frac{1}{2}}, t_k)$ 处考虑方程 (3.67) 和 (3.68) 有

$$u_t(x_i, t_{k+\frac{1}{2}}) + \gamma u(x_i, t_{k+\frac{1}{2}}) u_x(x_i, t_{k+\frac{1}{2}}) + v_{xx}(x_i, t_{k+\frac{1}{2}}) = 0, \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1,$$

$$v(x_{i+\frac{1}{2}}, t_k) = u_x(x_{i+\frac{1}{2}}, t_k), \quad 0 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n.$$

应用数值微分公式可得

$$\delta_t U_i^{k+\frac{1}{2}} + \gamma\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}})_i + \delta_x^2 V_i^{k+\frac{1}{2}} = R_i^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{3.71}$$

$$V_{i+\frac{1}{2}}^k = \delta_x U_{i+\frac{1}{2}}^k + Q_{i+\frac{1}{2}}^k, \quad 0 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n. \tag{3.72}$$

存在常数 c_5 使得

$$|R_i^{k+\frac{1}{2}}| \leqslant c_5(\tau^2 + h^2), \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{3.73}$$

$$|Q_{i+\frac{1}{2}}^k| \leqslant c_5 h^2, \quad 0 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n. \tag{3.74}$$

在 (3.71) 和 (3.72) 中略去小量项, 并注意到初边值条件

$$U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{3.75}$$

$$U_0^k = 0, \quad U_m^k = 0, \quad V_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{3.76}$$

对 (3.67)–(3.70) 建立如下差分格式

$$\delta_t u_i^{k+\frac{1}{2}} + \gamma\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i + \delta_x^2 v_i^{k+\frac{1}{2}} = 0, \quad 1 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{3.77}$$

$$v_{i+\frac{1}{2}}^k = \delta_x u_{i+\frac{1}{2}}^k, \quad 0 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n, \tag{3.78}$$

$$u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{3.79}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{3.80}$$

$$v_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{3.81}$$

由 (3.73) 和 (3.74) 知差分格式 (3.77), (3.78) 的截断误差关于空间步长和时间步长均是二阶的.

定理 3.8　差分格式 (3.77)–(3.81) 等价于

$$\delta_t u_{i+\frac{1}{2}}^{k+\frac{1}{2}} + \frac{\gamma}{2}[\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i + \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_{i+1}] + \delta_x^2(\delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}}) = 0,$$
$$1 \leqslant i \leqslant m-2,\ 0 \leqslant k \leqslant n-1, \tag{3.82}$$

$$\delta_t u_{m-1}^{k+\frac{1}{2}} + \gamma\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_{m-1} + \frac{2}{h^2}(\delta_x u_{m-\frac{3}{2}}^{k+\frac{1}{2}} - 3\delta_x u_{m-\frac{1}{2}}^{k+\frac{1}{2}}) = 0,$$
$$0 \leqslant k \leqslant n-1, \tag{3.83}$$

$$u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{3.84}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{3.85}$$

$$v_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{3.86}$$

$$v_i^k = 2\delta_x u_{i+\frac{1}{2}}^k - v_{i+1}^k, \quad i = m-1, m-2, \cdots, 1, 0,\ 0 \leqslant k \leqslant n. \tag{3.87}$$

证明 (3.78) 等价于

$$v_{i+\frac{1}{2}}^0 = \delta_x u_{i+\frac{1}{2}}^0, \quad 0 \leqslant i \leqslant m-1, \tag{3.88}$$

$$v_{i+\frac{1}{2}}^{k+\frac{1}{2}} = \delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}}, \quad 0 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{3.89}$$

(3.81) 等价于

$$v_m^0 = 0, \tag{3.90}$$

$$v_m^{k+\frac{1}{2}} = 0, \quad 0 \leqslant k \leqslant n-1. \tag{3.91}$$

(3.77) 等价于

$$\delta_t u_{i+\frac{1}{2}}^{k+\frac{1}{2}} + \frac{\gamma}{2}[\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i + \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_{i+1}]$$
$$+\delta_x^2\left(\frac{v_i^{k+\frac{1}{2}} + v_{i+1}^{k+\frac{1}{2}}}{2}\right) = 0, \quad 1 \leqslant i \leqslant m-2,\ 0 \leqslant k \leqslant n-1, \tag{3.92}$$

$$\delta_t u_{m-1}^{k+\frac{1}{2}} + \gamma\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_{m-1} + \delta_x^2 v_{m-1}^{k+\frac{1}{2}} = 0, \quad 0 \leqslant k \leqslant n-1. \tag{3.93}$$

由 (3.89) 和 (3.91) 并注意到

$$\delta_x^2 v_{m-1}^{k+\frac{1}{2}} = \frac{2}{h^2}(v_{m-\frac{3}{2}}^{k+\frac{1}{2}} - 3v_{m-\frac{1}{2}}^{k+\frac{1}{2}}),$$

可知 (3.92) 和 (3.93) 等价于

$$\delta_t u_{i+\frac{1}{2}}^{k+\frac{1}{2}} + \frac{\gamma}{2}[\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i + \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_{i+1}] + \delta_x^2(\delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}}) = 0,$$
$$1 \leqslant i \leqslant m-2, \quad 0 \leqslant k \leqslant n-1, \tag{3.94}$$

$$\delta_t u_{m-1}^{k+\frac{1}{2}} + \gamma\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_{m-1} + \frac{2}{h^2}(\delta_x u_{m-\frac{3}{2}}^{k+\frac{1}{2}} - 3\delta_x u_{m-\frac{1}{2}}^{k+\frac{1}{2}}) = 0,$$
$$0 \leqslant k \leqslant n-1. \tag{3.95}$$

此外由 (3.81) 可以将 (3.78) 改写为 (3.87). □

由定理 3.8, 我们对问题 (3.1)–(3.3) 建立差分格式 (3.82)–(3.85).

由对称性易知差分格式 (3.94) 的截断误差为 $O(\tau^2+h^2)$. 注意到由 (3.8) 可知差分格式 (3.95) 的截断误差也是 $O(\tau^2+h^2)$. 观察 (3.95) 和 (3.16) 发现它们的左端第三项相差一个常数. 差分格式 (3.16) 为与 (3.15) 在 $i=m-2$ 处的差分格式相"匹配"的差分格式, 差分格式 (3.95) 为与 (3.94) 在 $i=m-2$ 处的差分格式相"匹配"的差分格式. 这里的"匹配"是指保证差分格式解的守恒性、有界性和收敛性.

3.4.2　差分格式解的存在性

定理 3.9　差分格式 (3.77)–(3.81) 解是存在的.

证明　由 (3.79) 和 (3.80) 可得 $\{u_i^0\,|\,0\leqslant i\leqslant m\}$.

(3.78) 可以写成如下等价形式

$$v_{i+\frac{1}{2}}^0=\delta_x u_{i+\frac{1}{2}}^0,\quad 0\leqslant i\leqslant m-1,\tag{3.96}$$

$$v_{i+\frac{1}{2}}^{k+\frac{1}{2}}=\delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}},\quad 0\leqslant i\leqslant m-1,\quad 0\leqslant k\leqslant n-1.\tag{3.97}$$

现设第 k 层的解 $\{u^k,v^k\}$ 已求得. 记

$$w_i=\frac{1}{2}\big(u_i^{k+1}+u_i^k\big),\quad z_i=v_i^{k+\frac{1}{2}},\quad 0\leqslant i\leqslant m,$$

则有

$$u_i^{k+1}=2w_i-u_i^k,\quad v_i^{k+1}=2z_i-v_i^k,\quad 0\leqslant i\leqslant m.$$

由差分格式 (3.77), (3.78) 及 (3.80), (3.81) 可得关于 $\{w,z\}$ 的方程组

$$\frac{2}{\tau}\big(w_i-u_i^k\big)+\gamma\psi(w,w)_i+\delta_x^2 z_i=0,\quad 1\leqslant i\leqslant m-1,\tag{3.98}$$

$$z_{i+\frac{1}{2}}=\delta_x w_{i+\frac{1}{2}},\quad 0\leqslant i\leqslant m-1,\tag{3.99}$$

$$w_0=0,\quad w_m=0,\tag{3.100}$$

$$z_m=0.\tag{3.101}$$

由 (3.99) 和 (3.101) 可知

$$z_m=0;\quad z_i=2\delta_x w_{i+\frac{1}{2}}-z_{i+1},\quad i=m-1,m-2,\cdots,0.\tag{3.102}$$

易知 $z_i(i=m-1,m-2,\cdots,1,0)$ 可由 $\{w_i\,|\,0\leqslant i\leqslant m\}$ 线性表示:

$$z_{m-2p}=-2h\sum_{l=1}^{p}\delta_x^2 w_{m-2l+1},\quad p=0,1,2,\cdots,\left[\frac{m}{2}\right],$$

$$z_{m-2p-1}=2\delta_x w_{m-\frac{1}{2}}-2h\sum_{l=1}^{p}\delta_x^2 w_{m-2l},\quad p=0,1,2,\cdots,\left[\frac{m-1}{2}\right].$$

这里 $[A]$ 表示不大于 A 的正整数.

定义 $\mathring{\mathcal{U}}_h$ 上的算子

$$\Pi(w)_i = \frac{2}{\tau}(w_i - u_i^k) + \gamma\psi(w,w)_i + \delta_x^2 z_i, \quad 1 \leqslant i \leqslant m-1,$$

其中 $z_i(i=m,m-1,\cdots,0)$ 由 (3.99) 和 (3.101) 定义.

计算得到

$$\begin{aligned}(\Pi(w),w) &= \frac{2}{\tau}[(w,w)-(u^k,w)] + \gamma(\psi(w,w),w) + h\sum_{i=1}^{m-1}(\delta_x^2 z_i)w_i \\ &= \frac{2}{\tau}[(w,w)-(u^k,w)] - h\sum_{i=0}^{m-1}(\delta_x z_{i+\frac{1}{2}})(\delta_x w_{i+\frac{1}{2}}) \\ &= \frac{2}{\tau}[(w,w)-(u^k,w)] - h\sum_{i=1}^{m-1}(\delta_x z_{i+\frac{1}{2}})z_{i+\frac{1}{2}} \\ &= \frac{2}{\tau}[(w,w)-(u^k,w)] - \frac{1}{2}(v_m^2 - z_0^2) \\ &\geqslant \frac{2}{\tau}(\|w\|^2 - \|u^k\|\cdot\|w\|) + \frac{1}{2}z_0^2 \\ &\geqslant \frac{2}{\tau}(\|w\| - \|u^k\|)\|w\|.\end{aligned}$$

当 $\|w\| = \|u^k\|$ 时 $(\Pi(w),w) \geqslant 0$. 由 Browder 定理 (定理 1.3) 知方程组 (3.98)–(3.100) 存在解 $w \in \mathring{\mathcal{U}}_h$, 且 $\|w\| \leqslant \|u^k\|$, 使得 $\Pi(w)=0$. 当得到 w 后, 再由 (3.102) 得到 z. □

由定理 3.8 和定理 3.9, 我们间接证明了差分格式 (3.82)–(3.85) 解的存在性, 也即证明了如下非线性方程组解的存在性:

$$\frac{2}{\tau}(w_{i+\frac{1}{2}} - u_{i+\frac{1}{2}}^k) + \frac{\gamma}{2}[\psi(w,w)_i + \psi(w,w)_{i+1}] + \delta_x^2(\delta_x w_{i+\frac{1}{2}}) = 0, \quad 1 \leqslant i \leqslant m-2, \tag{3.103}$$

$$\frac{2}{\tau}(w_{m-1} - u_{m-1}^k) + \gamma\psi(w,w)_{m-1} + \frac{2}{h^2}(\delta_x w_{m-\frac{3}{2}} - 3\delta_x w_{m-\frac{1}{2}}) = 0, \tag{3.104}$$

$$w_0 = 0, \quad w_m = 0. \tag{3.105}$$

3.4.3 差分格式解的守恒性和有界性

定理 3.10 差分格式 (3.82)–(3.85) 的解满足

$$\|u^k\|^2 + \tau\sum_{l=0}^{k-1}(v_0^{l+\frac{1}{2}})^2 = \|u^0\|^2, \quad 0 \leqslant k \leqslant n, \tag{3.106}$$

其中 $v_0^{l+\frac{1}{2}}$ 由下式定义

$$v_m^{k+\frac{1}{2}} = 0, \quad v_i^{k+\frac{1}{2}} = 2\delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}} - v_{i+1}^{k+\frac{1}{2}}, \quad i = m-1, m-2, \cdots, 0.$$

证明　由定理 3.8 知只要证明 (3.77)–(3.80) 的解 $\{u^k\,|\,0\leqslant k\leqslant n\}$ 满足 (3.106). 用 $u^{k+\frac{1}{2}}$ 与 (3.77) 作内积, 可得

$$\frac{1}{2\tau}(\|u^{k+1}\|^2-\|u^k\|^2)+\gamma(\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}}),u^{k+\frac{1}{2}})+h\sum_{i=1}^{m-1}(\delta_x^2 v_i^{k+\frac{1}{2}})u_i^{k+\frac{1}{2}}=0.$$

注意到

$$(\psi(u^{k+\frac{1}{2}},u^{k+\frac{1}{2}}),u^{k+\frac{1}{2}})=0,$$

$$\begin{aligned}&h\sum_{i=1}^{m-1}(\delta_x^2 v_i^{k+\frac{1}{2}})u_i^{k+\frac{1}{2}}\\&=-h\sum_{i=0}^{m-1}(\delta_x v_{i+\frac{1}{2}}^{k+\frac{1}{2}})\delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}}\\&=-h\sum_{i=0}^{m-1}(\delta_x v_{i+\frac{1}{2}}^{k+\frac{1}{2}})(v_{i+\frac{1}{2}}^{k+\frac{1}{2}})\\&=-\frac{1}{2}[(v_m^{k+\frac{1}{2}})^2-(v_0^{k+\frac{1}{2}})^2],\end{aligned}$$

有

$$\frac{1}{2\tau}(\|u^{k+1}\|^2-\|u^k\|^2)+\frac{1}{2}(v_0^{k+\frac{1}{2}})^2=0,\quad 0\leqslant k\leqslant n.$$

于是,

$$\|u^k\|^2+\tau\sum_{l=0}^{k-1}(v_0^{l+\frac{1}{2}})^2=\|u^0\|^2,\quad 0\leqslant k\leqslant n.$$ □

由定理 3.10 易得

$$\|u^k\|\leqslant\|u^0\|,\quad 0\leqslant k\leqslant n.$$

3.5　空间二阶三层线性化差分格式

3.5.1　差分格式的建立

在点 (x_i,t_0) 处考虑方程 (3.1) 并注意到初边值条件 (3.2) 有

$$\begin{aligned}u_t(x_i,t_0)&=-\gamma u(x_i,t_0)u_x(x_i,t_0)-u_{xxx}(x_i,t_0)\\&=-\gamma\varphi(x_i)\varphi'(x_i)-\varphi'''(x_i),\quad 0\leqslant i\leqslant m.\end{aligned}$$

记

$$\hat{u}_i=\varphi(x_i)+\frac{\tau}{2}[-\gamma\varphi(x_i)\varphi'(x_i)-\varphi'''(x_i)],\quad 0\leqslant i\leqslant m.$$

在 $(x_i, t_{\frac{1}{2}})$ 处考虑方程 (3.67) 并应用 Taylor 展开式, 有

$$\delta_t U_i^{\frac{1}{2}} + \gamma\psi(\hat{u}, U^{\frac{1}{2}})_i + \delta_x^2 V_i^{\frac{1}{2}} = \hat{R}_i^0, \quad 1 \leqslant i \leqslant m-1, \tag{3.107}$$

存在正常数 c_5 使得

$$|\hat{R}_i^0| \leqslant c_5(\tau^2 + h^2), \quad 1 \leqslant i \leqslant m-1. \tag{3.108}$$

在点 (x_i, t_k) 处考虑方程 (3.67), 应用 Taylor 展开式, 得到

$$\Delta_t U_i^k + \gamma\psi(U^k, U^{\bar{k}})_i + \delta_x^2 V_i^{\bar{k}} = \hat{R}_i^k, \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{3.109}$$

且存在正常数 c_6 使得

$$|\hat{R}_i^k| \leqslant c_6(\tau^2 + h^2), \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1. \tag{3.110}$$

在点 $(x_{i+\frac{1}{2}}, t_k)$ 处考虑方程 (3.68), 可得

$$V_{i+\frac{1}{2}}^k = \delta_x U_{i+\frac{1}{2}}^k + \hat{Q}_{i+\frac{1}{2}}^k, \quad 0 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n, \tag{3.111}$$

存在常数 c_7 使得

$$|\hat{Q}_{i+\frac{1}{2}}^k| \leqslant c_7 h^2, \quad 0 \leqslant k \leqslant n. \tag{3.112}$$

注意到初边值条件 (3.69)–(3.70) 有

$$U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{3.113}$$

$$U_0^k = 0, \quad U_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{3.114}$$

$$V_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{3.115}$$

在 (3.107), (3.109), (3.111) 中略去小量项, 对 (3.67)–(3.70) 建立如下差分格式

$$\delta_t u_i^{\frac{1}{2}} + \gamma\psi(\hat{u}, u^{\frac{1}{2}})_i + \delta_x^2 v_i^{\frac{1}{2}} = 0, \quad 1 \leqslant i \leqslant m-1, \tag{3.116}$$

$$\Delta_t u_i^k + \gamma\psi(u^k, u^{\bar{k}})_i + \delta_x^2 v_i^{\bar{k}} = 0, \quad 1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{3.117}$$

$$v_{i+\frac{1}{2}}^k = \delta_x u_{i+\frac{1}{2}}^k, \quad 0 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n, \tag{3.118}$$

$$u_i^0 = \varphi(x_i) \quad 1 \leqslant i \leqslant m-1, \tag{3.119}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{3.120}$$

$$v_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{3.121}$$

由 (3.118) 和 (3.121) 可得

$$v_{i+\frac{1}{2}}^{\frac{1}{2}} = \delta_x u_{i+\frac{1}{2}}^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant m-1, \quad v_m^{\frac{1}{2}} = 0,$$

$$v_{i+\frac{1}{2}}^{\bar{k}} = \delta_x u_{i+\frac{1}{2}}^{\bar{k}}, \quad 0 \leqslant i \leqslant m-1, \quad v_m^{\bar{k}} = 0, \quad 1 \leqslant k \leqslant n-1.$$

定理 3.11　差分格式 (3.116)–(3.121) 等价于

$$\delta_t u_{i+\frac{1}{2}}^{\frac{1}{2}} + \frac{\gamma}{2}[\psi(\hat{u}, u^{\frac{1}{2}})_i + \psi(\hat{u}, u^{\frac{1}{2}})_{i+1}] + \delta_x^2(\delta_x u_{i+\frac{1}{2}}^{\frac{1}{2}}) = 0, \quad 1 \leqslant i \leqslant m-2, \tag{3.122}$$

$$\delta_t u_{m-1}^{\frac{1}{2}} + \gamma\psi(\hat{u}, u^{\frac{1}{2}})_{m-1} + \frac{2}{h^2}(\delta_x u_{m-\frac{3}{2}}^{\frac{1}{2}} - 3\delta_x u_{m-\frac{1}{2}}^{\frac{1}{2}}) = 0, \tag{3.123}$$

$$\Delta_t u_{i+\frac{1}{2}}^k + \frac{\gamma}{2}[\psi(u^k, u^{\bar{k}})_i + \psi(u^k, u^{\bar{k}})_{i+1}] + \delta_x^2(\delta_x u_{i+\frac{1}{2}}^{\bar{k}}) = 0, \quad 1 \leqslant i \leqslant m-2,\ 1 \leqslant k \leqslant n-1, \tag{3.124}$$

$$\Delta_t u_{m-1}^k + \gamma\psi(u^k, u^{\bar{k}})_{m-1} + \frac{2}{h^2}(\delta_x u_{m-\frac{3}{2}}^{\bar{k}} - 3\delta_x u_{m-\frac{1}{2}}^{\bar{k}}) = 0, \quad 1 \leqslant k \leqslant n-1, \tag{3.125}$$

$$u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, \tag{3.126}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{3.127}$$

$$v_m^k = 0, \quad v_i^k = 2\delta_x u_{i+\frac{1}{2}}^k - v_{i+1}^k, \quad i = m-1, m-2, \cdots, 0,\ 0 \leqslant k \leqslant n. \tag{3.128}$$

证明略. □

由定理 3.11 知, 我们对 (3.1)–(3.3) 建立差分格式 (3.122)–(3.127). 可以看到 (3.122)–(3.123) 为关于 u^1 的线性方程组; 当 u^{k-1}, u^k 已求得时, (3.124)–(3.125) 为关于 u^{k+1} 的线性方程组. 可以证明差分格式 (3.116)–(3.121) 的解是存在唯一的. 再由定理 3.11 知 (3.122)–(3.127) 的解也是存在唯一的.

3.5.2　差分格式解的守恒性和有界性

定理 3.12　差分格式 (3.122)–(3.127) 的解满足

$$\frac{1}{2}(\|u^{k+1}\|^2 + \|u^k\|^2) + \frac{1}{2}\tau(v_0^{\frac{1}{2}})^2 + \tau\sum_{l=1}^{k}\|v^{\bar{l}}\|^2 = \|u^0\|^2, \quad 1 \leqslant k \leqslant n-1. \tag{3.129}$$

证明　由定理 3.11 知只要证明差分格式 (3.116)–(3.121) 的解满足 (3.129).

(I) 用 $hu_i^{\frac{1}{2}}$ 乘以 (3.116) 的两边, 并对 i 从 1 到 $m-1$ 求和, 得

$$h\sum_{i=1}^{m-1} u_i^{\frac{1}{2}}\delta_t u_i^{\frac{1}{2}} + \gamma h\sum_{i=1}^{m-1} u_i^{\frac{1}{2}}\psi(\hat{u}, u^{\frac{1}{2}})_i + h\sum_{i=1}^{m-1} u_i^{\frac{1}{2}}\delta_x^2 v_i^{\frac{1}{2}} = 0. \tag{3.130}$$

现分析 (3.130) 中的每一项.

左端第一项

$$h\sum_{i=1}^{m-1} u_i^{\frac{1}{2}}\delta_t u_i^{\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\tau}\left[h\sum_{i=1}^{m-1}(u_i^1)^2 - h\sum_{i=1}^{m-1}(u_i^0)^2\right] = \frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2).$$

左端第二项

$$\gamma h\sum_{i=1}^{m-1}u_i^{\frac{1}{2}}\psi(\hat{u},u^{\frac{1}{2}})_i=\gamma\big(\psi(\hat{u},u^{\frac{1}{2}}),u^{\frac{1}{2}}\big)=0.$$

注意到 (3.120) 有 $u_0^{\frac{1}{2}}=0, u_m^{\frac{1}{2}}=0$. 左端第三项利用 (3.118), (3.121), 得到

$$\begin{aligned}
&h\sum_{i=1}^{m-1}u_i^{\frac{1}{2}}\delta_x^2v_i^{\frac{1}{2}}\\
=&-h\sum_{i=0}^{m-1}(\delta_xu_{i+\frac{1}{2}}^{\frac{1}{2}})\delta_xv_{i+\frac{1}{2}}^{\frac{1}{2}}\\
=&-h\sum_{i=0}^{m-1}v_{i+\frac{1}{2}}^{\frac{1}{2}}\delta_xv_{i+\frac{1}{2}}^{\frac{1}{2}}\\
=&-\frac{1}{2}\sum_{i=0}^{m-1}[(v_{i+1}^{\frac{1}{2}})^2-(v_i^{\frac{1}{2}})^2]\\
=&\frac{1}{2}[(v_0^{\frac{1}{2}})^2-(v_m^{\frac{1}{2}})^2]=\frac{1}{2}(v_0^{\frac{1}{2}})^2.
\end{aligned}$$

将以上三式代入 (3.130), 得到

$$\frac{1}{2\tau}(\|u^1\|^2-\|u^0\|^2)+\frac{1}{2}(v_0^{\frac{1}{2}})^2=0,$$

或

$$\frac{1}{2}\big(\|u^1\|^2+\|u^0\|^2\big)+\frac{1}{2}\tau(v_0^{\frac{1}{2}})^2=\|u^0\|^2. \tag{3.131}$$

(II) 用 $hu_i^{\bar{k}}$ 乘以 (3.117) 的两边, 并对 i 从 1 到 $m-1$ 求和, 得

$$\frac{1}{4\tau}(\|u^{k+1}\|^2-\|u^{k-1}\|^2)+\frac{1}{2}(v_0^{\bar{k}})^2=0,\quad 1\leqslant k\leqslant n-1,$$

或

$$\frac{1}{\tau}\left(\frac{\|u^{k+1}\|^2+\|u^k\|^2}{2}-\frac{\|u^k\|^2+\|u^{k-1}\|^2}{2}\right)+\big(v_0^{\bar{k}}\big)^2=0,\quad 1\leqslant k\leqslant n-1.$$

将以上式中 k 换为 l, 再对 l 从 1 到 k 求和, 得

$$\frac{\|u^{k+1}\|^2+\|u^k\|^2}{2}+\tau\sum_{l=1}^{k}(v_0^{\bar{l}})^2=\frac{\|u^1\|^2+\|u^0\|^2}{2},\quad 1\leqslant k\leqslant n-1. \tag{3.132}$$

由 (3.131) 和 (3.132) 可得 (3.129). □

3.6 小结与延拓

KdV 方程是空间三阶的发展型微分方程. KdV 方程初边值问题 (3.1)–(3.3) 的解满足守恒律 (3.4). 本章对其建立了 4 个差分格式. 前两个关于空间步长是一阶的, 后两个关于空间步长是二阶的.

引进新变量 $v = u_x$, 将原方程 (3.1) 写为等价的方程组 (3.67)–(3.68). (3.67) 是一个空间二阶方程, (3.68) 是一个空间一阶方程. 然后对 (3.67)–(3.70) 建立差分格式 (3.77)–(3.81). 如此建立差分格式比较自然. 引进的中间变量 $\{v_i^k\}$ 不必实际参加计算. 可以将差分格式 (3.77)–(3.81) 作变量分离, 得到仅含变量 $\{u_i^k\}$ 的差分方程组 (3.82)–(3.85). 我们证明了差分格式 (3.82)–(3.85) 解的存在性和有界性.

我们用 Browder 定理证明了差分格式 (3.116)–(3.121) 解的存在性, 从而间接地证明了差分格式 (3.82)–(3.85) 解的存在性. 直接证明差分格式 (3.82)–(3.85) 解的存在性, 需要证明非线性方程组 (3.103)–(3.105) 解的存在性. 引进新变量 z, 定义算子 $\Pi(w)$, 再用不动点定理进行证明.

可以对方程 (3.67)–(3.68) 离散化, 得到

$$\begin{aligned}
&\delta_t U_i^{k+\frac{1}{2}} + \gamma\psi(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}})_i + \delta_x^2 V_i^{k+\frac{1}{2}} = \widetilde{P}_i^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n-1,\\
&V_i^k = \delta_x U_{i+\frac{1}{2}}^k + \widetilde{Q}_i^k, \quad 0 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n,
\end{aligned}$$

存在常数 c_8 使得

$$\begin{aligned}
&|\widetilde{P}_i^{k+\frac{1}{2}}| \leqslant c_8(\tau^2 + h^2), \quad 1 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n-1,\\
&|\widetilde{Q}_i^k| \leqslant c_8 h, \quad 0 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n.
\end{aligned}$$

略去小量项, 对 (3.67)–(3.70) 建立如下空间一阶差分格式

$$\begin{aligned}
&\delta_t u_i^{k+\frac{1}{2}} + \gamma\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i + \delta_x^2 v_i^{k+\frac{1}{2}} = 0,\\
&\qquad\qquad 1 \leqslant i \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, &(3.133)\\
&v_i^k = \delta_x u_{i+\frac{1}{2}}^k, \quad 0 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n, &(3.134)\\
&u_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant m-1, &(3.135)\\
&u_0^k = 0, \quad u_m^k = 0, \quad v_m^k = 0, \quad 0 \leqslant k \leqslant n, &(3.136)
\end{aligned}$$

可以证明差分格式 (3.133)–(3.136) 等价于

$$\begin{aligned}
&\delta_t u_i^{k+\frac{1}{2}} + \gamma\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i + \delta_x^2(\delta_x u_{i+\frac{1}{2}}^{k+\frac{1}{2}}) = 0,\\
&\qquad\qquad 1 \leqslant i \leqslant m-2, \quad 0 \leqslant k \leqslant n-1, &(3.137)
\end{aligned}$$

$$\delta_t u_{m-1}^{k+\frac{1}{2}} + \gamma\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_{m-1} + \frac{1}{h^2}(\delta_x u_{m-\frac{3}{2}}^{k+\frac{1}{2}} - 2\delta_x u_{m-\frac{1}{2}}^{k+\frac{1}{2}}) = 0,$$
$$0 \leqslant k \leqslant n-1, \tag{3.138}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n \tag{3.139}$$

和

$$v_i^k = \delta_x u_{i+\frac{1}{2}}^k, \quad 0 \leqslant i \leqslant m-1,\ 0 \leqslant k \leqslant n, \tag{3.140}$$

$$v_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{3.141}$$

可以证明差分格式 (3.137)–(3.139) 解的存在性、有界性和守恒性. 比较 (3.138) 与 (3.16), 发现它们是不同的.

注意到 $u_{xxx}(L,t)=0$ 可知 (3.16) 的截断误差为 $O(\tau^2+h^2)$, 而 (3.138) 的截断误差为 $O(\tau^2+h^2)+O(h^{-1})u_{xx}(L,t_{k+\frac{1}{2}})$.

证明 KdV 方程初边值问题差分格式解的收敛性是一个不太容易的工作. 我们只证明了两个空间一阶差分格式解的收敛性.

第 4 章 Camassa-Holm 方程的差分方法

4.1 引　　言

Camassa-Holm (C-H) 方程具有带尖点的孤立波解使它成为浅水波理论研究的重要对象之一. 许多学者对 Camassa-Holm 方程多方面的性质做了深入的研究, 如孤波解、双 Hamiltonian 结构、完全对称性等.

本章考虑 Camassa-Holm 方程初边值问题

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad 0 < x < L,\ 0 < t \leqslant T, \tag{4.1}$$

$$u(x,0) = \varphi(x), \quad 0 < x < L, \tag{4.2}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad 0 < t \leqslant T, \tag{4.3}$$

其中 $\varphi(0) = \varphi(L) = 0$.

方程 (4.1) 可以写为

$$u_t - u_{xxt} + 3uu_x = u_x u_{xx} + (uu_{xx})_x. \tag{4.4}$$

在介绍差分方法之前, 我们先用能量方法给出 (4.1)–(4.3) 解的先验估计式.

定理 4.1　设 $u(x,t)$ 为问题 (4.1)–(4.3) 的解. 记

$$E(t) = \int_0^L u^2(x,t)\mathrm{d}x + \int_0^L u_x^2(x,t)\mathrm{d}x, \tag{4.5}$$

则有

$$E(t) \equiv E(0), \quad 0 < t \leqslant T. \tag{4.6}$$

证明　将 (4.1) 的两边同乘以 u, 并对 x 从 0 到 L 积分, 得

$$\int_0^L uu_t\mathrm{d}x - \int_0^L uu_{xxt}\mathrm{d}x + 3\int_0^L u^2u_x\mathrm{d}x = \int_0^L (2uu_xu_{xx} + u^2u_{xxx})\mathrm{d}x.$$

将上式应用分部求积公式, 可得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u^2\mathrm{d}x - uu_{xt}\big|_0^L + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L u_x^2\mathrm{d}x + u^3\big|_{x=0}^L = u^2u_{xx}\big|_{x=0}^L.$$

再由 (4.3) 得到

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0, \quad 0 < t \leqslant T.$$

因而 (4.6) 成立. □

由 (4.5) 知问题 (4.1)–(4.3) 的解在 H^1 范数下是守恒的.

4.2 二层非线性差分格式

本章记

$$c_0 = \max_{0\leqslant x\leqslant L, 0\leqslant t\leqslant T}|u(x,t)|, \qquad c_1 = \max_{0\leqslant x\leqslant L, 0\leqslant t\leqslant T}|u_x(x,t)|,$$
$$c_2 = \max_{0\leqslant x\leqslant L, 0\leqslant t\leqslant T}|u_{xx}(x,t)|, \quad c_3 = \max_{0\leqslant x\leqslant L, 0\leqslant t\leqslant T}|u_t(x,t)|.$$

4.2.1 差分格式的建立

在点 $(x_i, t_{k+\frac{1}{2}})$ 处考虑方程 (4.4) 有

$$\begin{aligned}&u_t(x_i,t_{k+\frac{1}{2}}) - u_{xxt}(x_i,t_{k+\frac{1}{2}}) + 3u(x_i,t_{k+\frac{1}{2}})u_x(x_i,t_{k+\frac{1}{2}})\\ =\ &u_x(x_i,t_{k+\frac{1}{2}})u_{xx}(x_i,t_{k+\frac{1}{2}}) + (uu_{xx})_x(x_i,t_{k+\frac{1}{2}}).\end{aligned}$$

应用数值微分公式, 可得到

$$\begin{aligned}&\delta_t U_i^{k+\frac{1}{2}} - \delta_t\delta_x^2 U_i^{k+\frac{1}{2}} + 3\psi\big(U^{k+\frac{1}{2}}, U^{k+\frac{1}{2}}\big)_i\\ =\ &\big(\Delta_x U_i^{k+\frac{1}{2}}\big)\delta_x^2 U_i^{k+\frac{1}{2}} + \Delta_x\big(U_i^{k+\frac{1}{2}}\delta_x^2 U_i^{k+\frac{1}{2}}\big) + R_i^{k+\frac{1}{2}},\\ &1\leqslant i\leqslant m-1, \quad 0\leqslant k\leqslant n-1,\end{aligned} \tag{4.7}$$

存在常数 c_4 使得

$$\big|R_i^{k+\frac{1}{2}}\big| \leqslant c_4(\tau^2+h^2), \quad 1\leqslant i\leqslant m-1,\ 0\leqslant k\leqslant n-1. \tag{4.8}$$

注意到初边值条件有

$$U_i^0 = \varphi(x_i), \quad 1\leqslant i\leqslant m-1, \tag{4.9}$$

$$U_0^k = 0, \quad U_m^k = 0, \quad 0\leqslant k\leqslant n. \tag{4.10}$$

在 (4.7) 中略去小量项 $R_i^{k+\frac{1}{2}}$, 对 (4.1)–(4.3) 建立如下差分格式

$$\begin{aligned}&\delta_t u_i^{k+\frac{1}{2}} - \delta_t\delta_x^2 u_i^{k+\frac{1}{2}} + 3\psi\big(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\big)_i = \big(\Delta_x u_i^{k+\frac{1}{2}}\big)\delta_x^2 u_i^{k+\frac{1}{2}}\\ &\quad+\Delta_x\big(u_i^{k+\frac{1}{2}}\delta_x^2 u_i^{k+\frac{1}{2}}\big), \quad 1\leqslant i\leqslant m-1, \quad 0\leqslant k\leqslant n-1,\end{aligned} \tag{4.11}$$

$$u_i^0 = \varphi(x_i), \quad 1\leqslant i\leqslant m-1, \tag{4.12}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0\leqslant k\leqslant n. \tag{4.13}$$

差分格式 (4.11) 的右端可以写为

$$2(\Delta_x u_i^{k+\frac{1}{2}})\delta_x^2 u_i^{k+\frac{1}{2}} + \frac{1}{2}\Big[u_{i-1}^{k+\frac{1}{2}}\frac{\delta_x^2 u_i^{k+\frac{1}{2}} - \delta_x^2 u_{i-1}^{k+\frac{1}{2}}}{h} + u_{i+1}^{k+\frac{1}{2}}\frac{\delta_x^2 u_{i+1}^{k+\frac{1}{2}} - \delta_x^2 u_i^{k+\frac{1}{2}}}{h}\Big].$$

4.2.2 差分格式解的守恒性

定理 4.2 设 $\{u_i^k \mid 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (4.11)–(4.13) 的解. 记

$$E^k = \|u^k\|^2 + |u^k|_1^2, \quad 0 \leqslant k \leqslant n,$$

则有

$$E^k \equiv E^0, \quad 0 \leqslant k \leqslant n.$$

证明 用 $u^{k+\frac{1}{2}}$ 与 (4.11) 作内积, 得

$$\begin{aligned}&\left(\delta_t u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) - \left(\delta_t \delta_x^2 u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) + 3\left(\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}), u^{k+\frac{1}{2}}\right)\\ =& \left((\Delta_x u^{k+\frac{1}{2}})\delta_x^2 u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) + \left(\Delta_x(u^{k+\frac{1}{2}}\delta_x^2 u^{k+\frac{1}{2}}), u^{k+\frac{1}{2}}\right).\end{aligned} \tag{4.14}$$

现在来分析上式中的每一项:

$$\left(\delta_t u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) = \frac{1}{2\tau}\left(\|u^{k+1}\|^2 - \|u^k\|^2\right);$$

$$-\left(\delta_t \delta_x^2 u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) = \left(\delta_t \delta_x u^{k+\frac{1}{2}}, \delta_x u^{k+\frac{1}{2}}\right) = \frac{1}{2\tau}\left(|u^{k+1}|_1^2 - |u^k|_1^2\right);$$

$$\left(\psi\left(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right), u^{k+\frac{1}{2}}\right) = 0;$$

$$\begin{aligned}&\left(\left(\Delta_x u^{k+\frac{1}{2}}\right)\delta_x^2 u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) + \left(\Delta_x\left(u^{k+\frac{1}{2}}\delta_x^2 u^{k+\frac{1}{2}}\right), u^{k+\frac{1}{2}}\right)\\ =& \left(\Delta_x u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\delta_x^2 u^{k+\frac{1}{2}}\right) + \left(u^{k+\frac{1}{2}}, \Delta_x\left(u^{k+\frac{1}{2}}\delta_x^2 u^{k+\frac{1}{2}}\right)\right)\\ =& 0.\end{aligned}$$

将以上 4 式代入 (4.14), 得

$$\frac{1}{2\tau}\left(E^{k+1} - E^k\right) = 0, \quad 0 \leqslant k \leqslant n-1.$$

因而

$$E^k \equiv E^0, \quad 0 \leqslant k \leqslant n.$$

□

4.2.3 差分格式解的存在性和唯一性

定理 4.3 差分格式 (4.11)–(4.13) 存在解.

证明 设第 k 层的值 u^k 已求得, 则由 (4.11) 和 (4.13) 可得

$$\begin{aligned}&\frac{2}{\tau}(u_i^{k+\frac{1}{2}} - u_i^k) - \frac{2}{\tau}(\delta_x^2 u_i^{k+\frac{1}{2}} - \delta_x^2 u_i^k) + 3\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_i\\ &\quad = (\Delta_x u_i^{k+\frac{1}{2}})\delta_x^2 u_i^{k+\frac{1}{2}} + \Delta_x(u_i^{k+\frac{1}{2}}\delta_x^2 u_i^{k+\frac{1}{2}}), \quad 1 \leqslant i \leqslant m-1,\\ &u_0^{k+\frac{1}{2}} = 0, \quad u_m^{k+\frac{1}{2}} = 0.\end{aligned}$$

先求出 $u_i^{k+\frac{1}{2}}$, 则有

$$u_i^{k+1} = 2u_i^{k+\frac{1}{2}} - u_i^k.$$

记

$$w_i = u_i^{k+\frac{1}{2}}, \quad 0 \leqslant i \leqslant m,$$

则有

$$\begin{aligned} &\frac{2}{\tau}(w_i - u_i^k) - \frac{2}{\tau}(\delta_x^2 w_i - \delta_x^2 u_i^k) + 3\psi(w,w)_i \\ &\quad - \big[(\Delta_x w_i)(\delta_x^2 w_i) + \Delta_x(w_i \delta_x^2 w_i)\big] = 0, \quad 1 \leqslant i \leqslant m-1, \end{aligned} \tag{4.15}$$

$$w_0 = 0, \quad w_m = 0. \tag{4.16}$$

设 $w \in \mathring{\mathcal{U}}_h$. 定义

$$\begin{aligned} \Pi(w)_i =& \frac{2}{\tau}(w_i - u_i^k) - \frac{2}{\tau}(\delta_x^2 w_i - \delta_x^2 u_i^k) + 3\psi(w,w)_i \\ & - [(\Delta_x w_i)(\delta_x^2 w_i) + \Delta_x(w_i \delta_x^2 w_i)], \quad 1 \leqslant i \leqslant m-1. \end{aligned}$$

用 w 和 $\Pi(w)$ 作内积, 得

$$\begin{aligned} (\Pi(w), w) =& \frac{2}{\tau}\big[(w,w) - (u^k, w)\big] - \frac{2}{\tau}\big[(\delta_x^2 w, w) - (\delta_x^2 u^k, w)\big] + 3\big(\psi(w,w), w\big) \\ & - \big((\Delta_x w)(\delta_x^2 w) + \Delta_x(w \delta_x^2 w), w\big) \\ =& \frac{2}{\tau}\big[\|w\|^2 - (u^k, w)\big] + \frac{2}{\tau}\big[(\delta_x w, \delta_x w) - (\delta_x u^k, \delta_x w)\big] \\ \geqslant& \frac{2}{\tau}\left[\|w\|^2 - \frac{1}{2}(\|u^k\|^2 + \|w\|^2)\right] + \frac{2}{\tau}\left[\|\delta_x w\|^2 - \left(\|\delta_x w\|^2 + \frac{1}{4}\|\delta_x u^k\|^2\right)\right] \\ =& \frac{1}{\tau}\left[\|w\|^2 - \left(\|u^k\|^2 + \frac{1}{2}|u^k|_1^2\right)\right], \end{aligned}$$

当 $\|w\|^2 = \|u^k\|^2 + \dfrac{1}{2}|u^k|_1^2$ 时, $\big(\Pi(w), w\big) \geqslant 0$. 由 Browder 定理 (定理 1.3) 知方程组 (4.15)–(4.16) 存在解. □

定理 4.4　存在常数 c_5 当 $\dfrac{\tau}{h} < \dfrac{2}{c_5}$ 时差分格式 (4.11)–(4.13) 的解是唯一的.

证明　由定理 4.3 的证明过程知只需要证明 (4.15)–(4.16) 的解是唯一的. 设 (4.15)–(4.16) 还有另一组解 $\{v_i \,|\, 0 \leqslant i \leqslant m\}$, 即

$$\begin{aligned} &\frac{2}{\tau}(v_i - u_i^k) - \frac{2}{\tau}(\delta_x^2 v_i - \delta_x^2 u_i^k) + 3\psi(v,v)_i = (\Delta_x v_i)\delta_x^2 v_i + \Delta_x(v_i \delta_x^2 v_i), \\ &\qquad 1 \leqslant i \leqslant m-1, \end{aligned} \tag{4.17}$$

$$v_0 = 0, \quad v_m = 0. \tag{4.18}$$

令

$$z_i = w_i - v_i, \quad 0 \leqslant i \leqslant m.$$

将 (4.15)–(4.16) 分别和 (4.17)–(4.18) 相减, 得

$$\frac{2}{\tau}z_i - \frac{2}{\tau}\delta_x^2 z_i + 3[\psi(w,w)_i - \psi(v,v)_i]$$
$$= [(\Delta_x w_i)\delta_x^2 w_i + \Delta_x(w_i\delta_x^2 w_i)] - [(\Delta_x v_i)\delta_x^2 v_i + \Delta_x(v_i\delta_x^2 v_i)], \quad 1 \leqslant i \leqslant m-1, \tag{4.19}$$

$$z_0 = 0, \quad z_m = 0. \tag{4.20}$$

将 (4.19) 与 z 作内积, 得

$$\frac{2}{\tau}(z,z) - \frac{2}{\tau}(\delta_x^2 z, z) + 3(\psi(w,w) - \psi(v,v), z)$$
$$= \Big((\Delta_x w)\delta_x^2 w + \Delta_x(w\delta_x^2 w) - (\Delta_x v)\delta_x^2 v - \Delta_x(v\delta_x^2 v), z\Big). \tag{4.21}$$

注意到

$$\begin{aligned}
&\psi(w,w)_i - \psi(v,v)_i \\
=&\psi(w,w)_i - \psi(w-z, w-z)_i \\
=&\psi(w,z)_i + \psi(z,w)_i - \psi(z,z)_i,
\end{aligned}$$

有

$$\big(\psi(w,w)-\psi(v,v), z\big) = \big(\psi(w,z),z\big)+\big(\psi(z,w),z\big)-\big(\psi(z,z),z\big) = \big(\psi(z,w),z\big). \tag{4.22}$$

再注意到

$$\begin{aligned}
&\big[(\Delta_x w_i)(\delta_x^2 w_i) + \Delta_x(w_i\delta_x^2 w_i)\big] - \big[(\Delta_x v_i)\delta_x^2 v_i + \Delta_x(v_i\delta_x^2 v_i)\big] \\
=&\big[(\Delta_x w_i)(\delta_x^2 w_i) + \Delta_x(w_i\delta_x^2 w_i)\big] \\
&- \big[(\Delta_x(w_i - z_i))\delta_x^2(w_i - z_i) + \Delta_x((w_i - z_i)\delta_x^2(w_i - z_i))\big] \\
=&(\Delta_x z_i)\delta_x^2 w_i + (\Delta_x w_i)\delta_x^2 z_i - (\Delta_x z_i)\delta_x^2 z_i + \Delta_x\big(z_i\delta_x^2 w_i + w_i\delta_x^2 z_i - z_i\delta_x^2 z_i\big) \\
=&\big[(\Delta_x z_i)\delta_x^2 w_i + \Delta_x(z_i\delta_x^2 w_i)\big] + \big[(\Delta_x w_i)\delta_x^2 z_i + \Delta_x(w_i\delta_x^2 z_i)\big] \\
&- \big[(\Delta_x z_i)\delta_x^2 z_i + \Delta_x(z_i\delta_x^2 z_i)\big],
\end{aligned}$$

有

$$\begin{aligned}
&\big((\Delta_x w)\delta_x^2 w + \Delta_x(w\delta_x^2 w) - (\Delta_x v)\delta_x^2 v - \Delta_x(v\delta_x^2 v), z\big) \\
=& \big((\Delta_x z)\delta_x^2 w + \Delta_x(z\delta_x^2 w), z\big) + \big((\Delta_x w)\delta_x^2 z + \Delta_x(w\delta_x^2 z), z\big) \\
&-\big((\Delta_x z)\delta_x^2 z + \Delta_x(z\delta_x^2 z), z\big) \\
=& \big((\Delta_x w)\delta_x^2 z, z\big) + \big(\Delta_x(w\delta_x^2 z), z\big) \\
=& \big((\Delta_x w)\delta_x^2 z, z\big) - \big(w\delta_x^2 z, \Delta_x z\big).
\end{aligned} \tag{4.23}$$

将 (4.22), (4.23) 代入 (4.21), 得

$$\frac{2}{\tau}(\|z\|^2+|z|_1^2)=-3\big(\psi(z,w),z\big)+\big((\Delta_x w)\delta_x^2 z,z\big)-\big(w\delta_x^2 z,\Delta_x z\big). \tag{4.24}$$

现估计上式右端 3 项:

$$\big(-\psi(z,w),z\big)\leqslant\frac{1}{3}\big(2\|z\|_\infty\cdot\|z\|\cdot|w|_1+\|z\|_\infty|z|_1\|w\|\big);$$

$$\big((\Delta_x w)\delta_x^2 z,z\big)\leqslant\|z\|_\infty|w|_1\|\delta_x^2 z\|\leqslant\frac{2}{h}\|z\|_\infty|w|_1|z|_1;$$

$$-(w\delta_x^2 z,\Delta_x z)\leqslant\|w\|_\infty\|\delta_x^2 z\|\cdot|z|_1\leqslant\frac{2}{h}\|w\|_\infty|z|_1^2.$$

将以上 3 式代入 (4.24), 并利用定理 4.2, 可知存在常数 c_5 使得

$$\frac{2}{\tau}|z|_1^2\leqslant\frac{c_5}{h}|z|_1^2.$$

当 $\dfrac{\tau}{h}<\dfrac{2}{c_5}$ 时, $|z|_1^2=0$. □

4.2.4 差分格式解的收敛性

定理 4.5 设 $\{U_i^k\,|\,0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$ 为 (4.1)–(4.3) 的解, $\{u_i^k\,|\,0\leqslant i\leqslant m,0\leqslant k\leqslant n\}$ 为差分格式 (4.11)–(4.13) 的解. 记

$$e_i^k=U_i^k-u_i^k,\quad 0\leqslant i\leqslant m,\ 0\leqslant k\leqslant n.$$

则存在常数 c_6 使得

$$\|e^k\|_\infty\leqslant c_6(\tau^2+h^2),\quad 0\leqslant k\leqslant n. \tag{4.25}$$

证明 将 (4.7), (4.9)–(4.10) 和 (4.11)–(4.13) 分别相减, 得误差方程组

$$\begin{aligned}&\delta_t e_i^{k+\frac12}-\delta_t\delta_x^2 e_i^{k+\frac12}+3\big[\psi(U^{k+\frac12},U^{k+\frac12})_i-\psi(u^{k+\frac12},u^{k+\frac12})_i\big]\\=&\big[(\Delta_x U_i^{k+\frac12})\delta_x^2 U_i^{k+\frac12}+\Delta_x(U_i^{k+\frac12}\delta_x^2 U_i^{k+\frac12})\big]\\&-\big[(\Delta_x u_i^{k+\frac12})\delta_x^2 u_i^{k+\frac12}+\Delta_x(u_i^{k+\frac12}\delta_x^2 u_i^{k+\frac12})\big]+R_i^{k+\frac12},\\&\qquad 1\leqslant i\leqslant m-1,\quad 0\leqslant k\leqslant n-1,\end{aligned} \tag{4.26}$$

$$e_i^0=0,\quad 1\leqslant i\leqslant m-1, \tag{4.27}$$

$$e_0^k=0,\quad e_m^k=0,\quad 0\leqslant k\leqslant n. \tag{4.28}$$

用 $e^{k+\frac12}$ 与 (4.26) 作内积, 类似于 (4.24) 的推导, 可得

$$\begin{aligned}&\frac{1}{2\tau}(\|e^{k+1}\|^2-\|e^k\|^2)+\frac{1}{2\tau}(|e^{k+1}|_1^2-|e^k|_1^2)\\=&-3\big(\psi(e^{k+\frac12},U^{k+\frac12}),e^{k+\frac12}\big)+\big((\Delta_x U^{k+\frac12})\delta_x^2 e^{k+\frac12},e^{k+\frac12}\big)\\&-\big(U^{k+\frac12}\delta_x^2 e^{k+\frac12},e^{k+\frac12}\big)+\big(R^{k+\frac12},e^{k+\frac12}\big),\quad 0\leqslant k\leqslant n-1.\end{aligned} \tag{4.29}$$

现在估计上式右端每一项.

$$
\begin{aligned}
&-(\psi(e^{k+\frac{1}{2}},U^{k+\frac{1}{2}}),e^{k+\frac{1}{2}})\\
\leqslant&\frac{1}{3}\left(2c_1\|e^{k+\frac{1}{2}}\|^2+c_0\|e^{k+\frac{1}{2}}\|\cdot|e^{k+\frac{1}{2}}|_1\right);\\
&\left((\Delta_x U^{k+\frac{1}{2}})\delta_x^2 e^{k+\frac{1}{2}},e^{k+\frac{1}{2}}\right)\\
=&\left((\delta_x^2 e^{k+\frac{1}{2}}),(\Delta_x U^{k+\frac{1}{2}})e^{k+\frac{1}{2}}\right)\\
=&-h\sum_{i=0}^{m-1}(\delta_x e_{i+\frac{1}{2}}^{k+\frac{1}{2}})\frac{(\Delta_x U_{i+1}^{k+\frac{1}{2}})e_{i+1}^{k+\frac{1}{2}}-(\Delta_x U_i^{k+1})e_i^{k+\frac{1}{2}}}{h}\\
=&-h\sum_{i=0}^{m-1}(\delta_x e_{i+\frac{1}{2}}^{k+\frac{1}{2}})\left[(\Delta_x U_{i+1}^{k+\frac{1}{2}})\delta_x e_{i+\frac{1}{2}}^{k+\frac{1}{2}}+\frac{\Delta_x U_{i+1}^{k+\frac{1}{2}}-\Delta_x U_i^{k+\frac{1}{2}}}{h}e_i^{k+\frac{1}{2}}\right]\\
\leqslant&c_1|e^{k+\frac{1}{2}}|_1^2+c_2\|e^{k+\frac{1}{2}}\|\cdot|e^{k+\frac{1}{2}}|_1;\\
&-(U^{k+\frac{1}{2}}\delta_x^2 e^{k+\frac{1}{2}},e^{k+\frac{1}{2}})\\
=&-(\delta_x^2 e^{k+\frac{1}{2}},U^{k+\frac{1}{2}}e^{k+\frac{1}{2}})\\
=&-h\sum_{i=1}^{m-1}(\delta_x^2 e_i^{k+\frac{1}{2}})U_i^{k+\frac{1}{2}}e_i^{k+\frac{1}{2}}\\
=&h\sum_{i=0}^{m-1}(\delta_x e_{i+\frac{1}{2}}^{k+\frac{1}{2}})\frac{U_{i+1}^{k+\frac{1}{2}}e_{i+1}^{k+\frac{1}{2}}-U_i^{k+\frac{1}{2}}e_i^{k+\frac{1}{2}}}{h}\\
=&h\sum_{i=0}^{m-1}(\delta_x e_{i+\frac{1}{2}}^{k+\frac{1}{2}})\frac{U_{i+1}^{k+\frac{1}{2}}(e_{i+1}^{k+\frac{1}{2}}-e_i^{k+\frac{1}{2}})+(U_{i+1}^{k+\frac{1}{2}}-U_i^{k+\frac{1}{2}})e_i^{k+\frac{1}{2}}}{h}\\
\leqslant&c_0|e^{k+\frac{1}{2}}|_1^2+c_1|e^{k+\frac{1}{2}}|_1\cdot\|e^{k+\frac{1}{2}}\|;
\end{aligned}
$$

$$
(R^{k+\frac{1}{2}},e^{k+\frac{1}{2}})\leqslant\frac{1}{2}\|e^{k+\frac{1}{2}}\|^2+\frac{1}{2}\|R^{k+\frac{1}{2}}\|^2\leqslant\frac{1}{2}\|e^{k+\frac{1}{2}}\|^2+\frac{1}{2}Lc_4^2(\tau^2+h^2)^2.
$$

将以上 4 式代入 (4.29) 可知存在常数 c_6 使得

$$
\begin{aligned}
&\frac{1}{2\tau}[(\|e^{k+1}\|^2+|e^{k+1}|_1^2)-(\|e^k\|^2+|e^k|_1^2)]\\
\leqslant&\, c_6(\|e^{k+\frac{1}{2}}\|^2+|e^{k+\frac{1}{2}}|_1^2)+c_6(\tau^2+h^2)^2\\
\leqslant&\,\frac{1}{2}c_6[(\|e^{k+1}\|^2+|e^{k+1}|_1^2)+(\|e^k\|^2+|e^k|_1^2)]+c_6(\tau^2+h^2)^2,\\
&\qquad 0\leqslant k\leqslant n-1.
\end{aligned}\tag{4.30}
$$

记 $F^k=\|e^k\|^2+|e^k|_1^2$, 则 (4.30) 可写为

$$
\frac{1}{2\tau}(F^{k+1}-F^k)\leqslant\frac{1}{2}c_6(F^{k+1}+F^k)+c_6(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant n-1,
$$

或

$$(1-c_6\tau)F^{k+1}\leqslant(1+c_6\tau)F^k+2c_6\tau(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant n-1.$$

当 $c_6\tau\leqslant\dfrac{1}{3}$ 时

$$F^{k+1}\leqslant(1+3c_6\tau)F^k+3c_6\tau(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant n-1.$$

由 Gronwall 不等式得

$$F^{k+1}\leqslant e^{3c_6T}[F^0+(\tau^2+h^2)^2]=e^{3c_6T}(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant n-1.\qquad\square$$

4.3 三层线性化差分格式

4.3.1 差分格式的建立

在点 $(x_i,t_{\frac{1}{2}})$ 处考虑方程 (4.4), 有

$$u_t(x_i,t_{\frac{1}{2}})-u_{xxt}(x_i,t_{\frac{1}{2}})+3u(x_i,t_{\frac{1}{2}})u_x(x_i,t_{\frac{1}{2}})=u_x(x_i,t_{\frac{1}{2}})u_{xx}(x_i,t_{\frac{1}{2}})+(uu_{xx})_x(x_i,t_{\frac{1}{2}}).$$

应用 Taylor 展开式及数值微分公式, 可得到

$$\begin{aligned}&\delta_tU_i^{\frac{1}{2}}-\delta_t\delta_x^2U_i^{\frac{1}{2}}+3\psi(U^0,U^{\frac{1}{2}})_i=(\Delta_xU^{\frac{1}{2}})_i\delta_x^2U_i^0+\Delta_x(U_i^{\frac{1}{2}}\delta_x^2U_i^0)+P_i^0,\\&\qquad 1\leqslant i\leqslant m-1,\end{aligned}\tag{4.31}$$

且存在常数 c_7 使得

$$|P_i^0|\leqslant c_7(\tau+h^2),\quad 1\leqslant i\leqslant m-1.\tag{4.32}$$

在点 (x_i,t_k) 处考虑方程 (4.4), 有

$$\begin{aligned}&u_t(x_i,t_k)-u_{xxt}(x_i,t_k)+3u(x_i,t_k)u_x(x_i,t_k)\\=&u_x(x_i,t_k)u_{xx}(x_i,t_k)+(uu_{xx})_x(x_i,t_k),\quad 1\leqslant i\leqslant m-1,\ 1\leqslant k\leqslant n-1.\end{aligned}$$

应用 Taylor 展开式及数值微分公式, 可得到

$$\begin{aligned}&\Delta_tU_i^k-\Delta_t\delta_x^2U_i^k+3\psi(U^k,U^{\bar{k}})_i=(\Delta_xU_i^{\bar{k}})\delta_x^2U_i^k+\Delta_x(U^{\bar{k}}\delta_x^2U^k)_i+P_i^k,\\&\qquad 1\leqslant i\leqslant m-1,\ 1\leqslant k\leqslant n-1,\end{aligned}\tag{4.33}$$

且存在常数 c_8 使得

$$|P_i^k|\leqslant c_8(\tau^2+h^2),\quad 1\leqslant i\leqslant m-1,\ 1\leqslant k\leqslant n-1.\tag{4.34}$$

由初边值条件 (4.2)–(4.3) 可得

$$U_i^0=\varphi(x_i),\quad 1\leqslant i\leqslant m-1, \tag{4.35}$$

$$U_0^k=0,\quad U_m^k=0,\quad 0\leqslant k\leqslant n. \tag{4.36}$$

在 (4.31) 和 (4.33) 中略去小量项, 对 (4.1)–(4.3) 建立如下差分格式

$$\delta_t u_i^{\frac{1}{2}}-\delta_t\delta_x^2 u_i^{\frac{1}{2}}+3\psi(u^0,u^{\frac{1}{2}})_i=(\Delta_x u_i^{\frac{1}{2}})\delta_x^2 u_i^0+\Delta_x(u_i^{\frac{1}{2}}\delta_x^2 u_i^0),$$
$$1\leqslant i\leqslant m-1, \tag{4.37}$$

$$\Delta_t u_i^k-\Delta_t\delta_x^2 u_i^k+3\psi(u^k,u^{\bar{k}})_i=(\Delta_x u_i^{\bar{k}})\delta_x^2 u_i^k+\Delta_x(u_i^{\bar{k}}\delta_x^2 u_i^k),$$
$$1\leqslant i\leqslant m-1,\ 1\leqslant k\leqslant n-1, \tag{4.38}$$

$$u_i^0=\varphi(x_i),\quad 1\leqslant i\leqslant m-1, \tag{4.39}$$

$$u_0^k=0,\quad u_m^k=0,\quad 1\leqslant k\leqslant n. \tag{4.40}$$

在每一时间层, (4.37)–(4.40) 是三对角线性代数方程组, 可用追赶法求解.

4.3.2　差分格式解的守恒性和有界性

定理 4.6　设 $\{u_i^k\,|\,0\leqslant i\leqslant m, 0\leqslant k\leqslant n\}$ 为差分格式 (4.37)–(4.40) 的解. 记

$$E^k=\|u^k\|^2+|u^k|_1^2,$$

则有

$$E^k\equiv E^0,\quad 0\leqslant k\leqslant n. \tag{4.41}$$

证明　(I) 用 $u^{\frac{1}{2}}$ 与 (4.37) 作内积, 得

$$\frac{1}{2\tau}(\|u^1\|^2-\|u^0\|^2)+\frac{1}{2\tau}(|u^1|_1^2-|u^0|_1^2)=0,$$

即

$$\frac{1}{2\tau}(E^1-E^0)=0.$$

因而

$$E^1=E^0. \tag{4.42}$$

(II) 用 $u^{\bar{k}}$ 与 (4.38) 作内积, 得

$$\frac{1}{4\tau}(\|u^{k+1}\|^2-\|u^{k-1}\|^2)+\frac{1}{4\tau}(|u^{k+1}|_1^2-|u^{k-1}|_1^2)=0,\quad 1\leqslant k\leqslant n-1.$$

即

$$\frac{1}{4\tau}(E^{k+1}-E^{k-1})=0,\quad 1\leqslant k\leqslant n-1.$$

因而

$$E^{k+1} = E^{k-1}, \quad 1 \leqslant k \leqslant n-1. \tag{4.43}$$

综合 (4.42) 和 (4.43) 知 (4.41) 成立. □

4.3.3 差分格式解的存在性和唯一性

定理 4.7 差分格式 (4.37)–(4.40) 的解是存在且唯一的.

证明 由 (4.39)–(4.40) 知 u^0 已给定.

由 (4.37) 和 (4.40) 可得关于 u^1 的线性方程组. 考虑它的齐次线性方程组

$$\frac{1}{\tau}u_i^1 - \frac{1}{\tau}\delta_x^2 u_i^1 + \frac{3}{2}\psi(u^0, u^1)_i = \frac{1}{2}(\Delta_x u_i^1)\delta_x^2 u_i^0 + \frac{1}{2}\Delta_x(u_i^1 \delta_x^2 u_i^0),$$
$$1 \leqslant i \leqslant m-1, \tag{4.44}$$

$$u_0^1 = 0, \quad u_m^1 = 0, \tag{4.45}$$

用 u^1 与 (4.44) 作内积, 可得

$$\frac{1}{\tau}\|u^1\|^2 + \frac{1}{\tau}|u^1|_1^2 = 0,$$

于是 $u^1 = 0$. 因而 (4.37)–(4.40) 关于 u^1 的解是存在唯一的.

设 $u^l(1 \leqslant l \leqslant k)$ 已确定. 则由 (4.38) 和 (4.40) 可得关于 u^{k+1} 的线性代数方程组. 考虑它的齐次线性方程组

$$\frac{1}{2\tau}u_i^{k+1} - \frac{1}{2\tau}\delta_x^2 u_i^{k+1} + \frac{3}{2}\psi(u^k, u^{k+1})_i$$
$$= \frac{1}{2}(\Delta_x u_i^{k+1})\delta_x^2 u_i^k + \frac{1}{2}\Delta_x(u_i^{k+1}\delta_x^2 u_i^k), \quad 1 \leqslant i \leqslant m-1, \tag{4.46}$$

$$u_0^{k+1} = 0, \quad u_m^{k+1} = 0. \tag{4.47}$$

用 u^{k+1} 与 (4.46) 作内积, 可得

$$\frac{1}{2\tau}\|u^{k+1}\|^2 + \frac{1}{2\tau}|u^{k+1}|_1^2 = 0.$$

于是

$$u^{k+1} = 0.$$

因而 (4.38), (4.40) 关于 u^{k+1} 的解是存在唯一的. □

4.3.4 差分格式解的收敛性

定理 4.8 设 $\{U_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为 (4.1)–(4.3) 的解, $\{u_i^k \,|\, 0 \leqslant i \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (4.37)–(4.40) 的解. 记

$$e_i^k = U_i^k - u_i^k, \quad 0 \leqslant i \leqslant m, \ 0 \leqslant k \leqslant n.$$

设 $c_0\dfrac{\tau}{h}<1$, 其中 $c_0=\max\limits_{0\leqslant x\leqslant l,0\leqslant t\leqslant T}|u(x,t)|$, 则存在常数 c_9 使得

$$\|e^k\|_\infty\leqslant c_9(\tau^2+h^2),\quad 0\leqslant k\leqslant n.$$

证明　将 (4.31), (4.33), (4.35)–(4.36) 与 (4.37)–(4.40) 相减, 得误差方程

$$\begin{aligned}&\delta_t e_i^{\frac12}-\delta_t\delta_x^2 e_i^{\frac12}+3\psi(u^0,e^{\frac12})_i\\&=(\Delta_x U_i^{\frac12})\delta_x^2U_i^0+\Delta_x(U^{\frac12}\delta_x^2U_i^0)-\big[(\Delta_x u_i^{\frac12})\delta_x^2u_i^0+\Delta_x(u_i^{\frac12}\delta_x^2u_i^0)\big]+P_i^0,\\&\qquad 1\leqslant i\leqslant m-1,\end{aligned}\tag{4.48}$$

$$\begin{aligned}&\Delta_t e_i^k-\Delta_t\delta_x^2e_i^k+3\big[\psi(U^k,U^{\bar k})_i-\psi(u^k,u^{\bar k})_i\big]\\&=(\Delta_x U_i^{\bar k})\delta_x^2U_i^k+\Delta_x(U^{\bar k}\delta_x^2U^k)-\big[(\Delta_x u_i^{\bar k})\delta_x^2u_i^k+\Delta_x(u_i^{\bar k}\delta_x^2u_i^k)\big]+P_i^k,\\&\qquad 1\leqslant i\leqslant m-1,\quad 1\leqslant k\leqslant n-1,\end{aligned}\tag{4.49}$$

$$e_i^0=0,\quad 1\leqslant i\leqslant m-1,\tag{4.50}$$

$$e_0^k=0,\quad e_m^k=0,\quad 0\leqslant k\leqslant n.\tag{4.51}$$

由 (4.50) 和 (4.51), 得

$$\|e^0\|=0,\quad |e^0|_1=0.\tag{4.52}$$

(I) 用 $e^{\frac12}$ 与 (4.48) 作内积, 得

$$\begin{aligned}&\frac{1}{2\tau}(\|e^1\|-\|e^0\|^2)+\frac{1}{2\tau}(|e^1|_1^2-|e^0|_1^2)+3(\psi(u^0,e^{\frac12}),e^{\frac12})\\=&\big((\Delta_xU^{\frac12})\delta_x^2U^0+\Delta_x(U^{\frac12}\delta_x^2U^0)-(\Delta_xu^{\frac12})\delta_x^2u^0-\Delta_x(u^{\frac12}\delta_x^2u^0),e^{\frac12}\big)+(P^0,e^{\frac12})\\=&((\Delta_xe^{\frac12})\delta_x^2u^0+\Delta_x(e^{\frac12}\delta_x^2u^0),e^{\frac12})+(P^0,e^{\frac12})\\=&(P^0,e^{\frac12}).\end{aligned}$$

因而

$$\frac{1}{2\tau}(\|e^1\|^2+|e^1|_1^2)=\frac12(P^0,e^1)\leqslant\frac12\|P^0\|\cdot\|e^1\|.\tag{4.53}$$

由上式可得

$$\frac{1}{2\tau}\|e^1\|^2\leqslant\frac12\|P^0\|\cdot\|e^1\|.$$

因而

$$\|e^1\|\leqslant\tau\|P^0\|.$$

再由 (4.53) 得到

$$\|e^1\|^2+|e^1|_1^2\leqslant\tau\|P^0\|\cdot\|e^1\|\leqslant\tau^2\|P^0\|^2\leqslant Lc_7^2(\tau^2+h^2)^2.\tag{4.54}$$

(II) 注意到

$$\begin{aligned}&\psi(U^k,U^{\bar{k}})-\psi(u^k,u^{\bar{k}})\\=&\psi(U^k,U^{\bar{k}})-\psi(U^k-e^k,U^{\bar{k}}-e^{\bar{k}})\\=&\psi(U^k,e^{\bar{k}})+\psi(e^k,U^{\bar{k}})-\psi(e^k,e^{\bar{k}}),\end{aligned}$$

有

$$\begin{aligned}&\big(\psi(U^k,U^{\bar{k}})-\psi(u^k,u^{\bar{k}}),e^{\bar{k}}\big)\\=&\big(\psi(U^k,e^{\bar{k}}),e^{\bar{k}}\big)+\big(\psi(e^k,U^{\bar{k}}),e^{\bar{k}}\big)-\big(\psi(e^k,e^{\bar{k}}),e^{\bar{k}}\big)\\=&\big(\psi(e^k,U^{\bar{k}}),e^{\bar{k}}\big).\end{aligned}\tag{4.55}$$

再注意到

$$\begin{aligned}&\big[(\Delta_xU^{\bar{k}})\delta_x^2U^k+\Delta_x(U^{\bar{k}}\delta_x^2U^k)\big]-\big[(\Delta_xu^{\bar{k}})\delta_x^2u^k+\Delta_x(u^{\bar{k}}\delta_x^2u^k)\big]\\=&(\Delta_xU^{\bar{k}})\delta_x^2U^k+\Delta_x(U^{\bar{k}}\delta_x^2U^k)-(\Delta_x(U^{\bar{k}}-e^{\bar{k}}))\delta_x^2(U^k-e^k)\\&-\Delta_x((U^{\bar{k}}-e^{\bar{k}})\delta_x^2(U^k-e^k))\\=&(\Delta_xU^{\bar{k}})\delta_x^2e^k+(\Delta_xe^{\bar{k}})\delta_x^2U^k-(\Delta_xe^{\bar{k}})\delta_x^2e^k\\&+\Delta_x(U^{\bar{k}}\delta_x^2e^k)+\Delta_x(e^{\bar{k}}\delta_x^2U^k)-\Delta_x(e^{\bar{k}}\delta_x^2e^k)\\=&\big[(\Delta_xU^{\bar{k}})\delta_x^2e^k+\Delta_x(U^{\bar{k}}\delta_x^2e^k)\big]+\big[(\Delta_xe^{\bar{k}})\delta_x^2U^k+\Delta_x(e^{\bar{k}}\delta_x^2U^k)\big]\\&-\big[(\Delta_xe^{\bar{k}})\delta_x^2e^k+\Delta_x(e^{\bar{k}}\delta_x^2e^k)\big],\end{aligned}$$

得到

$$\begin{aligned}&\big((\Delta_xU^{\bar{k}})\delta_x^2U^k+\Delta_x(U^{\bar{k}}\delta_x^2U^k)-(\Delta_xu^{\bar{k}})\delta_x^2u^k-\Delta_x(u^{\bar{k}}\delta_x^2u^k),e^{\bar{k}}\big)\\=&\big((\Delta_xU^{\bar{k}})\delta_x^2e^k+\Delta_x(U^{\bar{k}}\delta_x^2e^k),e^{\bar{k}}\big)+\big((\Delta_xe^{\bar{k}})\delta_x^2U^k+\Delta_x(e^{\bar{k}}\delta_x^2U^k),e^{\bar{k}}\big)\\&-\big((\Delta_xe^{\bar{k}})\delta_x^2e^k+\Delta_x(e^{\bar{k}}\delta_x^2e^k),e^{\bar{k}}\big)\\=&\big((\Delta_xU^{\bar{k}})\delta_x^2e^k+\Delta_x(U^{\bar{k}}\delta_x^2e^k),e^{\bar{k}}\big)\\=&\big(\delta_x^2e^k,(\Delta_xU^{\bar{k}})e^{\bar{k}}\big)-\big(U^{\bar{k}}\delta_x^2e^k,\Delta_xe^{\bar{k}}\big).\end{aligned}\tag{4.56}$$

将 (4.49) 与 $e^{\bar{k}}$ 作内积, 并利用 (4.55) 和 (4.56), 可以得到

$$\begin{aligned}&\frac{1}{4\tau}\big(\|e^{k+1}\|^2-\|e^{k-1}\|^2\big)+\frac{1}{4\tau}\big(|e^{k+1}|_1^2-|e^{k-1}|_1^2\big)\\=&-3\big(\psi(e^k,U^{\bar{k}}),e^{\bar{k}}\big)+\big(\delta_x^2e^k,(\Delta_xU^{\bar{k}})e^{\bar{k}}\big)-\big(\delta_x^2e^k,U^{\bar{k}}\Delta_xe^{\bar{k}}\big)+\big(P^k,e^{\bar{k}}\big)\\\equiv&A_k+B_k+C_k+\big(P^k,e^{\bar{k}}\big).\end{aligned}\tag{4.57}$$

现在来估计 (4.57) 右端的每一项.

右端的第一项

$$\begin{aligned}A_k &= -[(e^k\Delta_x U^{\bar{k}}, e^{\bar{k}}) + (\Delta_x(e^k U^{\bar{k}}), e^{\bar{k}})]\\ &= -(e^k\Delta_x U^{\bar{k}}, e^{\bar{k}}) + (e^k U^{\bar{k}}, \Delta_x e^{\bar{k}})\\ &\leqslant c_1\|e^k\|\cdot\|e^{\bar{k}}\| + c_0\|e^k\|\cdot|e^{\bar{k}}|_1.\end{aligned}\tag{4.58}$$

右端的第二项

$$\begin{aligned}B_k &= (\delta_x^2 e^k, (\Delta_x U^{\bar{k}})e^{\bar{k}})\\ &= h\sum_{i=1}^{m-1}(\delta_x^2 e_i^k)(\Delta_x U_i^{\bar{k}})e_i^{\bar{k}}\\ &= \sum_{i=1}^{m-1}(\delta_x e_{i+\frac12}^k - \delta_x e_{i-\frac12}^k)(\Delta_x U_i^{\bar{k}})e_i^{\bar{k}}\\ &= \sum_{i=1}^{m-1}(\delta_x e_{i+\frac12}^k)(\Delta_x U_i^{\bar{k}})e_i^{\bar{k}} - \sum_{i=0}^{m-2}(\delta_x e_{i+\frac12}^k)(\Delta_x U_{i+1}^{\bar{k}})e_{i+1}^{\bar{k}}\\ &= \sum_{i=0}^{m-2}(\delta_x e_{i+\frac12}^k)\big[(\Delta_x U_i^{\bar{k}})e_i^{\bar{k}} - (\Delta_x U_{i+1}^{\bar{k}})e_{i+1}^{\bar{k}}\big] + (\delta_x e_{m-\frac12}^k)(\Delta_x U_{m-1}^{\bar{k}})e_{m-1}^{\bar{k}}\\ &= \sum_{i=0}^{m-2}(\delta_x e_{i+\frac12}^k)\big[(e_i^{\bar{k}} - e_{i+1}^{\bar{k}})\Delta_x U_i^{\bar{k}} + e_{i+1}^{\bar{k}}(\Delta_x U_i^{\bar{k}} - \Delta_x U_{i+1}^{\bar{k}})\big]\\ &\quad -h(\delta_x e_{m-\frac12}^k)(\Delta_x U_{m-1}^{\bar{k}})\delta_x e_{m-\frac12}^{\bar{k}}\\ &= -h\sum_{i=0}^{m-1}(\delta_x e_{i+\frac12}^k)(\delta_x e_{i+\frac12}^{\bar{k}})\Delta_x U_i^{\bar{k}} - h\sum_{i=0}^{m-2}(\delta_x e_{i+\frac12}^k)e_{i+1}^{\bar{k}}\frac{\delta_x U_{i+1}^{\bar{k}} - \delta_x U_i^{\bar{k}}}{h}\\ &\leqslant c_1|e^k|_1\cdot|e^{\bar{k}}|_1 + c_2|e^k|_1\cdot\|e^{\bar{k}}\|.\end{aligned}\tag{4.59}$$

对于 (4.57) 的第三项, 有

$$\begin{aligned}C_k &\equiv -(\delta_x^2 e^k, U^{\bar{k}}\Delta_x e^{\bar{k}})\\ &= -\frac12\sum_{i=1}^{m-1}U_i^{\bar{k}}(\delta_x e_{i+\frac12}^k - \delta_x e_{i-\frac12}^k)(\delta_x e_{i+\frac12}^{\bar{k}} + \delta_x e_{i-\frac12}^{\bar{k}})\\ &= -\frac12\sum_{i=1}^{m-1}U_i^{\bar{k}}\big[(\delta_x e_{i+\frac12}^k)(\delta_x e_{i+\frac12}^{\bar{k}}) + (\delta_x e_{i+\frac12}^k)(\delta_x e_{i-\frac12}^{\bar{k}})\\ &\quad -(\delta_x e_{i-\frac12}^k)(\delta_x e_{i+\frac12}^{\bar{k}}) - (\delta_x e_{i-\frac12}^k)(\delta_x e_{i-\frac12}^{\bar{k}})\big]\\ &= \frac12\sum_{i=1}^{m-1}U_i^{\bar{k}}\big[(\delta_x e_{i-\frac12}^k)(\delta_x e_{i-\frac12}^{\bar{k}}) - (\delta_x e_{i+\frac12}^k)(\delta_x e_{i+\frac12}^{\bar{k}})\big]\end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{m-1}U_i^{\bar{k}}\big[(\delta_x e_{i-\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{\bar{k}})-(\delta_x e_{i+\frac{1}{2}}^k)(\delta_x e_{i-\frac{1}{2}}^{\bar{k}})\big]$$

$$\equiv D_k+E_k. \tag{4.60}$$

对于上式中的 D_k 有

$$\begin{aligned}D_k&=\frac{1}{2}\Big[\sum_{i=0}^{m-2}U_{i+1}^{\bar{k}}(\delta_x e_{i+\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{\bar{k}})-\sum_{i=1}^{m-1}U_i^{\bar{k}}(\delta_x e_{i+\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{\bar{k}})\Big]\\&=\frac{1}{2}h\sum_{i=0}^{m-1}\frac{U_{i+1}^{\bar{k}}-U_i^{\bar{k}}}{h}(\delta_x e_{i+\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{\bar{k}})\\&\leqslant\frac{1}{2}c_1|e^k|_1\cdot|e^{\bar{k}}|_1.\end{aligned} \tag{4.61}$$

(4.60) 中 E_k 的估计比较复杂, 有

$$\begin{aligned}E_k=&\frac{1}{2}\sum_{i=1}^{m-1}U_i^{\bar{k}}\big[(\delta_x e_{i-\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{\bar{k}})-(\delta_x e_{i+\frac{1}{2}}^k)(\delta_x e_{i-\frac{1}{2}}^{\bar{k}})\big]\\=&\frac{1}{4}\sum_{i=1}^{m-1}U_i^{\bar{k}}\big[(\delta_x e_{i-\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{k+1}+\delta_x e_{i+\frac{1}{2}}^{k-1})-(\delta_x e_{i+\frac{1}{2}}^k)(\delta_x e_{i-\frac{1}{2}}^{k+1}+\delta_x e_{i-\frac{1}{2}}^{k-1})\big]\\=&\frac{1}{4}\sum_{i=1}^{m-1}U_i^{\bar{k}}\Big\{\big[(\delta_x e_{i-\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{k+1})-(\delta_x e_{i-\frac{1}{2}}^{k+1})(\delta_x e_{i+\frac{1}{2}}^k)\big]\\&-\big[(\delta_x e_{i-\frac{1}{2}}^{k-1})(\delta_x e_{i+\frac{1}{2}}^k)-(\delta_x e_{i-\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{k-1})\big]\Big\}\\=&\frac{1}{4}\Big\{\sum_{i=1}^{m-1}U_i^{k+\frac{1}{2}}\big[(\delta_x e_{i-\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{k+1})-(\delta_x e_{i-\frac{1}{2}}^{k+1})(\delta_x e_{i+\frac{1}{2}}^k)\big]\\&-\sum_{i=1}^{m-1}U_i^{k-\frac{1}{2}}\big[(\delta_x e_{i-\frac{1}{2}}^{k-1})(\delta_x e_{i+\frac{1}{2}}^k)-(\delta_x e_{i-\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{k-1})\big]\Big\}\\&+\frac{1}{4}\sum_{i=1}^{m-1}(U_i^{\bar{k}}-U_i^{k+\frac{1}{2}})\big[(\delta_x e_{i-\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{k+1})-(\delta_x e_{i-\frac{1}{2}}^{k+1})(\delta_x e_{i+\frac{1}{2}}^k)\big]\\&+\frac{1}{4}\sum_{i=1}^{m-1}(U_i^{k-\frac{1}{2}}-U_i^{\bar{k}})\big[(\delta_x e_{i-\frac{1}{2}}^{k-1})(\delta_x e_{i+\frac{1}{2}}^k)-(\delta_x e_{i-\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{k-1})\big].\end{aligned}$$

记

$$G^{k+\frac{1}{2}}=\frac{1}{2}\sum_{i=1}^{m-1}U_i^{k+\frac{1}{2}}\big[(\delta_x e_{i-\frac{1}{2}}^k)(\delta_x e_{i+\frac{1}{2}}^{k+1})-(\delta_x e_{i-\frac{1}{2}}^{k+1})(\delta_x e_{i+\frac{1}{2}}^k)\big],$$

则当 $c_0\dfrac{\tau}{h}\leqslant 1$ 时,

$$E_k \leqslant \frac{1}{2}(G^{k+\frac{1}{2}} - G^{k-\frac{1}{2}}) + \frac{c_3}{8}\tau \sum_{i=1}^{m-1} \left|(\delta_x e^k_{i-\frac{1}{2}})(\delta_x e^{k+1}_{i+\frac{1}{2}}) - (\delta_x e^{k+1}_{i-\frac{1}{2}})(\delta_x e^k_{i+\frac{1}{2}})\right|$$
$$+\frac{c_3}{8}\tau \sum_{i=1}^{m-1} \left|(\delta_x e^{k-1}_{i-\frac{1}{2}})(\delta_x e^k_{i+\frac{1}{2}}) - (\delta_x e^k_{i-\frac{1}{2}})(\delta_x e^{k-1}_{i+\frac{1}{2}})\right|$$
$$\leqslant \frac{1}{2}(G^{k+\frac{1}{2}} - G^{k-\frac{1}{2}}) + c_0\frac{\tau}{h}\cdot\frac{c_3}{8c_0}\cdot h\sum_{i=1}^{m-1}\Big[\left|\delta_x e^k_{i-\frac{1}{2}}\right|\cdot\left|\delta_x e^{k+1}_{i+\frac{1}{2}}\right| + \left|\delta_x e^{k+1}_{i-\frac{1}{2}}\right|\cdot\left|\delta_x e^k_{i+\frac{1}{2}}\right|$$
$$+\left|\delta_x e^{k-1}_{i-\frac{1}{2}}\right|\cdot\left|\delta_x e^k_{i+\frac{1}{2}}\right| + \left|\delta_x e^k_{i-\frac{1}{2}}\right|\cdot\left|\delta_x e^{k-1}_{i+\frac{1}{2}}\right|\Big]$$
$$\leqslant \frac{1}{2}\left(G^{k+\frac{1}{2}} - G^{k-\frac{1}{2}}\right) + \frac{c_3}{4c_0}\left(|e^k|_1\cdot|e^{k+1}|_1 + |e^k|_1\cdot|e^{k-1}|_1\right). \tag{4.62}$$

将 (4.61) 和 (4.62) 代入 (4.60), 得到

$$C_k \leqslant \frac{1}{2}c_1|e^k|_1\cdot|e^{\bar{k}}|_1 + \frac{1}{2}(G^{k+\frac{1}{2}} - G^{k-\frac{1}{2}})$$
$$+\frac{c_3}{4c_0}(|e^k|_1\cdot|e^{k+1}|_1 + |e^k|_1\cdot|e^{k-1}|_1). \tag{4.63}$$

将 (4.58), (4.59) 和 (4.63) 代入 (4.57), 得到

$$\frac{1}{4\tau}(\|e^{k+1}\|^2 - \|e^{k-1}\|^2) + \frac{1}{4\tau}(|e^{k+1}|_1^2 - |e^{k-1}|_1^2)$$
$$\leqslant c_0\|e^k\|\cdot|e^{\bar{k}}|_1 + c_1\|e^k\|\cdot\|e^{\bar{k}}\| + c_1|e^k|_1\cdot|e^{\bar{k}}|_1 + c_2|e^k|_1\cdot\|e^{\bar{k}}\| + \frac{1}{2}c_1|e^k|_1\cdot|e^{\bar{k}}|_1$$
$$+\frac{1}{2}(G^{k+\frac{1}{2}} - G^{k-\frac{1}{2}}) + \frac{c_3}{4c_0}(|e^k|_1\cdot|e^{k+1}|_1 + |e^k|_1\cdot|e^{k-1}|_1) + (P^k, e^{\bar{k}}),$$

或

$$\frac{1}{2\tau}\Bigg[\left(\frac{\|e^{k+1}\|^2 + \|e^k\|^2}{2} + \frac{|e^{k+1}|_1^2 + |e^k|_1^2}{2} - \tau G^{k+\frac{1}{2}}\right)$$
$$-\left(\frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2} + \frac{|e^k|_1^2 + |e^{k-1}|_1^2}{2} - \tau G^{k-\frac{1}{2}}\right)\Bigg]$$
$$\leqslant c_0\|e^k\|\cdot|e^{\bar{k}}|_1 + c_1\|e^k\|\cdot\|e^{\bar{k}}\| + \frac{3}{2}c_1|e^k|_1\cdot|e^{\bar{k}}|_1 + c_2|e^k|_1\cdot\|e^{\bar{k}}\|$$
$$+\frac{c_3}{4c_0}(|e^k|_1\cdot|e^{k+1}|_1 + |e^k|_1\cdot|e^{k-1}|_1) + (P^k, e^{\bar{k}}). \tag{4.64}$$

记

$$\lambda = c_0\frac{\tau}{h}.$$

则

$$\begin{aligned}
\tau|G^{k+\frac{1}{2}}| &\leqslant \frac{1}{2}c_0\tau\sum_{i=1}^{m-1}(|\delta_x e^k_{i-\frac{1}{2}}|\cdot|\delta_x e^{k+1}_{i+\frac{1}{2}}|+|\delta_x e^{k+\frac{1}{2}}_{i-\frac{1}{2}}|\cdot|\delta_x e^k_{i+\frac{1}{2}}|)\\
&=\frac{1}{2}\cdot\frac{c_0\tau}{h}\cdot h\sum_{i=1}^{m-1}(|\delta_x e^k_{i-\frac{1}{2}}|\cdot|\delta_x e^{k+1}_{i+\frac{1}{2}}|+|\delta_x e^{k+1}_{i-\frac{1}{2}}|\cdot|\delta_x e^k_{i+\frac{1}{2}}|)\\
&\leqslant \lambda|e^k|_1\cdot|e^{k+1}|_1\\
&\leqslant \lambda\frac{|e^k|_1^2+|e^{k+1}|_1^2}{2}.
\end{aligned}$$

记

$$F^k=\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2}+\frac{|e^{k+1}|_1^2+|e^k|_1^2}{2}-\tau G^{k+\frac{1}{2}}, \tag{4.65}$$

则

$$\begin{aligned}
&\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2}+(1-\lambda)\frac{|e^{k+1}|_1^2+|e^k|_1^2}{2}\\
\leqslant F^k \leqslant &\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2}+(1+\lambda)\frac{|e^k|_1^2+|e^{k+1}|_1^2}{2}.
\end{aligned} \tag{4.66}$$

由 (4.64) 知存在常数 c_{10} 使得

$$\frac{1}{2\tau}(F^k-F^{k-1})\leqslant c_{10}(F^k+F^{k-1})+c_{10}\|P^k\|^2,\quad 1\leqslant k\leqslant n-1.$$

注意到 (4.34), 由上式得

$$(1-2c_{10}\tau)F^k\leqslant(1+2c_{10}\tau)F^{k-1}+2c_{10}Lc_8^2(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n-1.$$

当 $2c_{10}\tau\leqslant 1/3$ 时,

$$F^k\leqslant(1+6c_{10}\tau)F^{k-1}+3c_{10}Lc_8^2(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n-1.$$

由 Gronwall 不等式得到

$$F^k\leqslant e^{6c_{10}T}\left[F^0+\frac{Lc_8^2}{2}(\tau^2+h^2)^2\right],\quad 1\leqslant k\leqslant n-1. \tag{4.67}$$

由 (4.52) 和 (4.54), 得

$$F^0\leqslant\frac{\|e^1\|^2}{2}+(1+\lambda)\frac{|e^1|_1^2}{2}\leqslant\frac{Lc_7^2(\tau^2+h^2)^2}{2}, \tag{4.68}$$

将 (4.68) 代入 (4.67), 并注意到 F^k 的定义, 有

$$\frac{1}{2}\left(\|e^k\|^2+(1-\lambda)|e^k|_1^2\right)\leqslant e^{6c_{10}T}\frac{L(c_7^2+c_8^2)}{2}(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n.$$

取正数 ε 使得

$$\frac{1}{\varepsilon}+\frac{1}{L}=\frac{\varepsilon}{1-\lambda},$$

即

$$\varepsilon=\frac{1-\lambda+\sqrt{(1-\lambda)^2+4L(1-\lambda)}}{2},$$

则由引理 1.1(e) 得

$$\begin{aligned}\|e^k\|_\infty^2 &\leqslant \varepsilon|e^k|_1^2+\left(\frac{1}{\varepsilon}+\frac{1}{L}\right)\|e^k\|^2\\ &\leqslant \varepsilon|e^k|_1^2+\frac{\varepsilon}{1-\lambda}\|e^k\|^2\\ &=\frac{\varepsilon}{1-\lambda}((1-\lambda)|e^k|_1^2+\|e^k\|^2)\\ &=\frac{\varepsilon}{1-\lambda}e^{6c_{10}T}L(c_7^2+c_8^2)(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n,\end{aligned}$$

即

$$\|e^k\|_\infty\leqslant e^{3c_{10}T}\sqrt{\frac{\varepsilon L(c_7^2+c_8^2)}{1-\lambda}}(\tau^2+h^2),\quad 1\leqslant k\leqslant n. \qquad \Box$$

4.4　小结与延拓

本章对 Camassa-Holm 方程初边值问题建立了二层非线性差分格式和三层线性化差分格式.

对于二层非线性差分格式, 用 Browder 定理证明了差分格式解的存在性, 用能量方法证明了差分格式解的守恒性、唯一性和在无穷范数下的收敛性.

对于三层线性化差分格式, 用能量分析法证明了差分格式解的存在性、唯一性、守恒性和在无穷范数下的条件收敛性. 收敛性条件为 $c_0\dfrac{\tau}{h}<1$. 数值算例验证了这一条件是必不可少的. 这一条件类似于对于波动方程 $u_{tt}=c_0^2u_{xx}$, 建立的显式差分格式 $\delta_t^2u_i^k=c_0^2\delta_x^2u_i^k$ 的稳定性条件 $c_0\dfrac{\tau}{h}<1$. 当 k 层的值已知时, 为求第 $k+1$ 层的值, 只需要解三对角线性代数方程组.

微分方程 (4.1) 形式上是空间三阶的方程. 由于三阶导数 u_{xxx} 前的系数 $u(x,t)$ 在边界上的值为 0, 所以实际上它是在边界上退化的三阶方程. 对于方程 (4.1) 只需在左、右边界上各加一个边界条件 (4.3).

观察差分格式 (4.38) 的右端可以发现在构造三层线性化差分格式时, 我们对高阶导数项作了线性化处理, 证明了差分格式的条件收敛性. 这个条件是必要的. 容

易想到我们可以对高阶导数项做线性化处理, 得到如下差分格式

$$\Delta_t u_i^k - \Delta_t \delta_x^2 u_i^k + 3\psi(u^k, u^{\bar{k}})_i = (\Delta_x u_i^k)\delta_x^2 u_i^{\bar{k}} + \Delta_x(u_i^k \delta_x^2 u_i^{\bar{k}}),$$
$$1 \leqslant i \leqslant m-1,\ 1 \leqslant k \leqslant n-1.$$

在每一时间层上需要解五对角线性方程组. 我们猜想以上差分格式是唯一可解的、解是守恒的且是无条件收敛的.

本章三层线性化差分格式的结果主要取自 [12]. 文 [8] 研究了 Camassa-Holm 方程周期问题的二层非线性差分格式, 证明了差分格式解的收敛性.

第 5 章　Schrödinger 方程的差分方法

5.1 引　　言

Schrödinger 方程奠定了近代量子力学的基础, 揭示了微观世界中物质运动的基本规律. Schrödinger 方程在量子力学中的地位如同牛顿三定律之于经典力学、麦克斯韦方程之于电磁学. Schrödinger 方程在等离子物理、非线性光子学、水波及双分子动力学等领域也有重要应用.

考虑如下 Schrödinger 方程初边值问题

$$\mathrm{i}u_t + u_{xx} + q|u|^2u = 0, \quad 0 < x < L,\ 0 \leqslant t \leqslant T, \tag{5.1}$$

$$u(x,0) = \varphi(x), \quad 0 \leqslant x \leqslant L, \tag{5.2}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad 0 < t \leqslant T, \tag{5.3}$$

其中 q 为实常数, $\mathrm{i} = \sqrt{-1}$ 为虚数单位, $\varphi(x)$ 为复值函数, $\varphi(0) = \varphi(L) = 0, u(x,t)$ 为未知复函数. 问题 (5.1)–(5.3) 的解具有如下两个守恒律.

定理 5.1　设 $u(x,t)$ 为问题 (5.1)–(5.3) 的解. 记

$$Q(t) = \int_0^L |u(x,t)|^2 \mathrm{d}x, \quad E(t) = \int_0^L \left(|u_x(x,t)|^2 - \frac{q}{2}|u(x,t)|^4\right)\mathrm{d}x,$$

则

$$Q(t) \equiv Q(0), \quad 0 \leqslant t \leqslant T, \tag{5.4}$$

$$E(t) \equiv E(0), \quad 0 \leqslant t \leqslant T. \tag{5.5}$$

证明　(I) 在 (5.1) 的两边同乘以 $\bar{u}(x,t)$, 并对 x 从 0 到 L 求积分, 得

$$\mathrm{i}\int_0^L u_t(x,t)\bar{u}(x,t)\mathrm{d}x + \int_0^L u_{xx}(x,t)\bar{u}(x,t)\mathrm{d}x + q\int_0^L |u(x,t)|^4\mathrm{d}x = 0. \tag{5.6}$$

利用边界条件 (5.3), 有

$$\begin{aligned}\int_0^L u_{xx}(x,t)\bar{u}(x,t)\mathrm{d}x &= \int_0^L \Big[(u_x(x,t)\bar{u}(x,t))_x - u_x(x,t)\bar{u}_x(x,t)\Big]\mathrm{d}x \\ &= -\int_0^L |u_x(x,t)|^2\mathrm{d}x.\end{aligned}$$

在 (5.6) 的两边取虚部, 并注意到

$$\mathrm{Re}\{u_t(x,t)\bar{u}(x,t)\} = \left(\frac{1}{2}|u(x,t)|^2\right)_t,$$

有

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{1}{2}|u(x,t)|^2\mathrm{d}x = 0.$$

因而 (5.4) 成立.

(II) 在 (5.1) 的两边同乘以 $-\bar{u}_t(x,t)$, 并对 x 从 0 到 L 积分, 得

$$-\mathrm{i}\int_0^L |u_t(x,t)|^2\mathrm{d}x - \int_0^L u_{xx}(x,t)\bar{u}_t(x,t)\mathrm{d}x - q\int_0^L |u(x,t)|^2 u(x,t)\bar{u}_t(x,t)\mathrm{d}x = 0. \tag{5.7}$$

利用 (5.3) 有

$$\begin{aligned}-\int_0^L u_{xx}(x,t)\bar{u}_t(x,t)\mathrm{d}x &= -\int_0^L [(u_x(x,t)\bar{u}_t(x,t))_x - u_x(x,t)\bar{u}_{xt}(x,t)]\mathrm{d}x \\ &= \int_0^L u_x(x,t)\bar{u}_{xt}(x,t)\mathrm{d}x.\end{aligned}$$

对上式取实部, 得

$$\begin{aligned}&-\mathrm{Re}\left\{\int_0^L u_{xx}(x,t)\bar{u}_t(x,t)\mathrm{d}x\right\} \\ &= \int_0^L \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}|u_x(x,t)|^2\right)\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u_x(x,t)|^2\mathrm{d}x.\end{aligned}$$

在 (5.7) 的两边取实部并利用上式, 得

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u_x(x,t)|^2\mathrm{d}x - \frac{q}{4}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u(x,t)|^4\mathrm{d}x = 0.$$

因而 (5.5) 成立. □

当 $q \leqslant 0$ 时, 由 (5.5) 得

$$|u(\cdot,t)|_1^2 \leqslant E(0). \tag{5.8}$$

现考虑 $q > 0$. 由 (5.4) 及引理 1.1(c), 得到

$$\begin{aligned}\int_0^L |u(x,t)|^4\mathrm{d}x &\leqslant \|u(\cdot,t)\|_\infty^2\|u(\cdot,t)\|^2 \\ &\leqslant \left(\varepsilon|u(\cdot,t)|_1^2 + \frac{1}{4\varepsilon}\|u(\cdot,t)\|^2\right)\|u(\cdot,t)\|^2 \\ &= \left[\varepsilon|u(\cdot,t)|_1^2 + \frac{1}{4\varepsilon}Q(0)\right]Q(0).\end{aligned}$$

由 (5.5) 得到

$$|u(\cdot,t)|_1^2=\frac{q}{2}\int_0^L|u(x,t)|^4\mathrm{d}x+E(0)\leqslant\frac{q}{2}\left[\varepsilon|u(x,t)|_1^2+\frac{1}{4\varepsilon}Q(0)\right]Q(0)+E(0).$$

当 $Q(0)=0$ 时,

$$|u(\cdot,t)|_1^2=0. \tag{5.9}$$

当 $Q(0)\neq 0$ 时, 取 $\varepsilon=\dfrac{1}{qQ(0)}$, 则有

$$|u(\cdot,t)|_1^2\leqslant\frac{q}{4\varepsilon}Q^2(0)+2E(0)=\frac{1}{4}q^2Q^3(0)+2E(0). \tag{5.10}$$

再次应用引理 1.1(b), (5.8)–(5.10) 知

$$\|u(\cdot,t)\|_\infty^2\leqslant\frac{L}{4}|u(\cdot,t)|_1^2\leqslant\frac{L}{4}\left(\frac{1}{4}q^2Q^3(0)+2E(0)\right).$$

本章记

$$c_0=\max_{0\leqslant t\leqslant T}\|u(\cdot,t)\|_\infty.$$

5.2　二层非线性差分格式

5.2.1　差分格式的建立

在点 $(x_j,t_{k+\frac{1}{2}})$ 处考虑方程 (5.1), 有

$$\mathrm{i}u_t(x_j,t_{k+\frac{1}{2}})+u_{xx}(x_j,t_{k+\frac{1}{2}})+q|u(x_j,t_{k+\frac{1}{2}})|^2u(x_j,t_{k+\frac{1}{2}})=0,$$
$$1\leqslant j\leqslant m-1,\quad 0\leqslant k\leqslant n-1.$$

应用 Taylor 展开式及微分公式, 有

$$\mathrm{i}\delta_tU_j^{k+\frac{1}{2}}+\delta_x^2U_j^{k+\frac{1}{2}}+\frac{q}{2}(|U_j^k|^2+|U_j^{k+1}|^2)U_j^{k+\frac{1}{2}}=R_j^k,$$
$$1\leqslant j\leqslant m-1,\quad 0\leqslant k\leqslant n-1, \tag{5.11}$$

且存在正常数 c_1 使得

$$|R_j^k|\leqslant c_1(\tau^2+h^2),\quad 1\leqslant j\leqslant m-1,\ 0\leqslant k\leqslant n-1, \tag{5.12}$$

$$|\delta_tR_j^{k+\frac{1}{2}}|\leqslant c_1(\tau^2+h^2),\quad 1\leqslant j\leqslant m-1,\ 0\leqslant k\leqslant n-2. \tag{5.13}$$

要得到 (5.13), 需要用到带积分余项的 Taylor 展开式.

注意到初边值条件 (5.2)–(5.3), 有

$$U_j^0=\varphi(x_j),\quad 1\leqslant j\leqslant m-1, \tag{5.14}$$

$$U_0^k=0,\quad U_m^k=0,\quad 0\leqslant k\leqslant n. \tag{5.15}$$

在 (5.11) 中略去小量项, 并用 u_j^k 代替 U_j^k, 得到如下差分格式

$$\mathrm{i}\delta_t u_j^{k+\frac{1}{2}} + \delta_x^2 u_j^{k+\frac{1}{2}} + \frac{q}{2}(|u_j^k|^2 + |u_j^{k+1}|^2)u_j^{k+\frac{1}{2}} = 0,$$
$$1 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{5.16}$$

$$u_j^0 = \varphi(x_j), \quad 1 \leqslant j \leqslant m-1, \tag{5.17}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{5.18}$$

5.2.2 差分格式解的守恒性和有界性

记

$$Q^k = h\sum_{j=1}^{m-1}|u_j^k|^2, \quad E^k = h\sum_{j=0}^{m-1}|\delta_x u_{j+\frac{1}{2}}^k|^2 - \frac{q}{2}h\sum_{j=1}^{m-1}|u_j^k|^4.$$

定理 5.2 设 $\{u_j^k\,|\,0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 是差分格式 (5.16)–(5.18) 的解, 则有

$$Q^k \equiv Q^0, \quad 1 \leqslant k \leqslant n, \tag{5.19}$$

$$E^k \equiv E^0, \quad 1 \leqslant k \leqslant n. \tag{5.20}$$

证明 (I) 在 (5.16) 两边同乘以 $h\bar{u}_j^{k+\frac{1}{2}}$, 并对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}(\delta_t u_j^{k+\frac{1}{2}})\bar{u}_j^{k+\frac{1}{2}} + h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})\bar{u}_j^{k+\frac{1}{2}}$$
$$+\frac{q}{2}h\sum_{j=1}^{m-1}(|u_j^k|^2 + |u_j^{k+1}|^2)|u_j^{k+\frac{1}{2}}|^2 = 0, \quad 0 \leqslant k \leqslant n-1. \tag{5.21}$$

由 $u_0^{k+\frac{1}{2}} = 0, u_m^{k+\frac{1}{2}} = 0$, 得

$$h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})\bar{u}_j^{k+\frac{1}{2}} = -h\sum_{j=0}^{m-1}|\delta_x u_{j+\frac{1}{2}}^{k+\frac{1}{2}}|^2.$$

注意到

$$\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(\delta_t u_j^{k+\frac{1}{2}})\bar{u}_j^{k+\frac{1}{2}}\right\} = \frac{1}{2\tau}(\|u^{k+1}\|^2 - \|u^k\|^2),$$

在 (5.21) 两边取虚部, 得

$$\frac{1}{2\tau}(\|u^{k+1}\|^2 - \|u^k\|^2) = 0, \quad 0 \leqslant k \leqslant n-1,$$

即

$$\|u^{k+1}\|^2 = \|u^k\|^2, \quad 0 \leqslant k \leqslant n-1.$$

因而 (5.19) 成立.

(II) 在 (5.16) 两边同乘以 $-h\delta_t \bar{u}_j^{k+\frac{1}{2}}$, 并对 j 从 1 到 $m-1$ 求和, 得

$$\begin{aligned} & -\mathrm{i}h\sum_{j=1}^{m-1}|\delta_t u_j^{k+\frac{1}{2}}|^2 - h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})\delta_t \bar{u}_j^{k+\frac{1}{2}} \\ & -\frac{q}{2}h\sum_{j=1}^{m-1}(|u_j^k|^2 + |u_j^{k+1}|^2)u_j^{k+\frac{1}{2}}\delta_t \bar{u}_j^{k+\frac{1}{2}} = 0. \end{aligned} \tag{5.22}$$

注意到 $\delta_t u_0^{k+\frac{1}{2}} = 0, \delta_t u_m^{k+\frac{1}{2}} = 0$, 可得

$$-h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})\delta_t \bar{u}_j^{k+\frac{1}{2}} = h\sum_{j=0}^{m-1}(\delta_x u_{j+\frac{1}{2}}^{k+\frac{1}{2}})(\delta_x \delta_t \bar{u}_{j+\frac{1}{2}}^{k+\frac{1}{2}}).$$

由 (5.22) 两边取实部, 可得

$$\frac{1}{2\tau}(|u^{k+1}|_1^2 - |u^k|_1^2) - \frac{q}{2}\cdot\frac{1}{2\tau}\left(h\sum_{j=1}^{m-1}|u_j^{k+1}|^4 - h\sum_{j=1}^{m-1}|u_j^k|^4\right) = 0, \quad 0 \leqslant k \leqslant n-1.$$

因而

$$|u^{k+1}|_1^2 - \frac{q}{2}h\sum_{j=1}^{m-1}|u_j^{k+1}|^4 = |u^k|_1^2 - \frac{1}{2}h\sum_{j=1}^{m-1}|u_j^k|^4, \quad 0 \leqslant k \leqslant n-1,$$

即 (5.20) 成立. □

对于 $\mathcal{U}_h$ 中的网格函数 $u = \{u_i \,|\, 0 \leqslant i \leqslant m\}$, 定义下列范数

$$\|u\|_p = \sqrt[p]{h\left(\frac{1}{2}|u_0|^p + \sum_{i=1}^{m-1}|u_i|^p + \frac{1}{2}|u_m|^p\right)}.$$

定理 5.3　设 $\{u_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (5.16)–(5.18) 的解, 则有

$$\|u^k\|_\infty^2 \leqslant \frac{L}{2}\left(\frac{1}{8}q^2\|u^0\|^6 + |u^0|_1^2 - \frac{q}{2}\|u^0\|_4^4\right), \quad 1 \leqslant k \leqslant n. \tag{5.23}$$

证明　由定理 5.2 有

$$\|u^k\|^2 = \|u^0\|^2, \quad 1 \leqslant k \leqslant n, \tag{5.24}$$

$$|u^k|_1^2 - \frac{q}{2}\|u^k\|_4^4 = |u^0|_1^2 - \frac{q}{2}\|u^0\|_4^4, \quad 1 \leqslant k \leqslant n. \tag{5.25}$$

当 $q \leqslant 0$ 时, 由 (5.25), 有 $|u^k|_1^2 \leqslant |u^k|_1^2 - \frac{q}{2}\|u\|_4^4 = |u^0|_1^2 - \frac{q}{2}\|u^0\|_4^2$. 于是

$$\|u^k\|_\infty^2 \leqslant \frac{L}{4}|u^k|_1^2 \leqslant \frac{L}{4}\left(|u^0|_1^2 - \frac{q}{2}\|u^0\|_4^2\right).$$

因而 (5.23) 成立.

当 $q > 0$ 时, 由 (5.25) 得

$$|u^k|_1^2 = |u^0|_1^2 - \frac{q}{2}\|u^0\|_4^4 + \frac{q}{2}\|u^k\|_4^4, \quad 1 \leqslant k \leqslant n. \tag{5.26}$$

易知

$$\|u^k\|_4^4 = h\sum_{i=1}^{m-1}|u_i^k|^4 \leqslant \|u^k\|_\infty^2\|u^k\|^2.$$

于是

$$\|u^k\|_4^4 \leqslant \left(\varepsilon|u^k|_1^2 + \frac{1}{4\varepsilon}\|u^k\|^2\right)\|u^k\|^2. \tag{5.27}$$

在 (5.26) 的右端利用 (5.27) 得

$$\begin{aligned}|u^k|_1^2 &\leqslant |u^0|_1^2 - \frac{q}{2}\|u^0\|_4^4 + \frac{q}{2}\left(\varepsilon|u^k|_1^2 + \frac{1}{4\varepsilon}\|u^k\|^2\right)\|u^k\|^2 \\ &= |u^0|_1^2 - \frac{q}{2}\|u^0\|_4^4 + \frac{q}{2}\left(\varepsilon|u^k|_1^2 + \frac{1}{4\varepsilon}\|u^0\|^2\right)\|u^0\|^2.\end{aligned}$$

当 $\|u^0\|^2 = 0$ 时

$$|u^k|_1^2 = 0.$$

当 $\|u^0\|^2 \neq 0$ 时, 取 $\varepsilon = 1/(|q| \cdot \|u^0\|^2)$, 有

$$|u^k|_1^2 \leqslant 2\left(\frac{1}{8}q^2\|u^0\|^6 + |u^0|_1^2 - \frac{q}{2}\|u^0\|_4^4\right).$$

因而

$$\|u^k\|_\infty^2 \leqslant \frac{L}{4}|u^k|_1^2 \leqslant \frac{L}{2}\left(\frac{1}{8}q^2\|u^0\|^6 + |u^0|_1^2 - \frac{q}{2}\|u^0\|_4^4\right), \quad 1 \leqslant k \leqslant n. \qquad \square$$

记 (5.23) 的右端为 c_2^2, 则

$$\|u^k\|_\infty \leqslant c_2, \quad 1 \leqslant k \leqslant n.$$

5.2.3 差分格式解的存在性和唯一性

定理 5.4 (I) 差分格式 (5.16)–(5.18) 的解是存在的.

(II) 当 $\tau < 1/(3c_2^2|q|)$ 时, 差分格式 (5.16)–(5.18) 的解是唯一的.

证明 (I) 设 $\{u_j^k \,|\, 0 \leqslant j \leqslant m\}$ 已经求出. 可将 (5.16)–(5.18) 改写为

$$\mathrm{i}\frac{2}{\tau}(u_j^{k+\frac{1}{2}} - u_j^k) + \delta_x^2 u_j^{k+\frac{1}{2}} + \frac{q}{2}(|u_j^k|^2 + |2u_j^{k+\frac{1}{2}} - u_j^k|^2)u_j^{k+\frac{1}{2}} = 0, \quad 1 \leqslant j \leqslant m-1, \tag{5.28}$$

$$u_0^{k+\frac{1}{2}} = 0, \quad u_m^{k+\frac{1}{2}} = 0. \tag{5.29}$$

如果能求得 $\{u_j^{k+\frac{1}{2}} \,|\, 0 \leqslant j \leqslant m\}$, 则

$$u_j^{k+1} = 2u_j^{k+\frac{1}{2}} - u_j^k, \quad 0 \leqslant j \leqslant m.$$

令

$$w_j = u_j^{k+\frac{1}{2}}, \quad 0 \leqslant j \leqslant m,$$

则有

$$\mathrm{i}\frac{2}{\tau}(w_j - u_j^k) + \delta_x^2 w_j + \frac{q}{2}(|u_j^k|^2 + |2w_j - u_j^k|^2)w_j = 0, \quad 1 \leqslant j \leqslant m-1, \tag{5.30}$$

$$w_0 = 0, \quad w_m = 0. \tag{5.31}$$

可将 (5.30) 改写为

$$\frac{2}{\tau}(w_j - u_j^k) - \mathrm{i}\delta_x^2 w_j - \mathrm{i}\cdot\frac{q}{2}(|u_j^k|^2 + |2w_j - u_j^k|^2)w_j = 0, \quad 1 \leqslant j \leqslant m-1.$$

作映射 $\Pi : \mathring{\mathcal{U}}_h \to \mathring{\mathcal{U}}_h$,

$$\Pi(w)_j = \begin{cases} \dfrac{2}{\tau}(w_j - u_j^k) - \mathrm{i}\delta_x^2 w_j - \mathrm{i}\dfrac{q}{2}(|u_j^k|^2 + |2w_j - u_j^k|^2)w_j, & 1 \leqslant j \leqslant m-1, \\ 0, & j = 0, m, \end{cases}$$

将 $\Pi(w)$ 和 w 作内积, 得

$$\begin{aligned}\big(\Pi(w), w\big) &= \frac{2}{\tau}\big((w,w) - (u^k, w)\big) - \mathrm{i}(\delta_x^2 w, w) - \mathrm{i}\frac{q}{2}h\sum_{i=1}^{m-1}(|u_j^k|^2 + |2w_j - u_j^k|^2)|w_j|^2 \\ &= \frac{2}{\tau}(\|w\|^2 - (u^k, w)) + \mathrm{i}|w|_1^2 - \mathrm{i}\cdot\frac{q}{2}h\sum_{i=1}^{m-1}(|u_j^k|^2 + |2w_j - u_j^k|^2)|w_j|^2.\end{aligned}$$

因而

$$\mathrm{Re}\{\big(\Pi(w), w\big)\} = \frac{2}{\tau}(\|w\|^2 - \mathrm{Re}(u^k, w)) \geqslant \frac{2}{\tau}\|w\|(\|w\| - \|u^k\|).$$

当 $\|w\| = \|u^k\|$ 时, $\mathrm{Re}\{(\pi(w), w)\} \geqslant 0$.

由 Browder 定理 (定理 1.3) 知存在 $w^* \in \mathring{\mathcal{U}}_h, \|w^*\| \leqslant \|u^k\|$, 使得 $\Pi(w^*) = 0$. 因而差分格式 (5.16)–(5.18) 存在解.

(II) 设 (5.30)–(5.31) 另有解 $v = \{v_j \,|\, 0 \leqslant j \leqslant m\}$, 即有 $v \in \mathring{\mathcal{U}}_h$ 使得

$$\mathrm{i}\frac{2}{\tau}(v_j - u_j^k) + \delta_x^2 v_j + \frac{q}{2}(|u_j^k|^2 + |2v_j - u_j^k|^2)v_j = 0, \quad 1 \leqslant j \leqslant m-1, \tag{5.32}$$

$$v_0 = 0, \quad v_m = 0. \tag{5.33}$$

记

$$\theta_j = w_j - v_j, \quad 0 \leqslant j \leqslant m.$$

将 (5.30)–(5.31) 和 (5.32)–(5.33) 相减, 得

$$\mathrm{i}\frac{2}{\tau}\theta_j + \delta_x^2\theta_j + \frac{q}{2}(|u_j^k|^2\theta_j + |2w_j - u_j^k|^2 w_j - |2v_j - u_j^k|^2 v_j) = 0, \quad 1 \leqslant j \leqslant m-1, \tag{5.34}$$

$$\theta_0 = 0, \quad \theta_m = 0. \tag{5.35}$$

用 $-\mathrm{i}h\bar{\theta}_j$ 乘以 (5.34) 的两边, 并对 j 从 1 到 $m-1$ 求和, 得

$$\begin{aligned}&\frac{2}{\tau}h\sum_{j=1}^{m-1}|\theta_j|^2 - \mathrm{i}h\sum_{j=1}^{m-1}(\delta_x^2\theta_j)\bar{\theta}_j - \mathrm{i}\cdot\frac{q}{2}h\sum_{j=1}^{m-1}|u_j^k|^2\cdot|\theta_j|^2 \\ &-\mathrm{i}\frac{q}{2}h\sum_{j=1}^{m-1}(|2w_j - u_j^k|^2 w_j - |2v_j - u_j^k|^2 v_j)\bar{\theta}_j = 0.\end{aligned} \tag{5.36}$$

注意到

$$\begin{aligned}&|2w_j - u_j^k|^2 w_j - |2v_j - u_j^k|^2 v_j \\ =&|2w_j - u_j^k|^2(w_j - v_j) + (|2w_j - u_j^k|^2 - |2v_j - u_j^k|^2)v_j \\ =&|2w_j - u_j^k|^2\theta_j + [(2w_j - u_j^k)\overline{2w_j - u_j^k} - (2v_j - u_j^k)\overline{2v_j - u_j^k}]v_j \\ =&|2w_j - u_j^k|^2\theta_j + [2(w_j - v_j)\overline{2w_j - u_j^k} + (2v_j - u_j^k)2\overline{w_j - v_j}]v_j \\ =&|2w_j - u_j^k|^2\theta_j + 2[\theta_j\overline{2w_j - u_j^k} + (2v_j - u_j^k)\bar{\theta}_j]v_j,\end{aligned}$$

在 (5.36) 两边取实部, 有

$$\begin{aligned}\frac{2}{\tau}\|\theta\|^2 &\leqslant |q|h\sum_{j=1}^{m-1}(|\theta_j|^2|2w_j - u_j^k| + |2v_j - u_j^k||\theta_j|^2)|v_j| \\ &\leqslant 6c_2^2|q|\cdot\|\theta\|^2.\end{aligned}$$

当 $\tau < 1/(3c_2^2|q|)$ 时 $\|\theta\|^2 = 0$, 即 $\theta_j = 0, 0 \leqslant j \leqslant m$. 因而差分格式 (5.16)–(5.18) 是唯一可解的. □

5.2.4　差分格式解的收敛性

定理 5.5　设 $\{U_j^k\,|\,0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ 是问题 (5.1)–(5.3) 的解, $\{u_j^k\,|\,0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ 是差分格式 (5.16)–(5.18) 的解. 记

$$e_j^k=U_j^k-u_j^k,\quad 0\leqslant j\leqslant m,\ 0\leqslant k\leqslant n.$$

则存在常数 c_3 使得

$$\|e^k\|\leqslant c_3(\tau^2+h^2),\quad 1\leqslant k\leqslant n.$$

证明　(I) 将 (5.11), (5.14)–(5.15) 和 (5.16)–(5.18) 相减, 得误差方程组

$$\mathrm{i}\delta_t e_j^{k+\frac12}+\delta_x^2 e_j^{k+\frac12}+\frac{q}{2}[(|U_j^k|^2+|U_j^{k+1}|^2)U_j^{k+\frac12}-(|u_j^k|^2+|u_j^{k+1}|^2)u_j^{k+\frac12}]=R_j^k,$$
$$1\leqslant j\leqslant m-1,\quad 0\leqslant k\leqslant n-1,\tag{5.37}$$

$$e_j^0=0,\quad 1\leqslant j\leqslant m-1,\tag{5.38}$$

$$e_0^k=0,\quad e_m^k=0,\quad 0\leqslant k\leqslant n.\tag{5.39}$$

记

$$G(U^{k+\frac12})_j=(|U_j^k|^2+|U_j^{k+1}|^2)U_j^{k+\frac12}.$$

用 $h\bar{e}_j^{k+\frac12}$ 乘以方程 (5.37) 的两边, 并对 j 从 1 到 $m-1$ 求和, 得到

$$\mathrm{i}h\sum_{j=1}^{m-1}(\delta_t e_j^{k+\frac12})\bar{e}_j^{k+\frac12}+h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{k+\frac12})\bar{e}_j^{k+\frac12}+\frac{q}{2}h\sum_{j=1}^{m-1}[G(U^{k+\frac12})_j-G(u^{k+\frac12})_j]\bar{e}_j^{k+\frac12}$$
$$=h\sum_{j=1}^{m-1}R_j^k\bar{e}_j^{k+\frac12},\quad 0\leqslant k\leqslant n-1.\tag{5.40}$$

注意到

$$h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{k+\frac12})\bar{e}_j^{k+\frac12}=-h\sum_{j=0}^{m-1}|\delta_x e_{j+\frac12}^{k+\frac12}|^2,$$

$$\begin{aligned}&G(U^{k+\frac12})_i-G(u^{k+\frac12})_i\\=&(|U_j^k|^2+|U_j^{k+1}|^2)(U_j^{k+\frac12}-u_j^{k+\frac12})+(|U_j^k|^2+|U_j^{k+1}|^2-|u_j^k|^2-|u_j^{k+1}|^2)u_j^{k+\frac12}\\=&(|U_j^k|^2+|U_j^{k+1}|^2)e_j^{k+\frac12}+(e_j^k\bar{U}_j^k+u_j^k\bar{e}_j^k+e_j^{k+1}\bar{U}_j^{k+1}+u_j^{k+1}\bar{e}_j^{k+1})u_j^{k+\frac12},\end{aligned}\tag{5.41}$$

在 (5.40) 两边取虚部, 得到

$$\begin{aligned}
&\frac{1}{2\tau}(\|e^{k+1}\|^2-\|e^k\|^2)\\
=&\text{Im}\left\{-\frac{q}{2}h\sum_{j=1}^{m-1}(e_j^k\bar{U}_j^k+u_j^k\bar{e}_j^k+e_j^{k+1}\bar{U}_j^{k+1}+u_j^{k+1}\bar{e}_j^{k+1})u_j^{k+\frac12}\bar{e}_j^{k+\frac12}+h\sum_{j=1}^{m-1}R_j^k\bar{e}_j^{k+\frac12}\right\}\\
\leqslant&\frac{|q|}{2}h\sum_{j=1}^{m-1}(c_0|e_j^k|+c_2|e_j^k|+c_0|e_j^{k+1}|+c_2|e_j^{k+1}|)c_2|\bar{e}_j^{k+\frac12}|+h\sum_{j=1}^{m-1}|R_j^k|\cdot|\bar{e}_j^{k+\frac12}|\\
\leqslant&\frac{|q|(c_0+c_2)c_2}{2}(\|e^k\|+\|e^{k+1}\|)\|e^{k+\frac12}\|+\|R^k\|\cdot\|e^{k+\frac12}\|\\
\leqslant&\frac{|q|(c_0+c_2)c_2}{2}(\|e^k\|+\|e^{k+1}\|)\cdot\frac{\|e^k\|+\|e^{k+1}\|}{2}+\|R^k\|\cdot\frac{\|e^k\|+\|e^{k+1}\|}{2},\\
&\qquad 0\leqslant k\leqslant n-1.
\end{aligned}$$

当 $\frac{1}{2}(\|e^{k+1}\|+\|e^k\|)\neq 0$ 时, 两边同时约去 $\frac{1}{2}(\|e^{k+1}\|+\|e^k\|)$, 得到

$$\frac{1}{\tau}(\|e^{k+1}\|-\|e^k\|)\leqslant\frac{|q|}{2}(c_0+c_2)c_2(\|e^k\|+\|e^{k+1}\|)+\|R^k\|.$$

当 $\frac{1}{2}(\|e^{k+1}\|+\|e^k\|)=0$ 时上式也成立. 于是

$$\left[1-\frac{|q|}{2}(c_0+c_2)c_2\tau\right]\|e^{k+1}\|\leqslant\left[1+\frac{|q|}{2}(c_0+c_2)c_2\tau\right]\|e^{k+1}\|+\tau\|R^k\|,\quad 0\leqslant k\leqslant n-1.$$

当 $\frac{|q|}{2}(c_0+c_2)c_2\tau\leqslant\frac{1}{3}$ 时, 注意到 (5.12), 可得

$$\begin{aligned}
\|e^{k+1}\|&\leqslant\left[1+\frac{3|q|}{2}(c_0+c_2)c_2\tau\right]\|e^k\|+\frac{3}{2}\tau\|R^k\|\\
&\leqslant\left[1+\frac{3|q|}{2}(c_0+c_2)c_2\tau\right]\|e^k\|+\frac{3}{2}\sqrt{L}c_1\tau(\tau^2+h^2),\quad 0\leqslant k\leqslant n-1.
\end{aligned}$$

当 $q=0$ 时, 由

$$\|e^{k+1}\|\leqslant\|e^k\|+\frac{3}{2}\sqrt{L}c_1\tau(\tau^2+h^2),\quad 0\leqslant k\leqslant n-1,$$

递推得到

$$\|e^k\|\leqslant\|e^0\|+\frac{3}{2}\sqrt{L}c_1\,k\tau(\tau^2+h^2)\leqslant\frac{3}{2}\sqrt{L}c_1T(\tau^2+h^2),\quad 1\leqslant k\leqslant n.$$

当 $q\neq 0$ 时, 由 Gronwall 不等式得

$$\|e^k\|\leqslant e^{\frac{3|q|}{2}(c_0+c_2)c_2T}\frac{\sqrt{L}c_1}{|q|(c_0+c_2)}(\tau^2+h^2),\quad 1\leqslant k\leqslant n.$$

□

定理 5.6　设 $\{U_j^k\,|\,0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ 是问题 (5.1)–(5.3) 的解, $\{u_j^k\,|\,0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ 是差分格式 (5.16)–(5.18) 的解. 记

$$e_j^k = U_j^k - u_j^k,\quad 0\leqslant j\leqslant m,\ 0\leqslant k\leqslant n,$$

则存在常数 c_4 使得

$$\left|e^k\right|_1 \leqslant c_4(\tau^2+h^2),\quad 1\leqslant k\leqslant n. \tag{5.42}$$

证明　用 $-h\delta_t\bar{e}_j^{k+\frac{1}{2}}$ 与 (5.37) 的两边相乘, 并对 j 从 1 到 $m-1$ 求和, 得

$$\begin{aligned}&-\mathrm{i}h\sum_{j=1}^{m-1}|\delta_t e_j^{k+1}|^2 - h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{k+\frac{1}{2}})\delta_t\bar{e}_j^{k+\frac{1}{2}} - \frac{q}{2}h\sum_{j=1}^{m-1}[G(U^{k+\frac{1}{2}})_j - G(u^{k+\frac{1}{2}})_j]\delta_t\bar{e}_j^{k+\frac{1}{2}}\\ =&-h\sum_{j=1}^{m-1}R_j^k\delta_t\bar{e}_j^{k+\frac{1}{2}},\quad 0\leqslant k\leqslant n-1.\end{aligned}$$

上式两边取实部, 可得

$$\begin{aligned}&\frac{1}{2\tau}(|e^{k+1}|_1^2 - |e^k|_1^2)\\ =&\mathrm{Re}\left\{\frac{q}{2}h\sum_{j=1}^{m-1}[G(U^{k+\frac{1}{2}})_j - G(u^{k+\frac{1}{2}})_j]\delta_t\bar{e}_j^{k+\frac{1}{2}}\right\}\\ &+\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}R_j^k\delta_t\bar{e}_j^{k+\frac{1}{2}}\right\},\quad 0\leqslant k\leqslant n-1.\end{aligned} \tag{5.43}$$

由 (5.37) 可得

$$\delta_t e_j^{k+\frac{1}{2}} = \mathrm{i}\delta_x^2 e_j^{k+\frac{1}{2}} + \mathrm{i}\frac{q}{2}[G(U^{k+\frac{1}{2}})_j - G(u^{k+\frac{1}{2}})_j] - \mathrm{i}R_j^k,$$

再取共轭, 得

$$\delta_t\bar{e}_j^{k+\frac{1}{2}} = -\mathrm{i}\delta_x^2\bar{e}_j^{k+\frac{1}{2}} - \mathrm{i}\cdot\frac{q}{2}\overline{G(U^{k+\frac{1}{2}})_j - G(u^{k+\frac{1}{2}})_j} + \mathrm{i}\bar{R}_j^k.$$

于是

$$\begin{aligned}&\mathrm{Re}\left\{\frac{q}{2}h\sum_{j=1}^{m-1}[G(U^{k+\frac{1}{2}})_j - G(u^{k+\frac{1}{2}})_j]\delta_t\bar{e}_j^{k+\frac{1}{2}}\right\}\\ =&\frac{q}{2}\mathrm{Re}\left\{h\sum_{j=1}^{m-1}[G(U^{k+\frac{1}{2}})_j - G(u^{k+\frac{1}{2}})_j][-\mathrm{i}\delta_x^2\bar{e}_j^{k+\frac{1}{2}}\right.\end{aligned}$$

$$
\begin{aligned}
&-\mathrm{i}\cdot\frac{q}{2}\overline{G(U^{k+\frac{1}{2}})_j-G(u^{k+\frac{1}{2}})_j+\mathrm{i}\bar{R}_j^k]}\Big\}\\
=&\frac{q}{2}\mathrm{Re}\Big\{-\mathrm{i}h\sum_{j=1}^{m-1}[G(U^{k+\frac{1}{2}})_j-G(u^{k+\frac{1}{2}})_j](\delta_x^2\bar{e}_j^{k+\frac{1}{2}})\\
&+\mathrm{i}h\sum_{j=1}^{m-1}\big[G(U^{k+\frac{1}{2}})_j-G(u^{k+\frac{1}{2}})_j\big]\bar{R}_j^k\Big\}\\
=&\frac{q}{2}\mathrm{Re}\Big\{\mathrm{i}h\sum_{j=0}^{m-1}\big[\delta_x(G(U^{k+\frac{1}{2}})-G(u^{k+\frac{1}{2}}))_{j+\frac{1}{2}}\big](\delta_x\bar{e}_{j+\frac{1}{2}}^{k+\frac{1}{2}})\\
&+\mathrm{i}h\sum_{j=1}^{m-1}[G(U^{k+\frac{1}{2}})_j-G(u^{k+\frac{1}{2}})_j]\bar{R}_j^k\Big\}\\
\leqslant&\frac{|q|}{2}|G(U^{k+\frac{1}{2}})-G(u^{k+\frac{1}{2}})|_1\cdot|e^{k+\frac{1}{2}}|_1+\|G(U^{k+\frac{1}{2}})-G(u^{k+\frac{1}{2}})\|\cdot\|R^k\|. \quad (5.44)
\end{aligned}
$$

由 (5.41) 可得存在常数 c_5,c_6 使得

$$\|G(U^{k+\frac{1}{2}})-G(u^{k+\frac{1}{2}})\|\leqslant c_5(\|e^k\|+\|e^{k+1}\|), \quad (5.45)$$

$$|G(U^{k+\frac{1}{2}})-G(u^{k+\frac{1}{2}})|_1\leqslant c_6(\|e^k\|+\|e^{k+1}\|+|e^k|_1+|e^{k+1}|_1). \quad (5.46)$$

将 (5.45)–(5.46) 代入 (5.44) 可得

$$
\begin{aligned}
&\mathrm{Re}\Big\{\frac{q}{2}h\sum_{j=1}^{m-1}[G(U^{k+\frac{1}{2}})_j-G(u^{k+\frac{1}{2}})_j]\delta_t\bar{e}_j^{k+\frac{1}{2}}\Big\}\\
\leqslant&\frac{|q|}{2}\cdot c_6(\|e^k\|+\|e^{k+1}\|+|e^k|_1+|e^{k+1}|_1)\cdot\frac{1}{2}(|e^k|_1+|e^{k+1}|_1)\\
&+c_5(\|e^k\|+\|e^{k+1}\|)\|R^k\|. \quad (5.47)
\end{aligned}
$$

将 (5.47) 代入 (5.43), 然后将 k 换为 l, 并对 l 从 0 到 k 求和, 得

$$
\begin{aligned}
\frac{1}{2\tau}|e^{k+1}|_1^2\leqslant&\sum_{l=0}^{k}\Big[\frac{|q|}{4}c_6(\|e^l\|+\|e^{k+1}\|+|e^l|_1+|e^{l+1}|_1)(|e^l|_1+|e^{l+1}|_1)\\
&+c_5(\|e^l\|+\|e^{l+1}\|)\|R^l\|\Big]+\mathrm{Re}\Big\{-h\sum_{j=1}^{m-1}\sum_{l=0}^{k}R_j^l\delta_t\bar{e}_j^{l+\frac{1}{2}}\Big\},\\
&0\leqslant k\leqslant n-1. \quad (5.48)
\end{aligned}
$$

现分析上式中的最后一项. 注意到

$$\begin{aligned}
&\sum_{l=0}^{k} R_j^l \delta_t \bar{e}_j^{l+\frac{1}{2}} \\
=&\sum_{l=0}^{k} R_j^l \frac{\bar{e}_j^{l+1}-\bar{e}_j^l}{\tau} \\
=&\frac{1}{\tau}\left(\sum_{l=0}^{k} R_j^l \bar{e}_j^{l+1}-\sum_{l=-1}^{k-1} R_j^{l+1} \bar{e}^{l+1}\right) \\
=&\frac{1}{\tau}\left[R_j^k \bar{e}_j^{k+1}-\sum_{l=0}^{k-1}(R_j^{l+1}-R_j^l) \bar{e}_j^{l+1}-R_j^0 \bar{e}^0\right] \\
=&\frac{1}{\tau} R_j^k \bar{e}_j^{k+1}-\sum_{l=0}^{k-1} \frac{R_j^{l+1}-R_j^l}{\tau} \bar{e}_j^{l+1},
\end{aligned}$$

有

$$\begin{aligned}
&\operatorname{Re}\left\{-h \sum_{j=1}^{m-1} \sum_{l=0}^{k} R_j^l \delta_t \bar{e}_j^{l+\frac{1}{2}}\right\} \\
=&\operatorname{Re}\left\{-h \sum_{j=1}^{m-1}\left[\frac{1}{\tau} R_j^k \bar{e}_j^{k+1}-\sum_{l=0}^{k-1} \frac{R_j^{l+1}-R_j^l}{\tau} \bar{e}_j^{l+1}\right]\right\} \\
\leqslant & \frac{1}{\tau}\|R^k\| \cdot\|e^{k+1}\|+\sum_{l=0}^{k-1}\|\delta_t R^{l+\frac{1}{2}}\| \cdot\|e^{l+1}\|.
\end{aligned} \tag{5.49}$$

将 (5.49) 代入 (5.48), 并将所得不等式两边同乘以 2τ, 得

$$\begin{aligned}
|e^{k+1}|_1^2 \leqslant & 2\tau \sum_{l=0}^{k}\left[\frac{|q|}{4} c_6(\|e^l\|+\|e^{l+1}\|+|e^l|_1+|e^{l+1}|_1)(|e^l|_1+|e^{l+1}|_1)\right. \\
&\left.+c_5(\|e^l\|+\|e^{l+1}\|)\|R^l\|\right]+2\|R^k\| \cdot\|e^{k+1}\|+2\tau \sum_{l=0}^{k-1}\|\delta_t R^{l+1}\| \cdot\|e^{l+1}\|.
\end{aligned}$$

应用 (5.12), (5.13), $\|e^l\| \leqslant \frac{L}{\sqrt{6}}|e^l|_1$ 及定理 5.5, 得

$$\begin{aligned}
|e^{k+1}|_1^2 \leqslant & 2\tau \sum_{l=0}^{k}\left[\frac{|q|}{4} c_6\left(1+\frac{L}{\sqrt{6}}\right)(|e^l|_1+|e^{l+1}|_1)^2+\sqrt{L} c_1 c_3 c_5(\tau^2+h^2)^2\right] \\
&+2\sqrt{L} c_1(\tau^2+h^2) \frac{L}{\sqrt{6}}|e^{k+1}|_1+2\tau \sum_{l=0}^{k-1} \sqrt{L} c_1(\tau^2+h^2) \cdot \frac{L}{\sqrt{6}}|e^{l+1}|_1, \\
& 0 \leqslant k \leqslant n-1.
\end{aligned}$$

存在常数 c_7 使得

$$|e^{k+1}|_1^2 \leqslant c_7\tau\sum_{l=1}^{k}|e^l|_1^2 + c_7(\tau^2+h^2)^2, \quad 0 \leqslant k \leqslant n-1.$$

由 Gronwall 不等式可知 (5.42) 成立. □

5.3 三层线性化差分格式

5.3.1 差分格式的建立

在点 $(x_j, t_{\frac{1}{2}})$ 处考虑方程 (5.1), 有

$$\mathrm{i}u_t(x_j, t_{\frac{1}{2}}) + u_{xx}(x_j, t_{\frac{1}{2}}) + q|u(x_j, t_{\frac{1}{2}})|^2 u(x_j, t_{\frac{1}{2}}) = 0, \quad 1 \leqslant j \leqslant m-1.$$

由微分公式可得

$$\mathrm{i}\delta_t U_j^{\frac{1}{2}} + \delta_x^2 U_j^{\frac{1}{2}} + q|\hat{u}_j|^2 U_j^{\frac{1}{2}} = P_j^0, \quad 1 \leqslant j \leqslant m-1, \tag{5.50}$$

其中

$$\hat{u}_j = u(x_j, 0) + \frac{\tau}{2}u_t(x_j, 0), \quad 1 \leqslant j \leqslant m-1,$$

且存在常数 c_8 使得

$$|P_j^0| \leqslant c_8(\tau^2+h^2), \quad 1 \leqslant j \leqslant m-1. \tag{5.51}$$

在点 (x_j, t_k) 处考虑方程 (5.1), 有

$$\mathrm{i}u_t(x_j, t_k) + u_{xx}(x_j, t_k) + q|u(x_j, t_k)|^2 u(x_j, t_k) = 0, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1.$$

由微分公式可得

$$\mathrm{i}\Delta_t U_j^k + \delta_x^2 U_j^{\bar{k}} + q|U_j^k|^2 U_j^{\bar{k}} = P_j^k, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{5.52}$$

且存在常数 c_9 使得

$$|P_j^k| \leqslant c_9(\tau^2+h^2), \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{5.53}$$

$$|\Delta_t P_j^k| \leqslant c_9(\tau^2+h^2), \quad 1 \leqslant j \leqslant m-1,\ 2 \leqslant k \leqslant n-2. \tag{5.54}$$

注意到初边值条件 (5.2) 和 (5.3), 有

$$U_j^0 = \varphi(x_j), \quad 1 \leqslant j \leqslant m-1, \tag{5.55}$$

$$U_0^k = 0, \quad U_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{5.56}$$

在 (5.50) 和 (5.52) 中略去小量项, 对问题 (5.1)–(5.3) 建立如下线性化差分格式

$$\mathrm{i}\delta_t u_j^{\frac{1}{2}} + \delta_x^2 u_j^{\frac{1}{2}} + q|\hat{u}_j|^2 u_j^{\frac{1}{2}} = 0, \quad 1 \leqslant j \leqslant m-1, \tag{5.57}$$

$$\mathrm{i}\Delta_t u_j^k + \delta_x^2 u_j^{\bar{k}} + q|u_j^k|^2 u_j^{\bar{k}} = 0, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{5.58}$$

$$u_j^0 = \varphi(x_j), \quad 1 \leqslant j \leqslant m-1, \tag{5.59}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{5.60}$$

5.3.2 差分格式解的守恒性和有界性

定理 5.7 设 $\{u_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (5.57)–(5.60) 的解. 记

$$E^k = \frac{1}{2}(|u^{k+1}|_1^2 + |u^k|_1^2) - \frac{q}{2}h\sum_{j=0}^{m-1}|u_j^k|^2 \cdot |u_j^{k+1}|^2, \quad 0 \leqslant k \leqslant n-1.$$

则有

$$\|u^k\|^2 = \|u^0\|^2, \quad 1 \leqslant k \leqslant n, \tag{5.61}$$

$$\frac{1}{2}(|u^1|_1^2 + |u^0|_1^2) - \frac{q}{2}h\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^1|^2 = |u^0|_1^2 - \frac{q}{2}h\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^0|^2, \tag{5.62}$$

$$E^k = E^0, \quad 1 \leqslant k \leqslant n-1. \tag{5.63}$$

证明 (I) 在 (5.57) 的两边同乘以 $h\bar{u}_j^{\frac{1}{2}}$, 并对 j 从 1 到 $m-1$ 求和, 得到

$$\mathrm{i}h\sum_{j=1}^{m-1}(\delta_t u_j^{\frac{1}{2}})\bar{u}_j^{\frac{1}{2}} + h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\frac{1}{2}})\bar{u}_j^{\frac{1}{2}} + qh\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^{\frac{1}{2}}|^2 = 0. \tag{5.64}$$

注意到 $u_0^{\frac{1}{2}} = 0, u_m^{\frac{1}{2}} = 0$, 有

$$h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\frac{1}{2}})\bar{u}_j^{\frac{1}{2}} = -|u^{\frac{1}{2}}|_1^2.$$

在 (5.64) 两边取虚部, 得到

$$\frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2) = 0,$$

即

$$\|u^1\|^2 = \|u^0\|^2. \tag{5.65}$$

在 (5.58) 两边同乘以 $h\bar{u}_j^{\bar{k}}$, 并对 j 从 1 到 $m-1$ 求和, 得到

$$\mathrm{i}h\sum_{j=1}^{m-1}(\Delta_t u_j^k)\bar{u}_j^{\bar{k}} + h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\bar{k}})\bar{u}_j^{\bar{k}} + qh\sum_{j=1}^{m-1}|u_j^k|^2|u_j^{\bar{k}}|^2 = 0. \tag{5.66}$$

注意到 $u_0^{\bar{k}} = 0, u_m^{\bar{k}} = 0$, 有

$$h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\bar{k}})\bar{u}_j^{\bar{k}} = -|u^{\bar{k}}|_1^2.$$

在 (5.66) 两边取虚部, 有

$$\frac{1}{4\tau}(\|u^{k+1}\|^2 - \|u^{k-1}\|^2) = 0, \quad 1 \leqslant k \leqslant n-1,$$

即

$$\|u^{k+1}\|^2 = \|u^{k-1}\|^2, \quad 1 \leqslant k \leqslant n-1. \tag{5.67}$$

综合 (5.66) 和 (5.67) 可得 (5.61).

(II) 在 (5.57) 的两边同乘以 $-h\delta_t\bar{u}_j^{\frac{1}{2}}$, 并对 j 从 1 到 $m-1$ 求和, 得到

$$-\mathrm{i}h\sum_{j=1}^{m-1}|\delta_t u_j^{\frac{1}{2}}|^2 - h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\frac{1}{2}})(\delta_t\bar{u}_j^{\frac{1}{2}}) - qh\sum_{j=1}^{m-1}|\hat{u}_j|^2 u_j^{\frac{1}{2}}\delta_t\bar{u}_j^{\frac{1}{2}} = 0. \tag{5.68}$$

注意到 $\delta_t u_0^{\frac{1}{2}} = 0, \delta_t u_m^{\frac{1}{2}} = 0$, 有

$$-h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\frac{1}{2}})\delta_t\bar{u}_j^{\frac{1}{2}} = h\sum_{j=0}^{m-1}(\delta_x u_{j+\frac{1}{2}}^{\frac{1}{2}})(\delta_t\delta_x\bar{u}_{j+\frac{1}{2}}^{\frac{1}{2}}).$$

在 (5.68) 两边取实部, 得

$$\frac{1}{2\tau}(|u^1|_1^2 - |u^0|_1^2) - qh\sum_{j=1}^{m-1}|\hat{u}_j|^2 \cdot \frac{1}{2\tau}(|u_j^1|^2 - |u_j^0|^2) = 0.$$

即

$$\frac{1}{2}(|u^1|_1^2 + |u^0|_1^2) - \frac{q}{2}h\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^1|^2 = |u^0|_1^2 - \frac{q}{2}h\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^0|^2. \tag{5.69}$$

因而 (5.62) 成立.

在 (5.58) 的两边同乘以 $-h\Delta_t\bar{u}_j^k$, 并对 j 从 1 到 $m-1$ 求和, 得到

$$-\mathrm{i}h\sum_{j=1}^{m-1}|\Delta_t u_j^k|^2 - h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\bar{k}})(\Delta_t\bar{u}_j^k) - qh\sum_{j=1}^{m-1}|u_j^k|^2 u_j^{\bar{k}}\Delta_t\bar{u}_j^k = 0.$$

将上式两边取实部, 可得

$$\frac{1}{4\tau}(|u^{k+1}|_1^2-|u^{k-1}|_1^2)-qh\sum_{j=1}^{m-1}|u_j^k|^2\cdot\frac{1}{4\tau}(|u_j^{k+1}|^2-|u_j^{k-1}|^2)=0,$$

于是

$$\begin{aligned}&\frac{1}{2}(|u^{k+1}|_1^2+|u^k|_1^2)-\frac{q}{2}h\sum_{j=1}^{m-1}|u_j^k|^2\cdot|u_j^{k+1}|^2\\=&\frac{1}{2}(|u^k|_1^2+|u^{k-1}|_1^2)-\frac{q}{2}h\sum_{j=1}^{m-1}|u_j^{k-1}|^2\cdot|u^k|^2,\quad 1\leqslant k\leqslant n-1.\end{aligned}\tag{5.70}$$

因而 (5.63) 成立. □

由定理 5.7 可知存在常数 c_{10} 使得

$$\|u^k\|_\infty\leqslant c_{10},\quad 1\leqslant k\leqslant n.$$

5.3.3 差分格式解的存在性和唯一性

定理 5.8 差分格式 (5.57)–(5.60) 是唯一可解的.

证明 (I) 差分格式 (5.57) 和 (5.60) 是关于 $\{u_j^1|0\leqslant j\leqslant m\}$ 的线性方程组. 考虑其齐次方程组

$$\mathrm{i}\frac{1}{\tau}u_j^1+\frac{1}{2}\delta_x^2u_j^1+\frac{1}{2}q|\hat{u}_j|^2u_j^1=0,\quad 1\leqslant j\leqslant m-1,\tag{5.71}$$

$$u_0^1=0,\quad u_m^1=0.\tag{5.72}$$

用 $h\bar{u}_j^1$ 乘以 (5.71) 的两边, 并对 j 从 1 到 $m-1$ 求和. 利用 (5.72), 然后取虚部, 得

$$\|u^1\|^2=0,$$

即

$$u_j^1=0,\quad 0\leqslant j\leqslant m.$$

因而差分格式唯一确定第 1 层的值 u^1.

(II) 设 $\{u_j^{k-1}\,|\,0\leqslant j\leqslant m\}$ 和 $\{u_j^k\,|\,0\leqslant j\leqslant m\}$ 已求得, 则由 (5.58) 和 (5.60) 可得关于 $\{u_j^{k+1}\,|\,0\leqslant j\leqslant m\}$ 的线性方程组. 考虑其齐次方程组

$$\mathrm{i}\cdot\frac{1}{2\tau}u_j^{k+1}+\frac{1}{2}\delta_x^2u_j^{k+1}+\frac{q}{2}|u_j^k|^2u_j^{k+1}=0,\quad 1\leqslant j\leqslant m-1,\tag{5.73}$$

$$u_0^{k+1}=0,\quad u_m^{k+1}=0.\tag{5.74}$$

在 (5.73) 的两边同时乘以 $h\bar{u}_j^{k+1}$, 并对 j 从 1 到 $m-1$ 求和, 取虚部, 可得

$$\frac{1}{2\tau}\|u^{k+1}\|^2 = 0.$$

因而

$$u_j^{k+1} = 0, \quad 0 \leqslant j \leqslant m.$$

于是 (5.58) 和 (5.60) 关于 $\{u_j^{k+1}\,|\,0 \leqslant j \leqslant m\}$ 有唯一解. □

5.3.4 差分格式解的收敛性

定理 5.9 设 $\{U_j^k\,|\,0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为问题 (5.1)–(5.3) 的解, $\{u_j^k\,|\,0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (5.57)–(5.60) 的解. 记

$$e_j^k = U_j^k - u_j^k, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n.$$

则存在常数 c_{11} 使得

$$\|e^k\| \leqslant c_{11}(\tau^2 + h^2), \quad 0 \leqslant k \leqslant n. \tag{5.75}$$

证明 将 (5.50), (5.52), (5.55), (5.56) 和 (5.57)–(5.60) 相减, 可得误差方程

$$\mathrm{i}\delta_t e_j^{\frac{1}{2}} + \delta_x^2 e_j^{\frac{1}{2}} + q|\hat{u}_j|^2 e_j^{\frac{1}{2}} = P_j^0, \quad 1 \leqslant j \leqslant m-1, \tag{5.76}$$

$$\mathrm{i}\Delta_t e_j^k + \delta_x^2 e_j^{\bar{k}} + q(|U_j^k|^2 U_j^{\bar{k}} - |u_j^k|^2 u_j^{\bar{k}}) = P_j^k,$$
$$1 \leqslant j \leqslant m-1, \quad 1 \leqslant k \leqslant n-1, \tag{5.77}$$

$$e_j^0 = 0, \quad 1 \leqslant j \leqslant m-1, \tag{5.78}$$

$$e_0^k = 0, \quad e_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{5.79}$$

由 (5.79) 可得

$$e_0^{\frac{1}{2}} = 0, \quad e_m^{\frac{1}{2}} = 0,$$
$$e_0^{\bar{k}} = 0, \quad e_m^{\bar{k}} = 0, \quad 1 \leqslant k \leqslant n-1.$$

(I) 用 $h\bar{e}_j^{\frac{1}{2}}$ 乘以 (5.76) 的两边, 并对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}(\delta_t e_j^{\frac{1}{2}})\bar{e}_j^{\frac{1}{2}} + h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\frac{1}{2}})\bar{e}_j^{\frac{1}{2}} + qh\sum_{j=1}^{m-1}|\hat{u}_j|^2|e_j^{\frac{1}{2}}|^2 = h\sum_{j=1}^{m-1}P_j^0\bar{e}_j^{\frac{1}{2}}. \tag{5.80}$$

注意到

$$h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\frac{1}{2}})\bar{e}_j^{\frac{1}{2}} = -h\sum_{j=0}^{m-1}|\delta_x e_{j+\frac{1}{2}}^{\frac{1}{2}}|^2,$$

在 (5.80) 两边取虚部, 可得

$$\frac{1}{2\tau}(\|e^1\|^2-\|e^0\|^2)=\mathrm{Im}\left\{h\sum_{j=1}^{m-1}P_j^0\bar{e}_j^{\frac{1}{2}}\right\}\leqslant\|P^0\|\cdot\|e^{\frac{1}{2}}\|.$$

注意到 $e^0=0$, 有

$$\frac{1}{2\tau}\|e^1\|^2\leqslant\|P^0\|\cdot\frac{1}{2}\|e^1\|.$$

于是

$$\|e^1\|\leqslant\tau\|P^0\|\leqslant\tau c_8\sqrt{L}(\tau^2+h^2)\leqslant c_8\sqrt{L}(\tau^2+h^2). \tag{5.81}$$

(II) 用 $h\bar{e}_j^{\bar{k}}$ 乘以 (5.77) 的两边, 并对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}(\Delta_t e_j^k)\bar{e}_j^{\bar{k}}+h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\bar{k}})\bar{e}_j^{\bar{k}}+qh\sum_{j=1}^{m-1}(|U_j^k|^2U_j^{\bar{k}}-|u_j^k|^2u_j^{\bar{k}})\bar{e}_j^{\bar{k}}=h\sum_{j=1}^{m-1}P_j^k\bar{e}_j^{\bar{k}}. \tag{5.82}$$

注意到

$$h\sum_{j=1}^{m-1}(\delta_x^2e_j^{\bar{k}})\bar{e}_j^{\bar{k}}=-h\sum_{j=0}^{m-1}|\delta_x e_{j+\frac{1}{2}}^{\bar{k}}|^2,$$

$$\begin{aligned}(|U_j^k|^2U_j^{\bar{k}}-|u_j^k|^2u_j^{\bar{k}})\bar{e}_j^{\bar{k}}&=[|U_j^k|^2e_j^{\bar{k}}+(|U_j^k|^2-|u_j^k|^2)u_j^{\bar{k}}]\bar{e}_j^{\bar{k}}\\&=|U_j^k|^2|e_j^{\bar{k}}|^2+(e_j^k\bar{U}_j^k+u_j^k\bar{e}_j^k)u_j^{\bar{k}}\bar{e}_j^{\bar{k}},\end{aligned}$$

在 (5.82) 两边取虚部, 可得

$$\begin{aligned}&\frac{1}{4\tau}(\|e^{k+1}\|^2-\|e^{k-1}\|^2)\\&=-q\mathrm{Im}\left\{h\sum_{j=1}^{m-1}(e_j^k\bar{U}_j^k+u_j^k\bar{e}_j^k)u_j^{\bar{k}}\bar{e}_j^{\bar{k}}\right\}+\mathrm{Im}\left\{h\sum_{j=1}^{m-1}P_j^k\bar{e}_j^{\bar{k}}\right\}\\&\leqslant|q|(c_0+c_{10})c_{10}\|e^k\|\cdot\|e^{\bar{k}}\|+\|P^k\|\cdot\|e^{\bar{k}}\|\\&\leqslant\Big(|q|(c_0+c_{10})c_{10}\|e^k\|+\|P^k\|\Big)\frac{\|e^{k+1}\|+\|e^{k-1}\|}{2},\quad 1\leqslant k\leqslant n-1.\end{aligned} \tag{5.83}$$

由上式可得

$$\begin{aligned}&\frac{1}{2\tau}(\|e^{k+1}\|-\|e^{k-1}\|)\\&\leqslant|q|(c_0+c_{10})c_{10}\|e^k\|+\|P^k\|\\&\leqslant|q|(c_0+c_{10})c_{10}\|e^k\|+\sqrt{L}c_9(\tau^2+h^2),\quad 1\leqslant k\leqslant n-1.\end{aligned} \tag{5.84}$$

或

$$\|e^{k+1}\| \leqslant \|e^{k-1}\| + 2|q|(c_0+c_{10})c_{10}\tau\|e^k\| + 2\sqrt{L}c_9\tau(\tau^2+h^2), \quad 1\leqslant k\leqslant n-1.$$

由上式可得

$$\max\{\|e^{k+1}\|,\|e^k\|\} \leqslant \Big[1+2|q|(c_0+c_{10})c_{10}\tau\Big]\max\{\|e^k\|,\|e^{k-1}\|\} + 2\sqrt{L}c_9\tau(\tau^2+h^2), \quad 1\leqslant k\leqslant n-1.$$

当 $q=0$ 时,

$$\begin{aligned}\max\{\|e^{k+1}\|,\|e^k\|\} &\leqslant \max\{\|e^1\|,\|e^0\|\} + 2\sqrt{L}c_9k\tau(\tau^2+h^2)\\ &\leqslant (c_8+2c_9T)\sqrt{L}(\tau^2+h^2), \quad 1\leqslant k\leqslant n-1.\end{aligned}$$

当 $q\neq 0$ 时, 由 Gronwall 不等式, 得到

$$\begin{aligned}&\max\{\|e^{k+1}\|,\|e^k\|\}\\ \leqslant\ & e^{2|q|(c_0+c_{10})c_{10}T}\left[\max\{\|e^1\|,\|e^0\|\} + \frac{\sqrt{L}c_9}{|q|(c_0+c_{10})c_{10}}(\tau^2+h^2)\right]\\ \leqslant\ & e^{2|q|(c_0+c_{10})c_{10}T}\left[c_8+\frac{c_9}{|q|(c_0+c_{10})c_{10}}\right]\sqrt{L}(\tau^2+h^2), \quad 0\leqslant k\leqslant n-1.\end{aligned}$$ □

定理 5.10 设 $\{U_j^k\,|\,0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ 为问题 (5.1)–(5.3) 的解, $\{u_j^k\,|\,0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ 为差分格式 (5.57)–(5.60) 的解. 记

$$e_j^k = U_j^k - u_j^k, \quad 0\leqslant j\leqslant m,\ 0\leqslant k\leqslant n,$$

则存在常数 c_{12} 使得

$$\left|e^k\right|_1 \leqslant c_{12}(\tau^2+h^2), \quad 0\leqslant k\leqslant n. \tag{5.85}$$

证明 (I) 用 $-h\delta_t\bar{e}_j^{\frac12}$ 乘以 (5.76) 的两边, 对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}|\delta_t e_j^{\frac12}|^2 - h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\frac12})(\delta_t\bar{e}_j^{\frac12}) - qh\sum_{j=1}^{m-1}|\hat{u}_j|^2 e_j^{\frac12}\delta_t\bar{e}_j^{\frac12} = -h\sum_{j=1}^{m-1}P_j^0\delta_t\bar{e}_j^{\frac12}. \tag{5.86}$$

注意到

$$-h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\frac12})\delta_t\bar{e}_j^{\frac12} = h\sum_{j=0}^{m-1}(\delta_x e_{j+\frac12}^{\frac12})(\delta_t\delta_x\bar{e}_{j+\frac12}^{\frac12}),$$

取 (5.86) 的实部, 得

$$\frac{1}{2\tau}(|e^1|_1^2-|e^0|_1^2)-qh\sum_{j=0}^{m-1}|\hat{u}_j|^2\frac{|e_j^1|^2-|e_j^0|^2}{2\tau}=\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}P_j^0\delta_t\bar{e}_j^{\frac{1}{2}}\right\}$$
$$\leqslant\|P^0\|\cdot\|\delta_t e^{\frac{1}{2}}\|.$$

再注意到

$$e_j^0=0,\quad 0\leqslant j\leqslant m,$$

有

$$\frac{1}{2\tau}|e^1|_1^2-q\cdot\frac{1}{2\tau}h\sum_{j=0}^{m-1}|\hat{u}_j|^2|e_j^1|^2=\frac{1}{\tau}\|P^0\|\cdot\|e^1\|.$$

两边同乘以 2τ, 得

$$|e^1|_1^2\leqslant|q|h\sum_{j=0}^{m-1}|\hat{u}_j|^2|e_j^1|^2+2\|P^0\|\cdot\|e^1\|.\tag{5.87}$$

记

$$c_{13}=\max_{0\leqslant x\leqslant L}|u_t(x,0)|,$$

则

$$|\hat{u}_j|\leqslant c_0+\frac{\tau}{2}c_{13}=c_0+c_{13},\quad 1\leqslant j\leqslant m-1.$$

由 (5.87) 和 (5.51), (5.81) 可得

$$\begin{aligned}|e^1|_1^2&\leqslant|q|(c_0+c_{13})^2\|e^1\|^2+2\|P^0\|\cdot\|e^1\|\\&\leqslant|q|(c_0+c_{13})^2[c_8\sqrt{L}(\tau^2+h^2)]^2+2\sqrt{L}c_8(\tau^2+h^2)c_8\sqrt{L}(\tau^2+h^2)\\&=[|q|(c_0+c_{13})^2+2][c_8\sqrt{L}(\tau^2+h^2)]^2,\end{aligned}$$

即

$$|e^1|_1\leqslant\sqrt{|q|(c_0+c_{13})^2+2}\,c_8\sqrt{L}(\tau^2+h^2).\tag{5.88}$$

(II) 用 $-\Delta_t\bar{e}_j^k$ 乘以 (5.77), 并对 j 从 1 到 $m-1$ 求和, 得

$$\begin{aligned}&-\mathrm{i}h\sum_{j=1}^{m-1}|\Delta_t e_j^k|^2-h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\bar{k}})(\Delta_t\bar{e}_j^k)\\&=qh\sum_{j=1}^{m-1}(|U_j^k|^2U_j^{\bar{k}}-|u_j^k|^2u_j^{\bar{k}})\Delta_t\bar{e}_j^k-h\sum_{j=1}^{m-1}P_j^k\Delta_t\bar{e}_j^k,\quad 1\leqslant k\leqslant n-1.\end{aligned}\tag{5.89}$$

对于 (5.89) 左端的第 2 项, 有

$$-h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\bar{k}})\Delta_t \bar{e}_j^k = h\sum_{j=0}^{m-1}(\delta_x e_{j+\frac{1}{2}}^{\bar{k}})\Delta_t \delta_x \bar{e}_{j+\frac{1}{2}}^k,$$

取实部, 有

$$\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\bar{k}})\Delta_t \bar{e}_j^k\right\} = \frac{1}{4\tau}(|e^{k+1}|_1^2 - |e^{k-1}|_1^2). \tag{5.90}$$

现在来分析 (5.89) 右端第 1 项.

由 (5.77) 可得

$$\Delta_t e_j^k = \mathrm{i}\delta_x^2 e_j^{\bar{k}} + \mathrm{i}q\Big(|U_j^k|^2 U_j^{\bar{k}} - |u_j^k|^2 u_j^{\bar{k}}\Big) - \mathrm{i}P_j^k.$$

于是

$$\begin{aligned}
&\mathrm{Re}\left\{qh\sum_{j=1}^{m-1}(|U_j^k|^2 U_j^{\bar{k}} - |u_j^k|^2 u_j^{\bar{k}})\Delta_t \bar{e}_j^k\right\}\\
=&\mathrm{Re}\left\{qh\sum_{j=1}^{m-1}(|U_j^k|^2 \bar{U}_j^{\bar{k}} - |u_j^k|^2 \bar{u}_j^{\bar{k}})\Delta_t e_j^k\right\}\\
=&\mathrm{Re}\left\{qh\sum_{j=1}^{m-1}\left(|U_j^k|^2 \bar{U}_j^{\bar{k}} - |u_j^k|^2 \bar{u}_j^{\bar{k}}\right)\left[\mathrm{i}\delta_x^2 e_j^{\bar{k}} + \mathrm{i}q(|U_j^k|^2 U_j^{\bar{k}} - |u_j^k|^2 u_j^{\bar{k}}) - \mathrm{i}P_j^k\right]\right\}.
\end{aligned}$$

存在常数 c_{14} 使得

$$\begin{aligned}
&\mathrm{Re}\left\{qh\sum_{j=1}^{m-1}\left(|U_j^k|^2 U_j^{\bar{k}} - |u_j^k|^2 u_j^{\bar{k}}\right)\Delta_t \bar{e}_j^k\right\}\\
=&\mathrm{Re}\left\{qh\sum_{j=1}^{m-1}\left(|U_j^k|^2 \bar{U}_j^{\bar{k}} - |u_j^k|^2 \bar{u}_j^{\bar{k}}\right)\left(\mathrm{i}\delta_x^2 e_j^{\bar{k}} - \mathrm{i}P_j^k\right)\right\}\\
\leqslant& c_{14}(\|e^{k-1}\|^2 + \|e^k\|^2 + \|e^{k+1}\|^2 + |e^{k-1}|_1^2\\
&+|e^k|_1^2 + |e^{k+1}|_1^2 + \|P^k\|^2).
\end{aligned} \tag{5.91}$$

在 (5.89) 两边取实部, 并利用 (5.90) 和 (5.91), 得

$$\begin{aligned}
&\frac{1}{4\tau}(|e^{k+1}|_1^2 - |e^{k-1}|_1^2)\\
\leqslant& c_{14}(\|e^{k-1}\|^2 + \|e^k\|^2 + \|e^{k+1}\|^2 + |e^{k-1}|_1^2 + |e^k|_1^2 + |e^{k+1}|_1^2 + \|P^k\|^2)\\
&+\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}P_j^k\Delta_t \bar{e}_j^k\right\}, \quad 1\leqslant k\leqslant n-1.
\end{aligned}$$

将上式中 k 换成 l, 并对 l 从 1 到 k 求和, 得

$$\begin{aligned}&\frac{1}{2\tau}(|e^{k+1}|_1^2+|e^k|_1^2-|e^1|_1^2-|e^0|_1^2)\\ \leqslant&c_{14}\sum_{l=1}^{k}\left(\|e^{l-1}\|^2+\|e^l\|^2+\|e^{l+1}\|^2+|e^{l-1}|_1^2+|e^l|_1^2+|e^{l+1}|_1^2+\|P^k\|^2\right)\\ &+\left|h\sum_{j=1}^{m-1}\sum_{l=1}^{k}P_j^l\Delta_t\bar{e}_j^l\right|,\end{aligned}\tag{5.92}$$

注意到

$$\begin{aligned}&\sum_{l=1}^{k}P_j^l\Delta_t\bar{e}_j^l\\ =&\frac{1}{2\tau}\sum_{l=1}^{k}P_j^l(\bar{e}_j^{l+1}-\bar{e}_j^{l-1})\\ =&\frac{1}{2\tau}\left(\sum_{l=2}^{k+1}P_j^{l-1}\bar{e}_j^l-\sum_{l=0}^{k-1}P_j^{l+1}\bar{e}_j^l\right)\\ =&\frac{1}{2\tau}\left[P_j^k\bar{e}_j^{k+1}+P_j^{k-1}e_j^k-\sum_{l=2}^{k-1}(P_j^{l+1}-P_j^{l-1})\bar{e}_j^l-P_j^2\bar{e}_j^1\right],\end{aligned}$$

可得

$$\begin{aligned}&\left|h\sum_{j=1}^{m-1}\sum_{l=1}^{k-1}P_j^l\Delta_t\bar{e}_j^l\right|\\ =&\left|\frac{1}{2\tau}h\sum_{j=1}^{m-1}P_j^k\bar{e}_j^{k+1}+\frac{1}{2\tau}h\sum_{j=1}^{m-1}P_j^{k-1}\bar{e}_j^k\right.\\ &\left.-\sum_{l=2}^{k-1}h\sum_{j=1}^{m-1}\frac{P_j^{l+1}-P_j^{l-1}}{2\tau}\bar{e}_j^l-\frac{1}{2\tau}\cdot h\sum_{j=1}^{m-1}P_j^2\bar{e}_j^1\right|\\ \leqslant&\frac{1}{2\tau}\|P^k\|\cdot\|e^{k+1}\|+\frac{1}{2\tau}\|P^{k-1}\|\cdot\|e^k\|\\ &+\sum_{l=2}^{k-1}\|\Delta_tP^l\|\cdot\|e^l\|+\frac{1}{2\tau}\|P^2\|\cdot\|e^1\|.\end{aligned}\tag{5.93}$$

将 (5.93) 代入 (5.92) 后, 两边同乘以 2τ, 得

$$\begin{aligned}&|e^{k+1}|_1^2+|e^k|_1^2-|e^1|_1^2-|e^0|_1^2\\ \leqslant&2c_{14}\tau\sum_{l=1}^{k}(\|e^{l-1}\|^2+\|e^l\|^2+\|e^{l+1}\|^2+|e^{l-1}|_1^2+|e^l|_1^2+|e^{l+1}|_1^2+\|P^l\|^2)\end{aligned}$$

$$+\|P^k\|\cdot\|e^{k+1}\|+\|P^{k-1}\|\cdot\|e^k\|+2\tau\sum_{l=2}^{k-1}\|\Delta_t P^l\|\cdot\|e^l\|$$
$$+\|P^2\|\cdot\|e^1\|,\quad 1\leqslant k\leqslant n-1.$$

注意到

$$\|P^k\|\cdot\|e^{k+1}\|\leqslant\frac{3}{L^2}\|e^{k+1}\|^2+\frac{L^2}{12}\|P^k\|^2\leqslant\frac{1}{2}|e^{k+1}|_1^2+\frac{L^2}{12}\|P^k\|^2,$$
$$\|P^{k-1}\|\cdot\|e^k\|\leqslant\frac{3}{L^2}\|e^k\|^2+\frac{L^2}{12}\|P^{k-1}\|^2\leqslant\frac{1}{2}|e^k|_1^2+\frac{L^2}{12}\|P^{k-1}\|^2,$$
$$\|P^2\|\cdot\|e^1\|\leqslant\frac{1}{2}\|P^2\|^2+\frac{1}{2}\|e^1\|^2,$$

以及 (5.75), (5.88), (5.53), (5.54), 知存在常数 c_{14} 使得

$$|e^{k+1}|_1^2+|e^k|_1^2$$
$$\leqslant c_{14}\tau\sum_{l=1}^{k}(|e^{l+1}|_1^2+2|e^l|_1^2+|e^{l-1}|_1^2)+c_{14}(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n-1.$$

即

$$(1-c_{14}\tau)(|e^{k+1}|_1^2+|e^k|_1^2)$$
$$\leqslant 2c_{14}\tau\sum_{l=0}^{k-1}(|e^{l+1}|_1^2+|e^l|_1^2)+c_{14}(\tau^2+h^2),\quad 1\leqslant k\leqslant n-1,$$

当 $c_{14}\tau\leqslant\dfrac{1}{3}$ 时

$$|e^{k+1}|_1^2+|e^k|_1^2$$
$$\leqslant 3c_{14}\tau\sum_{l=0}^{k-1}(|e^{l+1}|_1^2+|e^l|_1^2)+\frac{3}{2}c_{14}(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n-1.$$

由 Gronwall 不等式并注意到 (5.88), 得

$$|e^{k+1}|_1^2+|e^k|_1^2$$
$$\leqslant e^{3c_{14}\tau}\left[|e^1|_1^2+|e^0|_1^2+\frac{3}{2}c_{14}(\tau^2+h^2)^2\right]$$
$$\leqslant e^{3c_{14}T}\left[(|q|(c_0+c_{13})^2+2)c_8^2L+\frac{3}{2}c_{14}\right](\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n-1.$$ □

5.4 空间四阶三层线性化差分格式

5.4.1 几个数值微分公式

引理 5.1 设 $f(x)\in C^6[x_{j-2},x_{j+2}]$, 则

$$\begin{aligned}f''(x_j)=&\frac{4}{3}\cdot\frac{f(x_{j+1})-2f(x_j)+f(x_{j-1})}{h^2}-\frac{1}{3}\cdot\frac{f(x_{j+2})-2f(x_j)+f(x_{j-2})}{(2h)^2}\\&-\frac{h^4}{90}\int_0^1[f^{(6)}(x_j+sh)+f^{(6)}(x_j-sh)](1-s)^5\mathrm{d}s\\&+\frac{2h^4}{45}\int_0^1[f^{(6)}(x_j+2sh)+f^{(6)}(x_j-2sh)](1-s)^5\mathrm{d}s.\end{aligned}$$

证明 由带积分余项的 Taylor 展开式

$$f(x_j+h)=\sum_{l=0}^{k}\frac{h^l}{l!}f^{(l)}(x_j)+\frac{h^{k+1}}{k!}\int_0^1 f^{(k+1)}(x_j+sh)(1-s)^k\mathrm{d}s, \tag{5.94}$$

得到

$$\begin{aligned}f(x_j+h)=&f(x_j)+hf'(x_j)+\frac{h^2}{2}f''(x_j)+\frac{h^3}{6}f'''(x_j)+\frac{h^4}{24}f^{(4)}(x_j)\\&+\frac{h^5}{120}f^{(5)}(x_j)+\frac{h^6}{120}\int_0^1 f^{(6)}(x_j+sh)(1-s)^5\mathrm{d}s,\\f(x_j-h)=&f(x_j)-hf'(x_j)+\frac{h^2}{2}f''(x_j)-\frac{h^3}{6}f'''(x_j)+\frac{h^4}{24}f^{(4)}(x_j)\\&-\frac{h^5}{120}f^{(5)}(x_j)+\frac{h^6}{120}\int_0^1 f^{(6)}(x_j-sh)(1-s)^5\mathrm{d}s.\end{aligned}$$

将以上两式相加, 可得

$$\begin{aligned}f''(x_j)=&\frac{f(x_{j+1})-2f(x_j)+f(x_{j-1})}{h^2}-\frac{h^2}{12}f^{(4)}(x_j)\\&-\frac{h^4}{120}\int_0^1[f^{(6)}(x_j+sh)+f^{(6)}(x_j-sh)](1-s)^5\mathrm{d}s.\end{aligned} \tag{5.95}$$

同理可得

$$\begin{aligned}f''(x_j)=&\frac{f(x_{j+2})-2f(x_j)+f(x_{j-2})}{(2h)^2}-\frac{(2h)^2}{12}f^{(4)}(x_j)\\&-\frac{(2h)^4}{120}\int_0^1[f^{(6)}(x_j+2sh)+f^{(6)}(x_j-2sh)](1-s)^5\mathrm{d}s.\end{aligned} \tag{5.96}$$

将 (5.95) 乘以 $\frac{4}{3}$, 将 (5.96) 乘以 $\frac{1}{3}$, 然后将所得等式相减, 即得到所要结果. □

引理 5.2 设 $f(x)\in C^6[x_0,x_3]$, 则有

$$\begin{aligned}f''(x_1)=&\frac{7}{6}\times\frac{f(x_2)-f(x_1)+f(x_0)}{h^2}-\frac{1}{12}\times\frac{f(x_3)-2f(x_2)+f(x_1)}{h^2}\\&-\frac{1}{12}f''(x_0)-\frac{h^2}{144}f^{(4)}(x_0)-\frac{1}{12}(14\widetilde{R}_1-\widetilde{R}_2-\widetilde{R}_3-\widetilde{R}_4),\end{aligned}$$

其中

$$\begin{aligned}\widetilde{R}_1&=\frac{h^4}{120}\int_0^1[f^{(6)}(x_1+sh)+f^{(6)}(x_1-sh)](1-s)^5\mathrm{d}s,\\\widetilde{R}_2&=\frac{h^4}{120}\int_0^1[f^{(6)}(x_2+sh)+f^{(6)}(x_2-sh)](1-s)^5\mathrm{d}s,\\\widetilde{R}_3&=\frac{h^4}{6}\int_0^1[f^{(6)}(x_1+sh)+f^{(6)}(x_1-sh)](1-s)^3\mathrm{d}s,\\\widetilde{R}_4&=\frac{h^4}{12}\int_0^1[f^{(6)}(x_1+sh)+f^{(6)}(x_1-sh)](1-s)\mathrm{d}s.\end{aligned}$$

证明 由 (5.94) 及 (5.95) 有

$$f''(x_1)=\frac{f(x_2)-2f(x_1)+f(x_0)}{h^2}-\frac{h^2}{12}f^{(4)}(x_1)-\widetilde{R}_1,\tag{5.97}$$

$$f''(x_2)=\frac{f(x_3)-2f(x_2)+f(x_1)}{h^2}-\frac{h^2}{12}f^{(4)}(x_2)-\widetilde{R}_2,\tag{5.98}$$

$$f^{(4)}(x_1)=\frac{f''(x_2)-2f''(x_1)+f''(x_0)}{h^2}-\frac{1}{h^2}\widetilde{R}_3,\tag{5.99}$$

$$\begin{aligned}&\frac{f^{(4)}(x_2)-2f^{(4)}(x_1)+f^{(4)}(x_0)}{h^2}\\=&\int_0^1[f^{(6)}(x_1+sh)+f^{(6)}(x_1-sh)](1-s)\mathrm{d}s=\frac{12}{h^4}\widetilde{R}_4.\end{aligned}\tag{5.100}$$

将 (5.97)–(5.98) 代入 (5.99), 得

$$\begin{aligned}f^{(4)}(x_1)=&\frac{1}{h^2}\left[\frac{f(x_3)-2f(x_2)+f(x_1)}{h^2}-\frac{h^2}{12}f^{(4)}(x_2)-\widetilde{R}_2\right.\\&\left.-2\left(\frac{f(x_2)-2f(x_1)+f(x_0)}{h^2}-\frac{h^2}{12}f^{(4)}(x_1)-\widetilde{R}_1\right)+f''(x_0)\right]-\frac{1}{h^2}\widetilde{R}_3\\=&\frac{1}{h^2}\left[\frac{f(x_3)-2f(x_2)+f(x_1)}{h^2}-2\cdot\frac{f(x_2)-2f(x_1)+f(x_0)}{h^2}\right]\\&+\frac{1}{h^2}f''(x_0)+\frac{1}{h^2}(2\widetilde{R}_1-\widetilde{R}_2-\widetilde{R}_3)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{12}h^2\frac{f^{(4)}(x_2)-2f^{(4)}(x_1)+f^{(4)}(x_0)}{h^2}+\frac{1}{12}f^{(4)}(x_0)\\
=&\frac{1}{h^2}\left[\frac{f(x_3)-2f(x_2)+f(x_1)}{h^2}-2\frac{f(x_2)-2f(x_1)+f(x_0)}{h^2}\right]\\
&+\frac{1}{h^2}f''(x_0)+\frac{1}{12}f^{(4)}(x_0)+\frac{1}{h^2}(2\widetilde{R}_1-\widetilde{R}_2-\widetilde{R}_3-\widetilde{R}_4). \qquad (5.101)
\end{aligned}
$$

再将上式代入 (5.97), 得到

$$
\begin{aligned}
f''(x_1)=&\frac{f(x_2)-2f(x_1)+f(x_0)}{h^2}\\
&-\frac{1}{12}\left[\frac{f(x_3)-2f(x_2)+f(x_1)}{h^2}-2\frac{f(x_2)-2f(x_1)+f(x_0)}{h^2}\right]\\
&-\frac{1}{12}f''(x_0)-\frac{h^2}{144}f^{(4)}(x_0)-\frac{1}{12}(2\widetilde{R}_1-\widetilde{R}_2-\widetilde{R}_3-\widetilde{R}_4)-\widetilde{R}_1\\
=&\frac{7}{6}\times\frac{f(x_2)-2f(x_1)+f(x_0)}{h^2}-\frac{1}{12}\times\frac{f(x_3)-2f(x_2)+f(x_1)}{h^2}\\
&-\frac{1}{12}f''(x_0)-\frac{h^2}{144}f^{(4)}(x_0)-\frac{1}{12}(14\widetilde{R}_1-\widetilde{R}_2-\widetilde{R}_3-\widetilde{R}_4).
\end{aligned}
$$
□

引理 5.3　设 $f(x)\in C^6[x_{m-3},x_m]$, 则

$$
\begin{aligned}
f''(x_{m-1})=&\frac{7}{6}\times\frac{f(x_m)-2f(x_{m-1})+f(x_{m-2})}{h^2}\\
&-\frac{1}{12}\times\frac{f(x_{m-1})-2f(x_{m-2})+f(x_{m-3})}{h^2}\\
&-\frac{1}{12}f''(x_m)-\frac{h^2}{144}f^{(4)}(x_m)-\frac{1}{12}(14\hat{R}_1-\hat{R}_2-\hat{R}_3-\hat{R}_4).
\end{aligned}
$$

证明　类似于引理 5.2. □

5.4.2　差分格式的建立

在方程 (5.1) 中令 $x=0$ 及 $x=L$, 并利用边界条件 (5.3) 可得

$$
u_{xx}(x_0,t)=0,\quad u_{xx}(x_m,t)=0,\quad 0\leqslant t\leqslant T. \qquad (5.102)
$$

将方程 (5.1) 关于 x 求二阶偏导数得

$$
\mathrm{i}u_{xt}+u_{xxx}+q(\bar{u}_xu^2+2\bar{u}uu_x)=0,
$$

$$
\mathrm{i}u_{xxt}+u_{xxxx}+q(\bar{u}_{xx}u^2+2\bar{u}_xuu_x+2\bar{u}_xuu_x+2\bar{u}u_x^2+2\bar{u}uu_{xx})=0.
$$

在上式中令 $x=0$ 及 $x=L$, 利用 (5.3) 和 (5.102), 得

$$
u_{xxxx}(x_0,t)=0,\quad u_{xxxx}(x_m,t)=0,\quad 0\leqslant t\leqslant T. \qquad (5.103)
$$

定义

$$\Delta_x u_j^k = \frac{1}{2h}(u_{j+1}^k - u_{j-1}^k), \quad \Delta_x^2 u_j^k = \frac{1}{(2h)^2}(u_{j+2}^k - 2u_j^k + u_{j-2}^k).$$

记

$$\hat{u}_j = u(x_j, 0) + \frac{\tau}{2}u_t(x_j, 0), \quad 1 \leqslant j \leqslant m-1.$$

在点 $(x_j, t_{\frac{1}{2}})$ 处考虑方程 (5.1), 有

$$\mathrm{i}u_t(x_j, t_{\frac{1}{2}}) + u_{xx}(x_j, t_{\frac{1}{2}}) + q|u(x_j, t_{\frac{1}{2}})|^2 u(x_j, t_{\frac{1}{2}}) = 0, \quad 1 \leqslant j \leqslant m-1.$$

在点 (x_j, t_k) 处考虑方程 (5.1), 有

$$\mathrm{i}u_t(x_j, t_k) + u_{xx}(x_j, t_k) + q|u(x_j, t_k)|^2 u(x_j, t_k) = 0, \quad 1 \leqslant j \leqslant m-1, \quad 1 \leqslant k \leqslant n-1.$$

对于 $2 \leqslant j \leqslant m-2$ 应用引理 5.1, 对于 $j = 1$ 应用引理 5.2, 对于 $j = m-1$ 应用引理 5.3, 可得

$$\mathrm{i}\delta_t U_1^{\frac{1}{2}} + \frac{7}{6}\delta_x^2 U_1^{\frac{1}{2}} - \frac{1}{12}\delta_x^2 U_2^{\frac{1}{2}} + q|\hat{u}_1|^2 U_1^{\frac{1}{2}} = R_1^0, \tag{5.104}$$

$$\mathrm{i}\delta_t U_j^{\frac{1}{2}} + \frac{4}{3}\delta_x^2 U_j^{\frac{1}{2}} - \frac{1}{3}\Delta_x^2 U_j^{\frac{1}{2}} + q|\hat{u}_j|^2 U_j^{\frac{1}{2}} = R_j^0, \quad 2 \leqslant j \leqslant m-2, \tag{5.105}$$

$$\mathrm{i}\delta_t U_{m-1}^{\frac{1}{2}} + \frac{7}{6}\delta_x^2 U_{m-1}^{\frac{1}{2}} - \frac{1}{12}\delta_x^2 U_{m-2}^{\frac{1}{2}} + q|\hat{u}_{m-1}|^2 U_{m-1}^{\frac{1}{2}} = R_{m-1}^0 \tag{5.106}$$

及

$$\mathrm{i}\Delta_t U_1^k + \frac{7}{6}\delta_x^2 U_1^{\bar{k}} - \frac{1}{12}\delta_x^2 U_2^{\bar{k}} + q|U^k|^2 U_1^{\bar{k}} = R_1^k, \quad 1 \leqslant k \leqslant n-1, \tag{5.107}$$

$$\mathrm{i}\Delta_t U_j^k + \frac{4}{3}\delta_x^2 U_j^{\bar{k}} - \frac{1}{3}\Delta_x^2 U_j^{\bar{k}} + q|U_j^k|^2 U_j^{\bar{k}} = R_j^k,$$
$$2 \leqslant j \leqslant m-2, \ 1 \leqslant k \leqslant n-1, \tag{5.108}$$

$$\mathrm{i}\Delta_t U_{m-1}^k + \frac{7}{6}\delta_x^2 U_{m-1}^{\bar{k}} - \frac{1}{12}\delta_x^2 U_{m-2}^{\bar{k}} + q|U_{m-1}^k|^2 U_{m-1}^{\bar{k}} = R_{m-1}^k,$$
$$1 \leqslant k \leqslant n-1. \tag{5.109}$$

存在常数 c_{15} 使得

$$|R_j^k| \leqslant c_{15}(\tau^2 + h^4), \quad 1 \leqslant j \leqslant m-1, \ 0 \leqslant k \leqslant n-1, \tag{5.110}$$

$$|\Delta_t R_j^k| \leqslant c_{15}(\tau^2 + h^4), \quad 1 \leqslant j \leqslant m-1, \ 2 \leqslant k \leqslant n-2. \tag{5.111}$$

注意到初边值条件,

$$U_j^0 = \varphi(x_j), \quad 1 \leqslant j \leqslant m-1 \tag{5.112}$$

$$U_0^k = 0, \quad U_m^k = 0, \quad 0 \leqslant k \leqslant n, \tag{5.113}$$

在 (5.104)–(5.109) 中略去小量项, 对问题 (5.1)–(5.3) 建立如下线性化差分格式

$$\mathrm{i}\delta_t u_1^{\frac{1}{2}} + \frac{7}{6}\delta_x^2 u_1^{\frac{1}{2}} - \frac{1}{12}\delta_x^2 u_2^{\frac{1}{2}} + q|\hat{u}_1|^2 u_1^{\frac{1}{2}} = 0, \tag{5.114}$$

$$\mathrm{i}\delta_t u_j^{\frac{1}{2}} + \frac{4}{3}\delta_x^2 u_j^{\frac{1}{2}} - \frac{1}{3}\Delta_x^2 u_j^{\frac{1}{2}} + q|\hat{u}_j|^2 u_j^{\frac{1}{2}} = 0, \quad 2 \leqslant j \leqslant m-2, \tag{5.115}$$

$$\mathrm{i}\delta_t u_{m-1}^{\frac{1}{2}} + \frac{7}{6}\delta_x^2 u_{m-1}^{\frac{1}{2}} - \frac{1}{12}\delta_x^2 u_{m-2}^{\frac{1}{2}} + q|\hat{u}_{m-1}|^2 u_{m-1}^{\frac{1}{2}} = 0, \tag{5.116}$$

$$\mathrm{i}\Delta_t u_1^k + \frac{7}{6}\delta_x^2 u_1^{\bar{k}} - \frac{1}{12}\delta_x^2 u_2^{\bar{k}} + q|u_1^k|^2 u_1^{\bar{k}} = 0, \quad 1 \leqslant k \leqslant n-1, \tag{5.117}$$

$$\mathrm{i}\Delta_t u_j^k + \frac{4}{3}\delta_x^2 u_j^{\bar{k}} - \frac{1}{3}\Delta_x^2 u_j^{\bar{k}} + q|u_j^k|^2 u_j^{\bar{k}} = 0,$$
$$2 \leqslant j \leqslant m-2,\ 1 \leqslant k \leqslant n-1, \tag{5.118}$$

$$\mathrm{i}\Delta_t u_{m-1}^k + \frac{7}{6}\delta_x^2 u_{m-1}^{\bar{k}} - \frac{1}{12}\delta_x^2 u_{m-2}^{\bar{k}} + q|u_{m-1}^k|^2 u_{m-1}^{\bar{k}} = 0,$$
$$1 \leqslant k \leqslant n-1, \tag{5.119}$$

$$u_j^0 = \varphi(x_j), \quad 1 \leqslant j \leqslant m-1, \tag{5.120}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{5.121}$$

5.4.3 差分格式解的存在性和唯一性

引理 5.4 设 $w \in \mathring{\mathcal{U}}_h$,

$$\mathcal{A}(w) \equiv \left(\frac{7}{6}\delta_x^2 w_1 - \frac{1}{12}\delta_x^2 w_2\right)\bar{w}_1 + \sum_{j=2}^{m-2}\left(\frac{4}{3}\delta_x^2 w_j - \frac{1}{3}\Delta_x^2 w_j\right)\bar{w}_j$$
$$+ \left(\frac{7}{6}\delta_x^2 w_{m-1} - \frac{1}{12}\delta_x^2 w_{m-2}\right)\bar{w}_{m-1},$$

则 $\mathcal{A}(w)$ 为实的, 且

$$h\mathcal{A}(w) = -|w|_1^2 - \frac{h^2}{12}|w|_2^2 \geqslant -\frac{4}{3}|w|_1^2,$$

其中

$$|w|_1^2 = h\sum_{j=0}^{m-1}|\delta_x w_{j+\frac{1}{2}}|^2, \quad |w|_2^2 = h\sum_{j=1}^{m-1}|\delta_x^2 w_j|^2.$$

证明

$$\Delta_x^2 w_j = \frac{1}{4}(\delta_x^2 w_{j-1} + 2\delta_x^2 w_j + \delta_x^2 w_{j+1}),$$

$$\frac{4}{3}\delta_x^2 w_j - \frac{1}{3}\Delta_x^2 w_j = -\frac{1}{12}\delta_x^2 w_{j-1} + \frac{7}{6}\delta_x^2 w_j - \frac{1}{12}\delta_x^2 w_{j+1}.$$

$$\begin{aligned}\mathcal{A}(w)=&\left(\frac{7}{6}\delta_x^2w_1-\frac{1}{12}\delta_x^2w_2\right)\bar{w}_1+\sum_{j=2}^{m-2}\left(-\frac{1}{12}\delta_x^2w_{j-1}+\frac{7}{6}\delta_x^2w_j-\frac{1}{12}\delta_x^2w_{j+1}\right)\bar{w}_j\\&+\left(\frac{7}{6}\delta_x^2w_{m-1}-\frac{1}{12}\delta_x^2w_{m-2}\right)\bar{w}_{m-1}\\=&\frac{7}{6}\sum_{j=1}^{m-1}(\delta_x^2w_j)\bar{w}_j-\frac{1}{12}\sum_{j=1}^{m-2}(\delta_x^2w_j)\bar{w}_{j+1}-\frac{1}{12}\sum_{j=2}^{m-1}(\delta_x^2w_j)\bar{w}_{j-1}\\=&\sum_{j=1}^{m-1}(\delta_x^2w_j)\bar{w}_j-\frac{1}{12}\sum_{j=1}^{m-1}(\delta_x^2w_j)(\bar{w}_{j+1}-2\bar{w}_j+\bar{w}_{j-1})\\=&-\sum_{j=0}^{m-1}|\delta_xw_{j+\frac{1}{2}}|^2-\frac{h^2}{12}\sum_{j=1}^{m-1}|\delta_x^2w_j|^2,\end{aligned}$$

$$h\mathcal{A}(w)=-|w|_1^2-\frac{h^2}{12}|w|_2^2\geqslant-|w|_1^2-\frac{h^2}{12}\times\frac{4}{h^2}|w|_1^2=-\frac{4}{3}|w|_1^2.$$ □

定理 5.11 差分格式 (5.114)–(5.121) 是唯一可解的.

证明 (I) 由 (5.120)–(5.121) 知第 0 层上的值 u^0 已给定. 由 (5.114)–(5.116) 和 (5.121) 可得关于第 1 层值 u^1 的线性方程组. 考虑它的齐次方程组

$$\mathrm{i}\frac{1}{\tau}u_1^1+\frac{1}{2}\left(\frac{7}{6}\delta_x^2u_1^1-\frac{1}{12}\delta_x^2u_2^1\right)+\frac{q}{2}|\hat{u}_1|^2u_1^1=0, \tag{5.122}$$

$$\mathrm{i}\frac{1}{\tau}u_j^1+\frac{1}{2}\left(\frac{4}{3}\delta_x^2u_j^1-\frac{1}{3}\Delta_x^2u_j^1\right)+\frac{q}{2}|\hat{u}_j|^2u_j^1=0,\quad 2\leqslant j\leqslant m-2, \tag{5.123}$$

$$\mathrm{i}\frac{1}{\tau}u_{m-1}^1+\frac{1}{2}\times\left(\frac{7}{6}\delta_x^2u_{m-1}^1-\frac{1}{12}\delta_x^2u_{m-2}^1\right)+\frac{q}{2}|\hat{u}_{m-1}|^2u_{m-1}^1=0. \tag{5.124}$$

$$u_0^1=0,\quad u_m^1=0. \tag{5.125}$$

用 $2h\bar{u}_1^1$ 乘以 (5.122), 用 $2h\bar{u}_j^1$ 乘以 (5.123), 用 $2h\bar{u}_{m-1}^1$ 乘以 (5.124), 并将结果相加, 得到

$$\mathrm{i}\frac{2}{\tau}\|u^1\|^2+hA(u^1)+q\cdot h\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^1|^2=0.$$

取上式的虚部, 并利用引理 5.4, 得

$$\|u^1\|^2=0.$$

因而 (5.114)–(5.116) 和 (5.121) 唯一确定 u^1.

(II) 设 u^{k-1}, u^k 已确定. 则由 (5.117)–(5.119) 和 (5.121) 可得关于第 $k+1$ 层值 u^{k+1} 的线性方程组. 考虑其齐次方程组

$$\mathrm{i}\frac{1}{2\tau}u_1^{k+1}+\frac{1}{2}\left(\frac{7}{6}\delta_x^2u_1^{k+1}-\frac{1}{12}\delta_x^2u_2^{k+1}\right)+\frac{q}{2}|u_1^k|^2u_1^{k+1}=0, \tag{5.126}$$

$$\mathrm{i}\frac{1}{2\tau}u_j^{k+1}+\frac{1}{2}\left(\frac{4}{3}\delta_x^2u_j^{k+1}-\frac{1}{3}\Delta_x^2u_j^{k+1}\right)+\frac{q}{2}|u_j^k|^2u_j^{k+1}=0, \quad 2\leqslant j\leqslant m-2, \tag{5.127}$$

$$\mathrm{i}\frac{1}{2\tau}u_{m-1}^{k+1}+\frac{1}{2}\left(\frac{7}{6}\delta_x^2u_{m-1}^{k+1}-\frac{1}{12}\delta_x^2u_{m-2}^{k+1}\right)+\frac{q}{2}|u_{m-1}^k|^2u_{m-1}^{k+1}=0. \tag{5.128}$$

$$u_0^{k+1}=0,\quad u_m^{k+1}=0. \tag{5.129}$$

用 $2h\bar{u}_1^{k+1}$ 与 (5.126) 相乘, 用 $2h\bar{u}_j^{k+1}$ 与 (5.127) 相乘, 用 $2h\bar{u}_{m-1}^{k+1}$ 与 (5.128) 相乘, 再将所得结果相加, 得

$$\mathrm{i}\frac{1}{\tau}\|u^{k+1}\|^2+h\mathcal{A}(u^{k+1})+qh\sum_{j=1}^{m-1}|u_j^k|^2|u_j^{k+1}|^2=0.$$

取虚部, 并利用引理 5.4, 得

$$\|u^{k+1}\|^2=0.$$

因而 (5.117)–(5.119) 和 (5.121) 唯一确定 u^{k+1}. □

5.4.4 差分格式解的守恒性和有界性

引理 5.5 设 $u^k=(u_0^k,u_1^k,\cdots,u_m^k)\in\mathring{\mathcal{U}}_h$,

$$\begin{aligned}\mathcal{B}(u^k,u^{k+1})=&\left(\frac{7}{6}\delta_x^2u_1^{k+\frac{1}{2}}-\frac{1}{12}\delta_x^2u_2^{k+\frac{1}{2}}\right)(-\delta_t\bar{u}_1^{k+\frac{1}{2}})\\&+\sum_{j=2}^{m-2}\left(\frac{4}{3}\delta_x^2u_j^{k+\frac{1}{2}}-\frac{1}{3}\Delta_x^2u_j^{k+\frac{1}{2}}\right)(-\delta_t\bar{u}_j^{k+\frac{1}{2}})\\&+\left(\frac{7}{6}\delta_x^2u_{m-1}^{k+\frac{1}{2}}-\frac{1}{12}\delta_x^2u_{m-2}^{k+\frac{1}{2}}\right)(-\delta_t\bar{u}_{m-1}^{k+\frac{1}{2}}),\end{aligned}$$

则有

$$\mathrm{Re}\{h\mathcal{B}(u^k,u^{k+1})\}=\frac{1}{2\tau}\left[\left(|u^{k+1}|_1^2+\frac{h^2}{12}|u^{k+1}|_2^2\right)-\left(|u^k|_1^2+\frac{h^2}{12}|u^k|_2^2\right)\right].$$

证明

$$\begin{aligned}
\mathcal{B}(u^k,u^{k+1}) &= \left(\frac{7}{6}\delta_x^2 u_1^{k+\frac{1}{2}} - \frac{1}{12}\delta_x^2 u_2^{k+\frac{1}{2}}\right)(-\delta_t \bar{u}_1^{k+\frac{1}{2}}) \\
&\quad + \sum_{j=2}^{m-2}\left(-\frac{1}{12}\delta_x^2 u_{j-1}^{k+\frac{1}{2}} + \frac{7}{6}\delta_x^2 u_j^{k+\frac{1}{2}} - \frac{1}{12}\delta_x^2 u_{j+1}^{k+\frac{1}{2}}\right)(-\delta_t \bar{u}_j^{k+\frac{1}{2}}) \\
&\quad + \left(\frac{7}{6}\delta_x^2 u_{m-1}^{k+\frac{1}{2}} - \frac{1}{12}\delta_x^2 u_{m-2}^{k+\frac{1}{2}}\right)(-\delta_t \bar{u}_{m-1}^{k+\frac{1}{2}}) \\
&= \sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})(-\delta_t \bar{u}_j^{k+\frac{1}{2}}) - \frac{1}{12}\sum_{j=2}^{m-1}(\delta_x^2 u_{j-1}^{k+\frac{1}{2}})(-\delta_t \bar{u}_j^{k+\frac{1}{2}}) \\
&\quad - \frac{1}{12}\sum_{j=1}^{m-2}(\delta_x^2 u_{j+1}^{k+\frac{1}{2}})(-\delta_t \bar{u}_j^{k+\frac{1}{2}}) + \frac{2}{12}\sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})(-\delta_t \bar{u}_j^{k+\frac{1}{2}}) \\
&= \sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})(-\delta_t \bar{u}_j^{k+\frac{1}{2}}) - \frac{1}{12}\left\{\sum_{j=1}^{m-2}(\delta_x^2 u_j^{k+\frac{1}{2}})(-\delta_t \bar{u}_{j+1}^{k+\frac{1}{2}})\right. \\
&\quad \left. - 2\sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})(-\delta_t \bar{u}_j^{k+\frac{1}{2}}) + \sum_{j=2}^{m-1}(\delta_x^2 \bar{u}^{k+\frac{1}{2}})(-\delta_t \bar{u}_{j-1}^{k+\frac{1}{2}})\right\} \\
&= \sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})(-\delta_t \bar{u}_j^{k+\frac{1}{2}}) + \frac{1}{12}\sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})\delta_t(\bar{u}_{j+1}^{k+\frac{1}{2}} - 2\bar{u}_j^{k+\frac{1}{2}} + \bar{u}_{j-1}^{k+\frac{1}{2}}) \\
&= \sum_{j=0}^{m-1}(\delta_x u_{j+\frac{1}{2}}^{k+\frac{1}{2}})(\delta_t\delta_x \bar{u}_{j+\frac{1}{2}}^{k+\frac{1}{2}}) + \frac{h^2}{12}\sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})\delta_t(\delta_x^2 \bar{u}_j^{k+\frac{1}{2}}).
\end{aligned}$$

因而

$$\begin{aligned}
&\mathrm{Re}\{h\mathcal{B}(u^k,u^{k+1})\} \\
&= \frac{1}{2\tau}(|u^{k+1}|_1^2 - |u^k|_1^2) + \frac{h^2}{12}\cdot\frac{1}{2\tau}(|u^{k+1}|_2^2 - |u^k|_2^2). \\
&= \frac{1}{2\tau}\left[\left(|u^{k+1}|_1^2 + \frac{h^2}{12}|u^{k+1}|_2^2\right) - \left(|u^k|_1^2 + \frac{h^2}{12}|u^k|_2^2\right)\right].
\end{aligned}$$ □

同理可证如下结论.

引理 5.6 设 $u^k = (u_0^k, u_1^k, \cdots, u_m^k) \in \mathring{\mathcal{U}}_h$,

$$\begin{aligned}
\mathcal{C}(u^{k-1},u^{k+1}) &= \left(\frac{7}{6}\delta_x^2 u_1^{\bar{k}} - \frac{1}{12}\delta_x^2 u_2^{\bar{k}}\right)(-\Delta_t u_1^{\bar{k}}) \\
&\quad + \sum_{j=2}^{m-2}\left(\frac{4}{3}\delta_x^2 u_j^{\bar{k}} - \frac{1}{3}\Delta_x^2 u_j^{\bar{k}}\right)(-\Delta_t u_j^k) \\
&\quad + \left(\frac{7}{6}\delta_x^2 u_{m-1}^{\bar{k}} - \frac{1}{12}\delta_x^2 u_{m-2}^{\bar{k}}\right)(-\Delta_t u_{m-1}^k),
\end{aligned}$$

则有

$$\begin{aligned}&\mathrm{Re}\{h\mathcal{C}(u^{k-1},u^{k+1})\}\\&=\frac{1}{4\tau}\left[\left(|u^{k+1}|_1^2+\frac{h^2}{12}|u^{k+1}|_2^2\right)-\left(|u^{k-1}|_1^2+\frac{h^2}{12}|u^{k-1}|_2^2\right)\right].\end{aligned}$$

定理 5.12　设 $\{u_j^k\,|\,0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ 是差分格式 (5.114)–(5.121) 的解. 记

$$E^k=\frac{1}{2}\left(|u^{k+1}|_1^2+\frac{h^2}{12}|u^{k+1}|_2^2+|u^k|_1^2+\frac{h^2}{12}|u^k|_2^2\right)-\frac{q}{2}h\sum_{j=1}^{m-1}|u_j^k|^2|u_j^{k+1}|^2,$$

则有

$$\|u^k\|=\|u^0\|,\quad 0\leqslant k\leqslant n, \tag{5.130}$$

$$\begin{aligned}&\frac{1}{2}\Big(|u^1|_1^2+\frac{h^2}{12}|u^1|_2^2+|u^0|_1^2+\frac{h^2}{12}|u^0|_2^2\Big)-\frac{q}{2}h\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^1|^2\\&=|u^0|_1^2+\frac{h^2}{12}|u^0|_2^2-\frac{q}{2}h\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^0|^2,\end{aligned} \tag{5.131}$$

$$E^k=E^0,\quad 1\leqslant k\leqslant n-1. \tag{5.132}$$

证明　(I) 用 $h\bar{u}_1^{\frac{1}{2}}$ 乘以 (5.114), 用 $h\bar{u}_j^{\frac{1}{2}}$ 乘以 (5.115), 用 $h\bar{u}_{m-1}^{\frac{1}{2}}$ 乘以 (5.116), 并将结果相加, 可得

$$\mathrm{i}(\delta_t u^{\frac{1}{2}},u^{\frac{1}{2}})+h\mathcal{A}(u^{\frac{1}{2}})+qh\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^{\frac{1}{2}}|^2=0.$$

取上式的虚部, 并应用引理 5.4, 得

$$\frac{1}{2\tau}(\|u^1\|^2-\|u^0\|^2)=0,$$

即

$$\|u^1\|=\|u^0\|. \tag{5.133}$$

用 $h\bar{u}_1^{\bar{k}}$ 乘以 (5.117), 用 $h\bar{u}_j^{\bar{k}}$ 乘以 (5.118), 用 $h\bar{u}_{m-1}^{\bar{k}}$ 乘以 (5.119), 并将所得结果相加, 得

$$\mathrm{i}(\Delta_t u^k,u^{\bar{k}})+h\mathcal{A}(u^{\bar{k}})+qh\sum_{j=1}^{m-1}|u_j^k|^2|u_j^{\bar{k}}|^2=0.$$

取上式的虚部, 并再次利用引理 5.4, 得

$$\frac{1}{4\tau}(\|u^{k+1}\|^2-\|u^{k-1}\|^2)=0,\quad 1\leqslant k\leqslant n-1. \tag{5.134}$$

综合 (5.133)–(5.134), 即得 (5.130).

(II) 用 $-h\delta_t\bar{u}_1^{\frac{1}{2}}$ 乘以 (5.114), 用 $-h\delta_t\bar{u}_j^{\frac{1}{2}}$ 乘以 (5.115), 用 $-h\delta_t\bar{u}_{m-1}^{\frac{1}{2}}$ 乘以 (5.116), 并将结果相加, 得

$$-\mathrm{i}\|\delta_t u^{\frac{1}{2}}\|^2+h\mathcal{B}(u^0,u^1)+qh\sum_{j=0}^{m-1}|\hat{u}_j|^2u_j^{\frac{1}{2}}(-\delta_t\bar{u}_j^{\frac{1}{2}})=0.$$

取上式的实部, 并利用引理 5.5, 得

$$\frac{1}{2\tau}\left[\left(|u^1|_1^2+\frac{h^2}{12}|u^1|_2^2\right)-\left(|u^0|_1^2+\frac{h^2}{12}|u^0|_2^2\right)\right]-qh\sum_{j=1}^{m-1}|\hat{u}_j|^2\frac{|u_j^1|^2-|u_j^0|^2}{2\tau}=0,$$

因而

$$|u^1|_1^2+\frac{h^2}{12}|u^1|_2^2-qh\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^1|^2=|u^0|_1^2+\frac{h^2}{12}|u^0|_2^2-qh\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^0|^2.$$

此式即为 (5.131).

(III) 用 $-h\Delta_t\bar{u}_1^k$ 乘以 (5.117), 用 $-h\Delta_t\bar{u}_j^k$ 乘以 (5.118), 用 $-h\Delta_t\bar{u}_{m-1}^k$ 乘以 (5.119), 并将所得结果相加, 得

$$-\mathrm{i}\|\Delta_t u^k\|^2+h\mathcal{C}(u^{k-1},u^{k+1})-qh\sum_{j=1}^{m-1}|u_j^k|^2u_j^{\bar{k}}\Delta_t\bar{u}_j^k=0,\quad 1\leqslant k\leqslant n-1.$$

取上式的实部, 并利用引理 5.6, 得到

$$\begin{aligned}&\frac{1}{4\tau}\left[\left(|u^{k+1}|_1^2+\frac{h^2}{12}|u^{k+1}|_2^2\right)-\left(|u^{k-1}|_1^2+\frac{h^2}{12}|u^{k-1}|_2^2\right)\right]\\&-\frac{1}{4\tau}qh\sum_{j=1}^{m-1}|u_j^k|^2(|u_j^{k+1}|^2-|u_j^{k-1}|^2)=0,\quad 1\leqslant k\leqslant n-1.\end{aligned}$$

将上式变形后可得

$$\begin{aligned}&\frac{1}{2\tau}\left[\left(\frac{|u^{k+1}|_1^2+\frac{h^2}{12}|u^{k+1}|_2^2+|u^k|_1^2+\frac{h^2}{12}|u^k|_2^2}{2}-\frac{q}{2}h\sum_{j=1}^{m-1}|u_j^k|^2|u_j^{k+1}|^2\right)\right.\\&\left.-\left(\frac{|u^k|_1^2+\frac{h^2}{12}|u^k|_2^2+|u^{k-1}|_1^2+\frac{h^2}{12}|u^{k-1}|_2^2}{2}-\frac{q}{2}h\sum_{j=1}^{m-1}|u_j^{k-1}|^2|u_j^k|^2\right)\right]=0,\\&\qquad 1\leqslant k\leqslant n-1.\end{aligned}$$

易知 (5.132) 成立. □

应用定理 5.12, 可得存在常数 c_{16} 使得

$$\|u^k\|_\infty \leqslant c_{16}, \quad 0 \leqslant k \leqslant n. \tag{5.135}$$

注 5.1 如果取

$$\hat{u}_j = u_j, 0 \leqslant j \leqslant m,$$

则有

$$E^k = |u^0|_1^2 + \frac{h^2}{12}|u^0|_2^2 - \frac{q}{2}h\sum_{j=1}^{m-1}|\hat{u}_j|^2|u_j^0|^2, \quad 0 \leqslant k \leqslant n-1.$$

可以得到差分格式解在 L_2 范数下的二阶收敛性.

5.4.5 差分格式解的收敛性

定理 5.13 设 $\{U_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为问题 (5.1)–(5.3) 的解, $\{u_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为 (5.114)–(5.121) 的解. 记

$$e_j^k = U_j^k - u_j^k, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n,$$

则存在常数 c_{17} 使得

$$\|e^k\| \leqslant c_{17}(\tau^2 + h^4), \quad 0 \leqslant k \leqslant n. \tag{5.136}$$

证明 将 (5.104)–(5.109), (5.112)–(5.113) 与 (5.114)–(5.121) 相减, 得到误差方程组

$$\mathrm{i}\delta_t e_1^{\frac{1}{2}} + \frac{7}{6}\delta_x^2 e_1^{\frac{1}{2}} - \frac{1}{12}\delta_x^2 e_2^{\frac{1}{2}} + q|\hat{u}_1|^2 e_1^{\frac{1}{2}} = R_1^0, \tag{5.137}$$

$$\mathrm{i}\delta_t e_j^{\frac{1}{2}} + \frac{4}{3}\delta_x^2 e_j^{\frac{1}{2}} - \frac{1}{3}\Delta_x^2 e_j^{\frac{1}{2}} + q|\hat{u}_j|^2 e_j^{\frac{1}{2}} = R_j^0, \quad 2 \leqslant j \leqslant m-2, \tag{5.138}$$

$$\mathrm{i}\delta_t e_{m-1}^{\frac{1}{2}} + \frac{7}{6}\delta_x^2 e_{m-1}^{\frac{1}{2}} - \frac{1}{12}\delta_x^2 e_{m-2}^{\frac{1}{2}} + q|\hat{u}_{m-1}|^2 e_{m-1}^{\frac{1}{2}} = R_{m-1}^0, \tag{5.139}$$

$$\mathrm{i}\Delta_t e_1^k + \frac{7}{6}\delta_x^2 e_1^{\bar{k}} - \frac{1}{12}\delta_x^2 e_2^{\bar{k}} + q(|U_1^k|^2 U_1^{\bar{k}} - |u_1^k|^2 u_1^{\bar{k}}) = R_1^k, \\ 1 \leqslant k \leqslant n-1, \tag{5.140}$$

$$\mathrm{i}\Delta_t e_j^k + \frac{4}{3}\delta_x^2 e_j^{\bar{k}} - \frac{1}{3}\Delta_x^2 e_j^{\bar{k}} + q(|U_j^k|^2 U_j^{\bar{k}} - |u_j^k|^2 u_j^{\bar{k}}) = R_j^k, \\ 2 \leqslant j \leqslant m-2,\ 1 \leqslant k \leqslant n-1, \tag{5.141}$$

$$\mathrm{i}\Delta_t e_{m-1}^k + \frac{7}{6}\delta_x^2 e_{m-1}^{\bar{k}} - \frac{1}{12}\delta_x^2 e_{m-2}^{\bar{k}} \\ +q(|U_{m-1}^k|^2 U_{m-1}^{\bar{k}} - |u_{m-1}^k|^2 u_{m-1}^{\bar{k}}) = R_{m-1}^k, \quad 1 \leqslant k \leqslant n-1, \tag{5.142}$$

$$e_j^0 = 0, \quad 1 \leqslant j \leqslant m-1, \tag{5.143}$$

$$e_0^k = 0, \quad e_m^k = 0, \quad 0 \leqslant k \leqslant n. \tag{5.144}$$

采用前面的记号及结论有

$$\begin{aligned}&c_0=\max_{0\leqslant x\leqslant L,0\leqslant t\leqslant T}|u(x,t)|,\quad c_{13}=\max_{0\leqslant x\leqslant L}|u_t(x,0)|,\\&\|\hat{u}\|_\infty\leqslant c_0+\frac{\tau}{2}c_{13}\leqslant c_0+c_{13},\\&\|u^k\|_\infty\leqslant c_{16}.\end{aligned}$$

(I) 用 $h\bar{e}_1^{\frac12}$ 乘以 (5.137), 用 $h\bar{e}_j^{\frac12}$ 乘以 (5.138), 用 $h\bar{e}_{m-1}^{\frac12}$ 乘以 (5.139), 并将结果相加, 得

$$\mathrm{i}(\delta_te^{\frac12},e^{\frac12})+h\mathcal{A}(e^{\frac12})+h\sum_{j=1}^{m-1}|\hat{u}_j|^2|e_j^{\frac12}|^2=(R^0,e^{\frac12}).$$

取上式的虚部并应用引理 5.4, 得

$$\frac{1}{2\tau}(\|e^1\|^2-\|e^0\|^2)=\mathrm{Im}\{(R^0,e^{\frac12})\}\leqslant\|R^0\|\cdot\|e^{\frac12}\|.$$

注意到 $e^0=0$, 有

$$\frac{1}{2\tau}\|e^1\|^2\leqslant\|R^0\|\cdot\frac12\|e^1\|.$$

因而

$$\|e^1\|\leqslant\tau\|R^0\|\leqslant\tau\sqrt{L}c_{15}(\tau^2+h^4).\tag{5.145}$$

(II) 用 $h\bar{e}_1^{\bar{k}}$ 乘以 (5.140), 用 $h\bar{e}_j^{\bar{k}}$ 乘以 (5.141), 用 $h\bar{e}_{m-1}^{\bar{k}}$ 乘以 (5.142), 并将所得结果相加, 得

$$\mathrm{i}(\Delta_te^k,e^{\bar{k}})+h\mathcal{A}(e^{\bar{k}})+q(|U^k|^2U^{\bar{k}}-|u^k|^2u^{\bar{k}},e^{\bar{k}})=(R^k,e^{\bar{k}}),\quad 1\leqslant k\leqslant n-1.\tag{5.146}$$

注意到

$$\begin{aligned}&|U_j^k|^2U_j^{\bar{k}}-|u_j^k|^2u_j^{\bar{k}}\\=&|u_j^k|^2(U_j^{\bar{k}}-u_j^{\bar{k}})+(|U_j^k|^2-|u_j^k|^2)U_j^{\bar{k}}\\=&|u_j^k|^2e_j^{\bar{k}}+(e_j^k\bar{U}_j^k+u_j^k\bar{e}_j^k)U_j^{\bar{k}},\end{aligned}$$

取 (5.146) 的虚部, 并再次应用引理 5.4, 得到

$$\begin{aligned}&\frac{1}{4\tau}(\|e^{k+1}\|^2-\|e^{k-1}\|^2)+\mathrm{Im}\Big\{qh\sum_{j=1}^{m-1}(e_j^k\bar{U}_j^k+u_j^k\bar{e}_j^k)U_j^{\bar{k}}\bar{e}_j^{\bar{k}}\Big\}\\=&\mathrm{Im}(R^k,e^{\bar{k}}),\quad 1\leqslant k\leqslant n-1.\end{aligned}$$

因而

$$\begin{aligned}
&\frac{1}{4\tau}(\|e^{k+1}\|^2-\|e^{k-1}\|^2)\\
&\leqslant |q|c_0(c_0+c_{16})\|e^k\|\cdot\|e^{\bar{k}}\|+\|R^k\|\cdot\|e^{\bar{k}}\|\\
&\leqslant (|q|c_0(c_0+c_{16})\|e^k\|+\|R^k\|)\frac{\|e^{k+1}\|+\|e^{k-1}\|}{2},\quad 1\leqslant k\leqslant n-1.
\end{aligned}$$

两边约去 $\frac{1}{2}(\|e^{k+1}\|+\|e^{k-1}\|)$, 得到

$$\frac{1}{2\tau}(\|e^{k+1}\|-\|e^{k-1}\|)\leqslant |q|c_0(c_0+c_{16})\|e^k\|+\|R^k\|,\quad 1\leqslant k\leqslant n-1,$$

或

$$\|e^{k+1}\|\leqslant\|e^{k-1}\|+2|q|c_0(c_0+c_{16})\tau\|e^k\|+2\tau\|R^k\|,\quad 1\leqslant k\leqslant n-1.$$

由上式, 得到

$$\begin{aligned}
&\max\{\|e^{k+1}\|,\|e^k\|\}\\
\leqslant&[1+2|q|c_0(c_0+c_{16})\tau]\max\{\|e^k\|,e^{k-1}\}+2\tau\|R^k\|\\
\leqslant&[1+2|q|c_0(c_0+c_{16})\tau]\max\{\|e^k\|,\|e^{k-1}\|\}+2\sqrt{L}c_{15}\tau(\tau^2+h^4),\quad 1\leqslant k\leqslant n-1.
\end{aligned}$$

当 $q=0$ 时, 由

$$\max\{\|e^{k+1}\|,\|e^k\|\}\leqslant\max\{\|e^k\|,\|e^{k-1}\|\}+2\sqrt{L}c_{15}\tau(\tau^2+h^4),\quad 1\leqslant k\leqslant n-1$$

递推可得

$$\max\{\|e^{k+1}\|,\|e^k\|\}\leqslant 2\sqrt{L}c_{15}T(\tau^2+h^4)\equiv c_{17}(\tau^2+h^4),\quad 0\leqslant k\leqslant n-1.$$

当 $q\neq 0$ 时, 由 Gronwall 不等式得到

$$\begin{aligned}
\max\{\|e^{k+1}\|,\|e^k\|\}&\leqslant e^{(1+2|q|c_0(c_0+c_{16}))T}\left[\max\{\|e^1\|,\|e^0\|\}+\frac{\sqrt{L}c_{15}(\tau^2+h^4)}{2|q|c_0(c_0+c_{16})}\right]\\
&\leqslant c_{17}(\tau^2+h^4),\quad 0\leqslant k\leqslant n-1.
\end{aligned}$$

因而 (5.136) 成立.　□

定理 5.14　设 $\{U_j^k\,|\,0\leqslant j\leqslant m,0\leqslant k\leqslant n\}$ 为问题 (5.1)–(5.3) 的解, $\{u_j^k\,|\,0\leqslant j\leqslant m,0\leqslant k\leqslant n\}$ 为 (5.114)–(5.121) 的解. 记

$$e_j^k=U_j^k-u_j^k,\quad 0\leqslant j\leqslant m,\ 0\leqslant k\leqslant n,$$

则存在常数 c_{18} 使得

$$|e^k|_1\leqslant c_{18}(\tau^2+h^4),\quad 0\leqslant k\leqslant n. \tag{5.147}$$

证明 (I) 用 $-h\delta_t\bar{e}_1^{\frac{1}{2}}$ 乘以 (5.137), 用 $-h\delta_t\bar{e}_j^{\frac{1}{2}}$ 乘以 (5.138), 用 $-h\delta_t\bar{e}_{m-1}^{\frac{1}{2}}$ 乘以 (5.139), 并将所得结果相加, 得

$$-\mathrm{i}\|\delta_t e^{\frac{1}{2}}\|^2 + h\mathcal{B}(e^0, e^1) - h\sum_{j=1}^{m-1}|\hat{u}_j|^2 e_j^{\frac{1}{2}}\delta_t\bar{e}_j^{\frac{1}{2}} = -h(R^0, \delta_t e^{\frac{1}{2}}).$$

两边取实部, 并应用引理 5.5, 得

$$\begin{aligned}&\frac{1}{2\tau}\left[\left(|e^1|_1^2 + \frac{h^2}{12}\|\delta_x^2 e^1\|^2\right) - \left(|e^0|_1^2 + \frac{h^2}{12}\|\delta_x^2 e^0\|^2\right)\right] - h\sum_{j=1}^{m-1}|\hat{u}_j|^2\frac{|e_j^1|^2 - |e_j^0|^2}{2\tau}\\ =&\mathrm{Re}\{-h(R^0, \delta_t e^{\frac{1}{2}})\}\\ \leqslant&\|R^0\|\cdot\|\delta_t e^{\frac{1}{2}}\|.\end{aligned}$$

注意到 $e^0 = 0$, 得

$$\frac{1}{2\tau}\left(|e^1|_1^2 + \frac{h^2}{12}\|\delta_x^2 e^1\|^2\right) \leqslant h\cdot\frac{1}{2\tau}\sum_{j=1}^{m-1}|\hat{u}_j|^2|e_j^1|^2 + \frac{1}{\tau}\|R^0\|\cdot\|e^1\|,$$

因而

$$\begin{aligned}|e^1|_1^2 \leqslant& h\sum_{j=1}^{m-1}|\hat{u}_j|^2|e_j^1|^2 + 2\|R^0\|\cdot\|e^1\|\\ \leqslant&(c_0 + c_{13})^2\|e^1\|^2 + 2\|R^0\|\cdot\|e^1\|\\ \leqslant&(c_0 + c_{13})^2 c_{17}^2(\tau^2 + h^4)^2 + 2c_{15}\sqrt{L}(\tau^2 + h^4)c_{17}(\tau^2 + h^4)\\ =&[(c_0 + c_{13})^2 c_{17}^2 + 2\sqrt{L}c_{15}c_{17}](\tau^2 + h^4)^2.\end{aligned} \tag{5.148}$$

(II) 用 $-h\Delta_t\bar{e}_1^k$ 乘以 (5.140), 用 $-h\Delta_t\bar{e}_j^k$ 乘以 (5.141), 用 $-h\Delta_t\bar{e}_{m-1}^k$ 乘以 (5.142), 并将所得结果相加, 得

$$\begin{aligned}&-\mathrm{i}\|\Delta_t e^k\|^2 + h\mathcal{C}(e^{k-1}, e^{k+1}) - qh\sum_{j=1}^{m-1}(|U_j^k|^2 U_j^{\bar{k}} - |u_j^k|^2 u_j^{\bar{k}})\Delta_t\bar{e}_j^k\\ =&-(R^k, \Delta_t e^k), \quad 1 \leqslant k \leqslant n-1.\end{aligned}$$

取上式的实部, 并应用引理 5.6, 得

$$\begin{aligned}&\frac{1}{4\tau}\left[\left(|e^{k+1}|_1^2 + \frac{h^2}{12}|e^{k+1}|_2^2\right) - \left(|u^{k-1}|_1^2 + \frac{h^2}{12}|e^{k-1}|_2^2\right)\right]\\ =&q\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(|U_j^k|^2 U_j^{\bar{k}} - |u_j^k|^2 u_j^{\bar{k}})\Delta_t\bar{e}_j^k\right\} + \mathrm{Re}\{-(R^k, \Delta_t e^k)\}, \quad 1 \leqslant k \leqslant n-1.\end{aligned} \tag{5.149}$$

应用 (5.140)-(5.142), 可得

$$
\begin{aligned}
&q\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(|U_j^k|^2U_j^{\bar{k}}-|u_j^k|^2u_j^{\bar{k}})\Delta_t\bar{e}_j^k\right\}\\
=&q\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(|U_j^k|^2\bar{U}_j^{\bar{k}}-|u_j^k|^2\bar{u}_j^{\bar{k}})\Delta_t e_j^k\right\}\\
=&q\mathrm{Im}\left\{h\sum_{j=1}^{m-1}(|U_j^k|^2\bar{U}_j^{\bar{k}}-|u_j^k|^2\bar{u}_j^{\bar{k}})\mathrm{i}\Delta_t e_j^k\right\}\\
=&qh\mathrm{Im}\left\{(|U_1^k|^2\bar{U}_1^{\bar{k}}-|u_1^k|^2\bar{u}_1^{\bar{k}})\mathrm{i}\Delta_t e_1^k+\sum_{j=2}^{m-2}(|U_j^k|^2\bar{U}_j^{\bar{k}}-|u_j^k|^2\bar{u}_j^{\bar{k}})\mathrm{i}\Delta_t e_j^k\right.\\
&\left.+(|U_{m-1}^k|^2\bar{U}_{m-1}^{\bar{k}}-|u_{m-1}^k|^2\bar{u}_{m-1}^{\bar{k}})\mathrm{i}\Delta_t e_{m-1}^k\right\}\\
=&qh\mathrm{Im}\left\{(|U_1^k|^2\bar{U}_1^{\bar{k}}-|u_1^k|^2\bar{u}_1^{\bar{k}})\left[R_1^k-\left(\frac{7}{6}\delta_x^2e_1^{\bar{k}}-\frac{1}{12}\delta_x^2e_2^{\bar{k}}+q(|U_1^k|^2U_1^{\bar{k}}-|u_1^k|^2u_1^{\bar{k}})\right)\right]\right.\\
&+\sum_{j=2}^{m-2}(|U_j^k|^2\bar{U}_j^{\bar{k}}-|u_j^k|^2\bar{u}_j^{\bar{k}})\left[R_j^k-\left(\frac{4}{3}\delta_x^2e_j^{\bar{k}}-\frac{1}{3}\Delta_x^2e_j^{\bar{k}}+q(|U_j^k|^2U_j^{\bar{k}}-|u_j^k|^2u_j^{\bar{k}})\right)\right]\\
&+(|U_{m-1}^k|^2\bar{U}_{m-1}^{\bar{k}}-|u_{m-1}^k|^2\bar{u}_{m-1}^{\bar{k}})\\
&\left.\cdot\left[R_{m-1}^k-\left(\frac{7}{6}\delta_x^2e_{m-1}^{\bar{k}}-\frac{1}{12}\delta_x^2e_{m-2}^{\bar{k}}+q(|U_{m-1}^k|^2U_{m-1}^{\bar{k}}-|u_{m-1}^k|^2u_{m-1}^{\bar{k}})\right)\right]\right\}\\
=&qh\mathrm{Im}\left\{(|U_1^k|^2\bar{U}_1^{\bar{k}}-|u_1^k|^2\bar{u}_1^{\bar{k}})\left[R_1^k-\left(\frac{7}{6}\delta_x^2e_1^{\bar{k}}-\frac{1}{12}\delta_x^2e_2^{\bar{k}}\right)\right]\right.\\
&+\sum_{j=2}^{m-2}(|U_j^k|^2\bar{U}_j^{\bar{k}}-|u_j^k|^2\bar{u}_j^{\bar{k}})\left[R_j^k-\left(\frac{4}{3}\delta_x^2e_j^{\bar{k}}-\frac{1}{3}\Delta_x^2e_j^{\bar{k}}\right)\right]\\
&\left.+(|U_{m-1}^k|^2\bar{U}_{m-1}^{\bar{k}}-|u_{m-1}^k|^2\bar{u}_{m-1}^{\bar{k}})\left[R_{m-1}^k-\left(\frac{7}{6}\delta_x^2e_{m-1}^{\bar{k}}-\frac{1}{12}\delta_x^2e_{m-2}^{\bar{k}}\right)\right]\right\}\\
\leqslant& c_{19}(\|e^{k+1}\|^2+\|e^k\|^2+\|e^{k-1}\|^2+|e^{k+1}|_1^2+|e^k|_1^2+|e^{k-1}|_1^2+\|R^k\|^2),\\
&\qquad 1\leqslant k\leqslant n-1. \qquad (5.150)
\end{aligned}
$$

类似于 (5.87) 的推导可得

$$
\left|\sum_{l=1}^{k}(R^l,\Delta_t e^l)\right|\leqslant\frac{1}{2\tau}\|R^k\|\cdot\|e^{k+1}\|+\frac{1}{2\tau}\|R^{k-1}\|\cdot\|e^k\|
$$

$$+\sum_{l=2}^{k-1}\|\Delta_t R^l\|\cdot\|e^l\|+\frac{1}{2\tau}\|R^2\|\cdot\|e^1\|. \tag{5.151}$$

将 (5.149) 中的 k 换为 l, 并对 l 从 1 到 k 求和, 再利用 (5.150) 和 (5.151), 得

$$\begin{aligned}
&\frac{1}{4\tau}\left[\left(|e^{k+1}|_1^2+\frac{h^2}{12}|e^{k+1}|_2^2+|e^k|_1^2+\frac{h^2}{12}|e^k|_2^2\right)\right.\\
&\left.-\left(|e^1|_1^2+\frac{h^2}{12}|e^1|_2^2+|e^0|_1^2+\frac{h^2}{12}|e^0|_2^2\right)\right]\\
\leqslant&\sum_{l=1}^{k}c_{19}(\|e^{l+1}\|^2+\|e^l\|^2+\|e^{l-1}\|^2+|e^{l+1}|_1^2+|e^l|_1^2+|e^{l-1}|_1^2+\|R^l\|^2)\\
&+\frac{1}{2\tau}\|R^k\|\cdot\|e^{k+1}\|+\frac{1}{2\tau}\|R^{k-1}\|\cdot\|e^k\|\\
&+\sum_{l=2}^{k-1}\|\Delta_t R^l\|\cdot\|e^l\|+\frac{1}{2\tau}\|R^2\|\cdot\|e^1\|,\quad 1\leqslant k\leqslant n-1.
\end{aligned}$$

注意到

$$2\|R^k\|\cdot\|e^{k+1}\|\leqslant\frac{3}{L^2}\|e^{k+1}\|^2+\frac{L^2}{3}\|R^k\|^2\leqslant\frac{1}{2}|e^{k+1}|_1^2+\frac{L^2}{3}\|R^k\|^2,$$

$$2\|R^{k-1}\|\cdot\|e^k\|\leqslant\frac{3}{L^2}\|e^k\|^2+\frac{L^2}{3}\|R^{k-1}\|^2\leqslant\frac{1}{2}|e^k|_1^2+\frac{L^2}{3}\|R^{k-1}\|^2,$$

以及 (5.110)–(5.111), (5.136), (5.148), 知存在常数 c_{19} 使得

$$|e^{k+1}|_1^2\leqslant c_{19}\tau\sum_{l=1}^{k}|e^l|_1^2+c_{19}(\tau^2+h^4)^2,\quad 0\leqslant k\leqslant n-1.$$

由 Gronwall 不等式得到

$$|e^k|_1^2\leqslant e^{c_{19}T}\cdot c_{19}(\tau^2+h^4)^2,\quad 0\leqslant k\leqslant n.$$

两边开方, 得

$$|e^k|_1\leqslant e^{\frac{1}{2}c_{19}T}\sqrt{c_{19}}(\tau^2+h^4),\quad 0\leqslant k\leqslant n.$$ □

由引理 1.1(b) 可得

$$\|e^k\|_\infty\leqslant\frac{\sqrt{L}}{2}|e^k|_1\leqslant\frac{\sqrt{L}}{2}c_{18}(\tau^2+h^4),\quad 0\leqslant k\leqslant n.$$

5.5　小结及延拓

本章对一维 Schrödinger 方程初边值问题建立了三个差分格式, 并作了相应的理论分析.

第一个差分格式是二层非线性差分格式; 证明了差分格式解满足两个能量守恒率, 给出了差分格式解的无穷模估计. 用 Browder 定理证明了差分格式解的存在性, 证明了差分格式解的唯一性, 证明了差分格式解在无穷模下关于时间步长和空间步长均是二阶无条件收敛的. 本工作可见 [9], [31].

第二个差分格式和第三个差分格式均是三层线性化差分格式. 证明了差分格式解满足两个能量守恒率, 给出了差分格式解的无穷模估计. 证明了差分格式的唯一可解性和收敛性.

第三个差分格式是在 [1] 和 [6] 工作的基础上加以发展得到的. 原文建立的是二层非线性差分格式. 这里我们建立的是三层线性化差分格式.

王廷春、郭柏灵在文 [7] 中对 Schrödinger 方程建立了二层非线性紧致差分格式和三层线性化紧致差分格式.

文 [22], [31], [32] 考虑耦合 Schrödinger 方程组的差分方法.

文 [36] 研究了二维 Schrödinger 方程周期边界值问题的差分方法.

第 6 章　Kuramoto-Tsuzuki 方程的差分方法

Kuramoto-Tsuzuki 方程 [19] 描述了在分歧点附近两个分支的行为状况, 它可以看成是 Ginzburg-Landau 方程的一维形式. 本章研究一维 Kuramoto-Tsuzuki 方程 Neumann 边界值问题的差分方法. 第 8 章将探讨二维 Ginzburg-Landau 方程 Dirichlet 边界值问题的差分方法.

6.1 引　　言

本章研究 Kuramoto-Tsuzuki 方程初边值问题

$$u_t = (1+\mathrm{i}c_1)u_{xx} + u - (1+\mathrm{i}c_2)\,|u|^2 u, \quad 0<x<L,\ 0<t\leqslant T, \tag{6.1}$$

$$u(x,0)=\varphi(x), \quad 0\leqslant x\leqslant L, \tag{6.2}$$

$$u_x(0,t)=0, \quad u_x(L,t)=0, \quad 0<t\leqslant T \tag{6.3}$$

的有限差分方法, 其中 c_1 和 c_2 为实常数, $\varphi(x), u(x,t)$ 为复值函数, $\varphi_x(0)=\varphi_x(L)=0$. 我们先给出问题 (6.1)–(6.3) 解的估计式.

设 $f(x)\in C[0,L]$, 记

$$\|f\|_p = \sqrt[p]{\int_0^L |f(x)|^p \mathrm{d}x}, \qquad \|f\| = \|f\|_2.$$

此外, 如果 $f(x)\in C^1[0,L]$, 记

$$|f|_1 = \sqrt{\int_0^L |f'(x)|^2 \mathrm{d}x}.$$

引理 6.1　*设 s_1 和 s_2 是两个非负常数, $f(x)\in C[0,L]$, $s_1<s_2$, $\theta\in(0,1)$, 则有*

$$\|f\|_{\theta s_1+(1-\theta)s_2}^{\theta s_1+(1-\theta)s_2} \leqslant \|f\|_{s_1}^{\theta s_1}\cdot\|f\|_{s_2}^{(1-\theta)s_2}.$$

证明　应用求积分的 Hölder 不等式, 得到

$$\|f\|_{\theta s_1+(1-\theta)s_2}^{\theta s_1+(1-\theta)s_2} = \int_0^L |f(x)|^{\theta s_1+(1-\theta)s_2}\mathrm{d}x$$

$$
\begin{aligned}
&\leqslant \left[\int_0^L \left(|f(x)|^{\theta s_1}\right)^{\frac{1}{\theta}}\mathrm{d}x\right]^{\theta}\cdot\left[\int_0^L \left(|f(x)|^{(1-\theta)s_2}\right)^{\frac{1}{1-\theta}}\mathrm{d}x\right]^{1-\theta}\\
&=\left[\int_0^L |f(x)|^{s_1}\mathrm{d}x\right]^{\theta}\cdot\left[\int_0^L |f(x)|^{s_2}\mathrm{d}x\right]^{1-\theta}\\
&=\|f\|_{s_1}^{\theta s_1}\cdot\|f\|_{s_2}^{(1-\theta)s_2}.
\end{aligned}
$$

□

推论 6.1　对任意的 $f(x)\in C[0,L]$ 及 $p>4$, 有

$$
\|f\|_{p-2}^{p-2}\leqslant\|f\|^{\frac{4}{p-2}}\cdot\|f\|_p^{\frac{p(p-4)}{p-2}},
$$

$$
\|f\|_{\frac{p}{2}}^{\frac{p}{2}}\leqslant\|f\|^{\frac{p}{p-2}}\cdot\|f\|_p^{\frac{p(p-4)}{2(p-2)}}.
$$

引理 6.2　对任意的 $f(x)\in C^1[0,L]$ 及 $p>2$, 则有

$$
\|f\|_\infty^{\frac{p}{2}}\leqslant\frac{p}{2}\|f\|_{p-2}^{\frac{p}{2}-1}\cdot|f|_1+\frac{1}{L}\|f\|_{\frac{p}{2}}^{\frac{p}{2}}.
$$

证明　设 $0\leqslant y<x\leqslant L$, 则有

$$
\begin{aligned}
&|f(x)|^{\frac{p}{2}}-|f(y)|^{\frac{p}{2}}\\
=&\int_y^x\frac{\mathrm{d}}{\mathrm{d}s}\left(|f(s)|^{\frac{p}{2}}\right)\mathrm{d}s\\
=&\frac{p}{2}\int_y^x|f(s)|^{\frac{p}{2}-1}\frac{\mathrm{d}(|f(s)|)}{\mathrm{d}s}\mathrm{d}s\\
\leqslant&\frac{p}{2}\int_0^L|f(s)|^{\frac{p}{2}-1}\left|\frac{\mathrm{d}(|f(s)|)}{\mathrm{d}s}\right|\mathrm{d}s\\
\leqslant&\frac{p}{2}\left[\int_0^L|f(s)|^{p-2}\mathrm{d}s\right]^{\frac{1}{2}}\cdot\left[\int_0^L\left(\frac{\mathrm{d}(|f(s)|)}{\mathrm{d}s}\right)^2\mathrm{d}s\right]^{\frac{1}{2}}\\
=&\frac{p}{2}\left[\int_0^L|f(s)|^{p-2}\mathrm{d}s\right]^{\frac{1}{2}}\cdot\left[\int_0^L|f'(s)|^2\mathrm{d}s\right]^{\frac{1}{2}}\\
=&\frac{p}{2}\|f\|_{p-2}^{\frac{p-2}{2}}\cdot|f|_1,
\end{aligned}
$$

即

$$
|f(x)|^{\frac{p}{2}}\leqslant\frac{p}{2}\|f\|_{p-2}^{\frac{p-2}{2}}\cdot|f|_1+|f(y)|^{\frac{p}{2}}.
$$

上式对 $0\leqslant x\leqslant y\leqslant L$ 也是成立的. 将上式关于 y 在 $[0,L]$ 上积分得到

$$
L|f(x)|^{\frac{p}{2}}\leqslant\frac{p}{2}L\|f\|_{p-2}^{\frac{p-2}{2}}\cdot|f|_1+\|f\|_{\frac{p}{2}}^{\frac{p}{2}}.
$$

因而

$$\|f\|_{\infty}^{\frac{p}{2}} \leqslant \frac{p}{2}\|f\|_{p-2}^{\frac{p-2}{2}} \cdot |f|_1 + \frac{1}{L}\|f\|_{\frac{p}{2}}^{\frac{p}{2}}.$$ □

注 6.1 在引理 6.2 的证明中, 我们默许了如下等式成立

$$\int_0^L \left[\frac{\mathrm{d}}{\mathrm{d}s}(|f(s)|)\right]^2 \mathrm{d}s = \int_0^L \left[f'(s)\right]^2 \mathrm{d}s.$$

如果 $f(x) \in C^1[0, L]$ 且在 $[0, L]$ 上只有可数个零点, 则上式是成立的.

引理 6.3 对任意的 $f \in C^1[0, L]$ 存在常数 κ 使得

$$\|f\|_p \leqslant \kappa\left(\|f\|^{1-\alpha} \cdot |f|_1^{\alpha} + \|f\|\right), \quad \alpha = \frac{1}{2} - \frac{1}{p}, \quad p \geqslant 2.$$

证明

$$\|f\|_p^p = \int_0^L |f(x)|^p \mathrm{d}x \leqslant \big(\max_{0 \leqslant x \leqslant L} |f(x)|\big)^{\frac{p}{2}} \int_0^L |f(x)|^{\frac{p}{2}} \mathrm{d}x = \|f\|_{\infty}^{\frac{p}{2}} \cdot \|f\|_{\frac{p}{2}}^{\frac{p}{2}}.$$

应用引理 6.2 得到

$$\|f\|_p^p \leqslant \left(\frac{p}{2}\|f\|_{p-2}^{\frac{p}{2}-1} \cdot |f|_1 + \frac{1}{L}\|f\|_{\frac{p}{2}}^{\frac{p}{2}}\right)\|f\|_{\frac{p}{2}}^{\frac{p}{2}}.$$

再应用推论 6.1 得到

$$\|f\|_p^p \leqslant \left(\frac{p}{2}\|f\|^{\frac{2}{p-2}} \cdot \|f\|_p^{\frac{p(p-4)}{2(p-2)}} \cdot |f|_1 + \frac{1}{L}\|f\|^{\frac{p}{p-2}} \cdot \|f\|_p^{\frac{p(p-4)}{2(p-2)}}\right)\|f\|^{\frac{p}{p-2}} \cdot \|f\|_p^{\frac{p(p-4)}{2(p-2)}},$$

即

$$\|f\|_p^{\frac{2p}{p-2}} \leqslant \frac{p}{2}\|f\|^{\frac{p+2}{p-2}} \cdot |f|_1 + \frac{1}{L}\|f\|^{\frac{2p}{p-2}}. \tag{6.4}$$

注意到当 $a \geqslant 0, b \geqslant 0, r \in (0, 1)$ 时有不等式

$$(a+b)^r \leqslant a^r + b^r.$$

由 (6.4) 得到

$$\|f\|_p \leqslant \left[\frac{p}{2}\|f\|^{\frac{p+2}{p-2}}|f|_1 + \frac{1}{L}\|f\|^{\frac{2p}{p-2}}\right]^{\frac{p-2}{2p}} \leqslant \left(\frac{p}{2}\right)^{\alpha}\|f\|^{1-\alpha} \cdot |f|_1^{\alpha} + L^{-\alpha}\|f\|.$$ □

定理 6.1 设 $u(x, t)$ 为 (6.1)–(6.3) 的解, 则有

$$\int_0^L |u(x,t)|^2 \mathrm{d}x \leqslant e^{2t}\|\varphi\|^2, \quad 0 < t \leqslant T, \tag{6.5}$$

$$\int_0^L |u_x(x,t)|^2 \mathrm{d}x \leqslant e^{c_3 t}\left(|\varphi|_1^2 + \frac{c_4}{c_3}\right), \quad 0 < t \leqslant T,$$

其中

$$c_3 = \left(1 + c_2^2\right)\kappa^6 2^5 e^{4T}\|\varphi\|^2, \quad c_4 = e^{2T}\|\varphi\|^2 + \left(1 + c_2^2\right)\kappa^6 2^5 e^{6T}\|\varphi\|^6. \tag{6.6}$$

证明　(I) 用 $\bar{u}$ 乘以 (6.1) 的两边, 并对 x 从 0 到 L 求积分, 得

$$\int_0^L u_t\bar{u}\mathrm{d}x-(1+\mathrm{i}c_1)\int_0^L u_{xx}\bar{u}\mathrm{d}x=\int_0^L |u|^2\mathrm{d}x-(1+\mathrm{i}c_2)\int_0^L |u|^4\mathrm{d}x. \tag{6.7}$$

由分部积分公式, 并利用 (6.3) 得

$$-\int_0^L u_{xx}\bar{u}\mathrm{d}x=-u_x\bar{u}|_{x=0}^L+\int_0^L |u_x|^2\mathrm{d}x=\int_0^L |u_x|^2\mathrm{d}x. \tag{6.8}$$

对 (6.7) 两边取实部, 并利用 (6.8), 得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u(x,t)|^2\mathrm{d}x+\int_0^L |u_x(x,t)|^2\,\mathrm{d}x=\int_0^l |u(x,t)|^2\mathrm{d}x-\int_0^L |u(x,t)|^4\mathrm{d}x.$$

于是

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u(x,t)|^2\mathrm{d}x\leqslant\int_0^L |u(x,t)|^2\mathrm{d}x,\quad 0<t\leqslant T.$$

由 Gronwall 不等式, 得

$$\int_0^L |u(x,t)|^2\mathrm{d}x\leqslant e^{2t}\int_0^L |u(x,0)|^2\mathrm{d}x=e^{2t}\int_0^L |\varphi(x)|^2\mathrm{d}x,\quad 0<t\leqslant T.$$

(II) 用 $\dfrac{1}{1+\mathrm{i}c_1}\bar{u}_t$ 乘以 (6.1) 的两边, 并对 x 从 0 到 L 求积分, 得

$$\frac{1}{1+\mathrm{i}c_1}\int_0^L |u_t|^2\,\mathrm{d}x-\int_0^L u_{xx}\bar{u}_t\mathrm{d}x=\frac{1}{1+\mathrm{i}c_1}\int_0^L u\bar{u}_t\mathrm{d}x-\frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\int_0^L |u|^2u\bar{u}_t\mathrm{d}x. \tag{6.9}$$

由分部积分公式, 并利用 (6.3), 得

$$-\int_0^L u_{xx}\bar{u}_t\mathrm{d}x=-u_x\bar{u}_t\Big|_{x=0}^L+\int_0^L u_x\bar{u}_{xt}\mathrm{d}x=\int_0^L u_x\bar{u}_{xt}\mathrm{d}x.$$

对 (6.9) 两边取实部, 并利用上式, 得到

$$\begin{aligned}&\frac{1}{1+c_1^2}\int_0^L |u_t|^2\mathrm{d}x+\frac{1}{2}\cdot\frac{d}{dt}\int_0^L |u_x|^2\mathrm{d}x\\ =&\mathrm{Re}\left\{\frac{1}{1+\mathrm{i}c_1}\int_0^L u\bar{u}_t\mathrm{d}x-\frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\int_0^L |u|^2u\bar{u}_t\mathrm{d}x\right\}\\ \leqslant&\frac{1}{\sqrt{1+c_1^2}}\|u\|\cdot\|u_t\|+\frac{\sqrt{1+c_2^2}}{\sqrt{1+c_1^2}}\||u|^2u\|\cdot\|u_t\|\\ \leqslant&\frac{1}{2(1+c_1^2)}\|u_t\|^2+\frac{1}{2}\|u\|^2+\frac{1}{2(1+c_2^2)}\|u_t\|^2+\frac{1+c_1^2}{2}\|u\|_6^6,\end{aligned}$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u_x|^2\mathrm{d}x \leqslant \|u\|^2 + \left(1+c_2^2\right)\|u\|_6^6, \quad 0<t\leqslant T.$$

应用定理 6.1 及不等式 $(x+y)^6 \leqslant 2^5(x^6+y^6)$ (其中 $x, y \geqslant 0$), 得

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|u(\cdot,t)|_1^2 \leqslant \|u(\cdot,t)\|^2 + (1+c_2^2)\left[\kappa(\|u(\cdot,t)\|^{\frac{2}{3}}|u(\cdot,t)|_1^{\frac{1}{3}} + \|u(\cdot,t)\|)\right]^6\\
&\leqslant \|u(\cdot,t)\|^2 + (1+c_2^2)\kappa^6 2^5\left(\|u(\cdot,t)\|^4|u(\cdot,t)|_1^2 + \|u(\cdot,t)\|^6\right)\\
&\leqslant e^{2t}\|\varphi\|^2 + (1+c_2^2)\kappa^6 2^5\left(e^{4t}\|\varphi\|^4|u(\cdot,t)|_1^2 + e^{6t}\|\varphi\|^6\right)\\
&= (1+c_2^2)\kappa^6 2^5\|\varphi\|^4 e^{4t}|u(\cdot,t)|_1^2 + e^{2t}\|\varphi\|^2 + (1+c_2^2)\kappa^6\cdot 2^5\|\varphi\|^6 e^{6t}\\
&\leqslant (1+c_2^2)k^6 2^5\|\varphi\|^4 e^{4T}|u(\cdot,t)|_1^2 + e^{2T}\|\varphi\|^2 + (1+c_2^2)\kappa^6 2^5\|\varphi\|^6 e^{6T}\\
&= c_3|u(\cdot,t)|_1^2 + c_4, \quad 0<t\leqslant T.
\end{aligned}$$

由 Gronwall 不等式, 得到

$$|u(\cdot,t)|_1^2 \leqslant e^{c_3 t}\left[|u(\cdot,0)|_1^2 + \frac{c_4}{c_3}\right], \quad 0<t\leqslant T.$$

□

由以上定理知, 存在常数 c_5 使得

$$\|u(\cdot,t)\|_\infty \leqslant c_5, \quad 0\leqslant t\leqslant T. \tag{6.10}$$

6.2 二层非线性差分格式

6.2.1 差分格式的建立

在点 $(x_j, t_{k+\frac{1}{2}})$ 处考虑方程 (6.1) 可得

$$\begin{aligned}
u_t(x_j,t_{k+\frac{1}{2}}) =& (1+\mathrm{i}c_1)u_{xx}(x_j,t_{k+\frac{1}{2}}) + u(x_j,t_{k+\frac{1}{2}})\\
&-(1+\mathrm{i}c_2)\left|u(x_j,t_{k+\frac{1}{2}})\right|^2 u(x_j,t_{k+\frac{1}{2}}),\\
&0\leqslant j\leqslant m, \quad 0\leqslant k\leqslant n-1.
\end{aligned} \tag{6.11}$$

应用数值微分公式得

$$\begin{aligned}
\delta_t U_j^{k+\frac{1}{2}} =& (1+\mathrm{i}c_1)\delta_x^2 U_j^{k+\frac{1}{2}} + U_j^{k+\frac{1}{2}} - (1+\mathrm{i}c_2)|U_j^{k+\frac{1}{2}}|^2 U_j^{k+\frac{1}{2}} + R_j^{k+\frac{1}{2}},\\
&1\leqslant j\leqslant m-1, \quad 0\leqslant k\leqslant n-1,
\end{aligned} \tag{6.12}$$

且存在常数 c_6 使得

$$|R_j^{k+\frac{1}{2}}| \leqslant c_6(\tau^2+h^2), \quad 1 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n-1. \tag{6.13}$$

对方程 (6.1) 两边关于 x 求导, 得

$$u_{xt} = (1+\mathrm{i}c_1)u_{xxx} + u_x - (1+\mathrm{i}c_2)\left(2|u|^2u_x + u^2\bar{u}_x\right).$$

应用边界条件 (6.3), 得到

$$u_{xxx}(0,t)=0, \quad u_{xxx}(L,t)=0, \quad 0<t\leqslant T. \tag{6.14}$$

应用 (6.3) 和 (6.14) 及引理 1.2 可得

$$u_{xx}(0,t) = \frac{2}{h^2}\left[u(x_1,t)-u(x_0,t)\right] + O(h^2), \tag{6.15}$$

$$u_{xx}(L,t) = -\frac{2}{h^2}\left[u(x_m,t)-u(x_{m-1},t)\right] + O(h^2). \tag{6.16}$$

于是

$$\begin{aligned}
u_{xx}(0,t_{k+\frac{1}{2}}) &= \frac{1}{2}\left[u_{xx}(0,t_k)+u_{xx}(0,t_{k+1})\right] + O(\tau^2)\\
&= \frac{1}{2}\left[\frac{2}{h^2}(U_1^k-U_0^k) + \frac{2}{h^2}(U_1^{k+1}-U_0^{k+1})\right] + O(\tau^2+h^2)\\
&= \frac{2}{h}\delta_x U_{\frac{1}{2}}^{k+\frac{1}{2}} + O(\tau^2+h^2),
\end{aligned} \tag{6.17}$$

$$\begin{aligned}
u_{xx}(x_m,t_{k+\frac{1}{2}}) &= \frac{1}{2}\left[u_{xx}(x_m,t_k)+u_{xx}(x_m,t_{k+1})\right] + O(\tau^2)\\
&= \frac{1}{2}\left[-\frac{2}{h^2}(U_m^k-U_{m-1}^k) - \frac{2}{h^2}(U_m^{k+1}-U_{m-1}^{k+1})\right] + O(\tau^2+h^2)\\
&= -\frac{2}{h}\delta_x U_{m-\frac{1}{2}}^{k+\frac{1}{2}} + O(\tau^2+h^2).
\end{aligned} \tag{6.18}$$

在 (6.11) 中应用 (6.17)–(6.18) 可得

$$\begin{aligned}
\delta_t U_0^{k+\frac{1}{2}} &= (1+\mathrm{i}c_1)\frac{2}{h}\delta_x U_{\frac{1}{2}}^{k+\frac{1}{2}} + U_0^{k+\frac{1}{2}} - (1+\mathrm{i}c_2)|U_0^{k+\frac{1}{2}}|^2U_0^{k+\frac{1}{2}} + R_0^{k+\frac{1}{2}},\\
&\qquad 0\leqslant k\leqslant n-1,
\end{aligned} \tag{6.19}$$

$$\begin{aligned}
\delta_t U_m^{k+\frac{1}{2}} &= (1+\mathrm{i}c_1)\left(-\frac{2}{h}\delta_x U_{m-\frac{1}{2}}^{k+\frac{1}{2}}\right) + U_m^{k+\frac{1}{2}} - (1+\mathrm{i}c_2)|U_m^{k+\frac{1}{2}}|^2U_m^{k+\frac{1}{2}} + R_m^{k+\frac{1}{2}},\\
&\qquad 0\leqslant k\leqslant n-1,
\end{aligned} \tag{6.20}$$

且存在常数 c_7 使得

$$\left|R_0^{k+\frac{1}{2}}\right| \leqslant c_7(\tau^2+h^2), \quad 0 \leqslant k \leqslant n-1, \tag{6.21}$$

$$\left|R_m^{k+\frac{1}{2}}\right| \leqslant c_7(\tau^2+h^2), \quad 0 \leqslant k \leqslant n-1. \tag{6.22}$$

在 (6.12), (6.19)–(6.20) 中略去小量项, 并注意到初值条件

$$U_j^0 = \varphi(x_j), \quad 0 \leqslant j \leqslant m. \tag{6.23}$$

对 (6.1)–(6.3) 建立如下差分格式

$$\delta_t u_0^{k+\frac{1}{2}} = (1+\mathrm{i}c_1)\frac{2}{h}\delta_x u_{\frac{1}{2}}^{k+\frac{1}{2}} + u_0^{k+\frac{1}{2}} - (1+\mathrm{i}c_2)\left|u_0^{k+\frac{1}{2}}\right|^2 u_0^{k+\frac{1}{2}},$$
$$0 \leqslant k \leqslant n-1, \tag{6.24}$$

$$\delta_t u_j^{k+\frac{1}{2}} = (1+\mathrm{i}c_1)\delta_x^2 u_j^{k+\frac{1}{2}} + u_j^{k+\frac{1}{2}} - (1+\mathrm{i}c_2)\left|u_j^{k+\frac{1}{2}}\right|^2 u_j^{k+\frac{1}{2}},$$
$$1 \leqslant j \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{6.25}$$

$$\delta_t u_m^{k+\frac{1}{2}} = (1+\mathrm{i}c_1)\left(-\frac{2}{h}\delta_x u_{m-\frac{1}{2}}^{k+\frac{1}{2}}\right) + u_m^{k+\frac{1}{2}} - (1+\mathrm{i}c_2)\left|u_m^{k+\frac{1}{2}}\right|^2 u_m^{k+\frac{1}{2}},$$
$$0 \leqslant k \leqslant n-1, \tag{6.26}$$

$$u_j^0 = \varphi(x_j), \quad 0 \leqslant j \leqslant m. \tag{6.27}$$

6.2.2 差分格式解的存在性

差分格式 (6.24)–(6.27) 是一个二层非线性差分格式. 当 k 层的值 u^k 已确定时, 可将其看成是关于平均值 $\{u_j^{k+\frac{1}{2}} \,|\, 0 \leqslant j \leqslant m\}$ 的非线性方程组. 当求得 $\{u_j^{k+\frac{1}{2}} \,|\, 0 \leqslant j \leqslant m\}$ 时, 有

$$u_j^{k+1} = 2u_j^{k+\frac{1}{2}} - u_j^k, \quad 0 \leqslant j \leqslant m.$$

定理 6.2 差分格式 (6.24)–(6.27) 存在解.

证明 差分格式 (6.24)–(6.26) 可统一写为

$$\delta_t u_j^{k+\frac{1}{2}} = (1+\mathrm{i}c_1)\delta_x^2 u_j^{k+\frac{1}{2}} + u_j^{k+\frac{1}{2}} - (1+\mathrm{i}c_2)\left|u_j^{k+\frac{1}{2}}\right|^2 u_j^{k+\frac{1}{2}},$$
$$0 \leqslant j \leqslant m, \quad 0 \leqslant k \leqslant n-1, \tag{6.28}$$

其中

$$\delta_x^2 u_j = \begin{cases} \dfrac{2}{h}\delta_x u_{\frac{1}{2}}, & j=0, \\ \dfrac{1}{h}\left(\delta_x u_{j+\frac{1}{2}} - \delta_x u_{j-\frac{1}{2}}\right), & 1 \leqslant j \leqslant m-1, \\ -\dfrac{2}{h}\delta_x u_{m-\frac{1}{2}}, & j=m. \end{cases} \tag{6.29}$$

设 u^k 已确定. 令

$$w_j = u_j^{k+\frac{1}{2}}, \quad 0 \leqslant j \leqslant m.$$

则 (6.28) 可写为

$$\frac{2}{\tau}\left(w_j - u_j^k\right) = (1+\mathrm{i}c_1)\delta_x^2 w_j + w_j - (1+\mathrm{i}c_2)|w_j|^2 w_j, \quad 0 \leqslant j \leqslant m. \tag{6.30}$$

定义 $\Pi(w): \mathcal{U}_h \longrightarrow \mathcal{U}_h$

$$\Pi(w_j) = \frac{2}{\tau}\left(w_j - u_j^k\right) - (1+\mathrm{i}c_1)\delta_x^2 w_j - w_j + (1+\mathrm{i}c_2)|w_j|^2 w_j, \quad 0 \leqslant j \leqslant m.$$

现在来计算 $(\Pi(w), w)$.

$$\begin{aligned}(\Pi(w), w) = &\frac{2}{\tau}\left[(w,w) - (u^k, w)\right] - (1+\mathrm{i}c_1)(\delta_x^2 w, w) - (w,w)\\ &+ (1+\mathrm{i}c_2)(|w|^2 w, w).\end{aligned}$$

注意到

$$\begin{aligned}-(\delta_x^2 w, w) &= -h\left[\frac{1}{2}(\delta_x^2 w_0)\bar{w}_0 + \sum_{j=1}^{m-1}(\delta_x^2 w_j)\bar{w}_j + \frac{1}{2}(\delta_x^2 w_m)\bar{w}_m\right]\\ &= -\left[\left(\delta_x w_{\frac{1}{2}}\right)\bar{w}_0 + \sum_{j=1}^{m-1}\left(\delta_x w_{j+\frac{1}{2}} - \delta_x w_{j-\frac{1}{2}}\right)\bar{w}_j - \left(\delta_x w_{m-\frac{1}{2}}\right)\bar{w}_m\right]\\ &= h\sum_{j=0}^{m-1}\left|\delta_x w_{j+\frac{1}{2}}\right|^2,\end{aligned}$$

有

$$\begin{aligned}\mathrm{Re}(\Pi(w), w) &\geqslant \frac{2}{\tau}\left[(w,w) - \mathrm{Re}(u^k, w)\right] + |w|_1^2 - \|w\|^2\\ &\geqslant \frac{2}{\tau}\left(\|w\|^2 - \|u^k\|\cdot\|w\|\right) - \|w\|^2\\ &= \frac{2-\tau}{\tau}\left(\|w\| - \frac{2}{2-\tau}\|u^k\|\right)\|w\|,\end{aligned}$$

当 $\tau < 2$, 且 $\|w\| = \dfrac{2}{2-\tau}\|u^k\|$ 时

$$\mathrm{Re}(\Pi(w), w) \geqslant 0.$$

由 Browder 定理 (定理 1.3) 知 (6.30) 存在解. □

6.2.3 差分格式解的有界性

记

$$w_j = \begin{cases} 1, & 1 \leqslant j \leqslant m-1, \\ \dfrac{1}{2}, & j = 0, m. \end{cases}$$

引理 6.4 设 s_1 和 s_2 是两个非负常数, $\theta \in (0,1)$, 对任意的 $u \in \mathcal{U}_h$, 有

$$\|u\|_{\theta s_1+(1-\theta)s_2}^{\theta s_1+(1-\theta)s_2} \leqslant \|u\|_{s_1}^{\theta s_1} \cdot \|u\|_{s_2}^{(1-\theta)s_2}.$$

证明 应用求和的 Hölder 不等式, 得到

$$\begin{aligned}
&\|u\|_{\theta s_1+(1-\theta)s_2}^{\theta s_1+(1-\theta)s_2} \\
=& h\sum_{j=0}^{m} w_j|u_j|^{\theta s_1+(1-\theta)s_2} \\
\leqslant& \left[h\sum_{j=0}^{m} w_j\left(|u_j|^{\theta s_1}\right)^{\frac{1}{\theta}}\right]^{\theta} \cdot \left[h\sum_{j=0}^{m} w_j\left(|u_j|^{(1-\theta)s_2}\right)^{\frac{1}{1-\theta}}\right]^{1-\theta} \\
=& \left[h\sum_{j=0}^{m} w_j\left(|u_j|^{s_1}\right)\right]^{\theta} \cdot \left[h\sum_{j=0}^{m} w_j\left(|u_j|^{s_2}\right)\right]^{1-\theta} \\
=& \|u\|_{s_1}^{\theta s_1} \cdot \|u\|_{s_2}^{(1-\theta)s_2}.
\end{aligned}$$

□

推论 6.2 对任意的 $u \in \mathcal{U}_h$ 及 $p > 4$, 有

$$\|u\|_{p-2}^{p-2} \leqslant \|u\|^{\frac{4}{p-2}} \cdot \|u\|_p^{\frac{p(p-4)}{p-2}},$$

$$\|u\|_{\frac{p}{2}}^{\frac{p}{2}} \leqslant \|u\|^{\frac{p}{p-2}} \cdot \|u\|_p^{\frac{p(p-4)}{2(p-2)}}.$$

引理 6.5 对任意的 $u \in \mathcal{U}_h$ 及 $p > 2$,

$$\|u\|_\infty^{\frac{p}{2}} \leqslant p\|u\|_{p-2}^{\frac{p}{2}-1} \cdot |u|_1 + \frac{1}{L}\|u\|_{\frac{p}{2}}^{\frac{p}{2}}.$$

证明 对于任意 $l(0 \leqslant l \leqslant m-1)$, 存在 ξ_l 介于 $|u_l|$ 和 $|u_{l+1}|$ 之间, 使得

$$|u_{l+1}|^{\frac{p}{2}} - |u_l|^{\frac{p}{2}} = \frac{p}{2}\xi_l^{\frac{p}{2}-1}\left(|u_{l+1}| - |u_l|\right).$$

因而

$$\begin{aligned}
\left||u_{l+1}|^{\frac{p}{2}} - |u_l|^{\frac{p}{2}}\right| &\leqslant \frac{p}{2}\left(|u_{l+1}|^{\frac{p}{2}-1} + |u_l|^{\frac{p}{2}-1}\right)\left||u_{l+1}| - |u_l|\right| \\
&\leqslant \frac{p}{2}\left(|u_{l+1}|^{\frac{p}{2}-1} + |u_l|^{\frac{p}{2}-1}\right)\left|u_{l+1} - u_l\right|.
\end{aligned} \tag{6.31}$$

设 $s>j$, 应用 (6.31), 有

$$\begin{aligned}&|u_s|^{\frac{p}{2}}-|u_j|^{\frac{p}{2}}\\&=\sum_{l=j}^{s-1}\left(|u_{l+1}|^{\frac{p}{2}}-|u_l|^{\frac{p}{2}}\right)\\&\leqslant\sum_{l=j}^{s-1}\frac{p}{2}\left[|u_{l+1}|^{\frac{p}{2}-1}+|u_l|^{\frac{p}{2}-1}\right]\cdot\left|u_{l+1}-u_l\right|\\&=h\sum_{l=j}^{s-1}\frac{p}{2}\left[|u_{l+1}|^{\frac{p}{2}-1}+|u_l|^{\frac{p}{2}-1}\right]\cdot\left|\delta_x u_{l+\frac{1}{2}}\right|\\&\leqslant p\|u\|_{p-2}^{\frac{p-2}{2}}\cdot|u|_1.\end{aligned}$$

上式对 $s\leqslant j$ 也是成立的. 因而

$$|u_s|^{\frac{p}{2}}\leqslant p\|u\|_{p-2}^{\frac{p-2}{2}}\cdot|u|_1+|u_j|^{\frac{p}{2}},\quad 0\leqslant s,j\leqslant m.$$

将上式乘以 hw_j, 并对 j 从 0 到 m 求和, 得

$$L|u_s|^{\frac{p}{2}}\leqslant Lp\|u\|_{p-2}^{\frac{p-2}{2}}\cdot|u|_1+\|u\|_{\frac{p}{2}}^{\frac{p}{2}},\quad 0\leqslant s\leqslant m.$$

易得

$$\|u\|_\infty^{\frac{p}{2}}\leqslant p\|u\|_{p-2}^{\frac{p}{2}-1}\cdot|u|_1+\frac{1}{L}\|u\|_{\frac{p}{2}}^{\frac{p}{2}}.$$

□

引理 6.6　对任意的 $u\in\mathcal{U}_h$ 有

$$\|u\|_p\leqslant\kappa\left(\|u\|^{1-\alpha}|u|_1^\alpha+\|u\|\right),\quad \alpha=\frac{1}{2}-\frac{1}{p},\quad p\geqslant 2,$$

其中 $\kappa=\max\{p^\alpha,L^{-\alpha}\}$.

证明

$$\|u\|_p^p=h\sum_{j=0}^m w_j|u_j|^p\leqslant\max_{0\leqslant j\leqslant m}|u_j|^{\frac{p}{2}}\cdot h\sum_{j=0}^m w_j|u_j|^{\frac{p}{2}}=\|u\|_\infty^{\frac{p}{2}}\|u\|_{\frac{p}{2}}^{\frac{p}{2}}.$$

应用引理 6.5 得到

$$\|u\|_p^p\leqslant\left(p\|u\|_{p-2}^{\frac{p}{2}-1}|u|_1+\frac{1}{L}\|u\|_{\frac{p}{2}}^{\frac{p}{2}}\right)\|u\|_{\frac{p}{2}}^{\frac{p}{2}}.$$

再应用推论 6.2 得到

$$\begin{aligned}\|u\|_p^p&\leqslant\left[p\|u\|^{\frac{2}{p-2}}\|u\|_p^{\frac{p(p-4)}{2(p-2)}}|u|_1+\frac{1}{L}\|u\|^{\frac{p}{p-2}}\|u\|_p^{\frac{p(p-4)}{2(p-2)}}\right]\|u\|^{\frac{p}{p-2}}\|u\|_p^{\frac{p(p-4)}{2(p-2)}}\\&=\left[p\|u\|^{\frac{p+2}{p-2}}|u|_1+\frac{1}{L}\|u\|^{\frac{2p}{p-2}}\right]\|u\|_p^{\frac{p(p-4)}{p-2}},\end{aligned}$$

即

$$\|u\|_p^{\frac{2p}{p-2}} \leqslant p\|u\|^{\frac{p+2}{p-2}}|u|_1 + \frac{1}{L}\|u\|^{\frac{2p}{p-2}}. \tag{6.32}$$

注意到当 $a \geqslant 0, b \geqslant 0, r \in (0,1)$ 时有不等式

$$(a+b)^r \leqslant a^r + b^r.$$

由 (6.32) 得到

$$\|u\|_p \leqslant \left[p\|u\|^{\frac{p+2}{p-2}}|u|_1 + \frac{1}{L}\|u\|^{\frac{2p}{p-2}}\right]^{\frac{p-2}{2p}} \leqslant p^\alpha \|u\|^{1-\alpha}|u|_1^\alpha + L^{-\alpha}\|u\|. \qquad \square$$

类似于定理 6.1, 关于差分格式 (6.24)–(6.27) 的解有如下结论.

定理 6.3 设 $\{u_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 是差分格式 (6.24)–(6.27) 的解, 有

$$\|u^k\| \leqslant e^{\frac{3}{2}T}\|u^0\|, \quad 1 \leqslant k \leqslant n, \tag{6.33}$$

$$|u^k|_1^2 \leqslant e^{3c_8T}\left(|u^0|_1^2 + \frac{c_9}{c_8}\right), \quad 1 \leqslant k \leqslant n, \tag{6.34}$$

其中 $c_8 = (1+c_2^2)\kappa^6 2^4 e^{6T}\|u^0\|^4$, $c_9 = \frac{1}{2}e^{3T}\|u^0\|^2 + (1+c_2^2)\kappa^6 2^4 e^{9T}\|u^0\|^6$.

证明 (I) 用 $u^{k+\frac{1}{2}}$ 与 (6.28) 作内积, 得

$$\begin{aligned}\left(\delta_t u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) =& (1+\mathrm{i}c_1)\left(\delta_x^2 u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) + \left(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) \\ &- (1+\mathrm{i}c_2)\left(|u^{k+\frac{1}{2}}|^2 u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}\right) \\ =& -(1+\mathrm{i}c_1)|u^{k+\frac{1}{2}}|_1^2 + \|u^{k+\frac{1}{2}}\|^2 - (1+\mathrm{i}c_2)\left(|u^{k+\frac{1}{2}}|^4, 1\right).\end{aligned}$$

对上式两边取实部, 得到

$$\begin{aligned}\frac{1}{2\tau}\left(\|u^{k+1}\|^2 - \|u^k\|^2\right) &= -|u^{k+\frac{1}{2}}|_1^2 + \|u^{k+\frac{1}{2}}\|^2 - \left(|u^{k+\frac{1}{2}}|^4, 1\right) \\ &\leqslant \|u^{k+\frac{1}{2}}\|^2 \leqslant \left(\frac{\|u^{k+1}\| + \|u^k\|}{2}\right)^2, \quad 0 \leqslant k \leqslant n-1,\end{aligned}$$

即

$$\left(1 - \frac{\tau}{2}\right)\|u^{k+1}\| \leqslant \left(1 + \frac{\tau}{2}\right)\|u^k\|, \quad 0 \leqslant k \leqslant n-1.$$

当 $\tau \leqslant \frac{2}{3}$ 时

$$\|u^{k+1}\| \leqslant \left(1 + \frac{3}{2}\tau\right)\|u^k\|, \quad 0 \leqslant k \leqslant n-1.$$

递推得

$$\|u^{k+1}\| \leqslant e^{\frac{3}{2}T}\|u^0\|, \quad 0 \leqslant k \leqslant n-1.$$

(II) 用 $\dfrac{1}{1-\mathrm{i}c_1}\delta_t u^{k+\frac{1}{2}}$ 与 (6.28) 作内积, 得

$$\begin{aligned}&\frac{1}{1+\mathrm{i}c_1}\left(\delta_t u^{k+\frac{1}{2}}, \delta_t u^{k+\frac{1}{2}}\right) - \left(\delta_x^2 u^{k+\frac{1}{2}}, \delta_t u^{k+\frac{1}{2}}\right)\\ =&\frac{1}{1+\mathrm{i}c_1}\left(u^{k+\frac{1}{2}}, \delta_t u^{k+\frac{1}{2}}\right) - \frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\left(\left|u^{k+\frac{1}{2}}\right|^2 u^{k+\frac{1}{2}}, \delta_t u^{k+\frac{1}{2}}\right).\end{aligned}$$

对上式两边取实部, 得到

$$\begin{aligned}&\frac{1}{1+c_1^2}\|\delta_t u^{k+\frac{1}{2}}\|^2 + \frac{1}{2\tau}\left(|u^{k+1}|_1^2 - |u^k|_1^2\right)\\ \leqslant&\frac{1}{\sqrt{1+c_1^2}}\|u^{k+\frac{1}{2}}\| \cdot \|\delta_t u^{k+\frac{1}{2}}\| + \frac{\sqrt{1+c_2^2}}{\sqrt{1+c_1^2}}\|u^{k+\frac{1}{2}}\|_6^3 \cdot \|\delta_t u^{k+\frac{1}{2}}\|\\ \leqslant&\frac{1}{2(1+c_1^2)}\|\delta_t u^{k+\frac{1}{2}}\|^2 + \frac{1}{2}\|u^{k+\frac{1}{2}}\|^2 + \frac{1}{2(1+c_1^2)}\|\delta_t u^{k+\frac{1}{2}}\|^2 + \frac{1+c_2^2}{2}\|u^{k+\frac{1}{2}}\|_6^6,\\ &\qquad 0 \leqslant k \leqslant n-1.\end{aligned}$$

应用引理 6.6，得

$$\begin{aligned}&\frac{1}{2\tau}\left(|u^{k+1}|_1^2 - |u^k|_1^2\right)\\ \leqslant&\ \frac{1}{2}\|u^{k+\frac{1}{2}}\|^2 + \frac{1+c_2^2}{2}\|u^{k+\frac{1}{2}}\|_6^6\\ \leqslant&\ \frac{1}{2}\|u^{k+\frac{1}{2}}\|^2 + \frac{1+c_2^2}{2}\left[\kappa\left(|u^{k+\frac{1}{2}}|_1^{\frac{1}{3}}\|u^{k+\frac{1}{2}}\|^{\frac{2}{3}} + \|u^{k+\frac{1}{2}}\|\right)\right]^6\\ \leqslant&\ \frac{1}{2}\|u^{k+\frac{1}{2}}\|^2 + \frac{1+c_2^2}{2}\kappa^6 2^5\left(|u^{k+\frac{1}{2}}|_1^2\|u^{k+\frac{1}{2}}\|^4 + \|u^{k+\frac{1}{2}}\|^6\right).\end{aligned} \tag{6.35}$$

由 (6.33) 可知

$$\|u^{k+\frac{1}{2}}\| \leqslant e^{\frac{3}{2}T}\|u^0\|, \quad 0 \leqslant k \leqslant n-1, \tag{6.36}$$

将 (6.36) 代入 (6.35), 得到

$$\begin{aligned}&\frac{1}{2\tau}\left(|u^{k+1}|_1^2 - |u^k|_1^2\right) \leqslant c_8|u^{k+\frac{1}{2}}|_1^2 + c_9\\ \leqslant&\ \frac{c_8}{2}\left(|u^{k+1}|_1^2 + |u^k|_1^2\right) + c_9, \quad 0 \leqslant k \leqslant n-1,\end{aligned}$$

或

$$(1-c_8\tau)\,|u^{k+1}|_1^2 \leqslant (1+c_8\tau)|u^k|_1^2 + 2c_9\tau, \quad 0 \leqslant k \leqslant n-1.$$

当 $c_8\tau \leqslant \dfrac{1}{3}$ 时,

$$|u^{k+1}|_1^2 \leqslant (1+3c_8\tau)\,|u^k|_1^2 + 3c_9\tau, \quad 0 \leqslant k \leqslant n-1.$$

由 Gronwall 不等式, 得

$$|u^{k+1}|_1^2 \leqslant e^{3c_8T}\left(|u^0|_1^2+\frac{c_9}{c_8}\right),\quad 0\leqslant k\leqslant n-1. \qquad \square$$

由定理 6.3 及引理 1.1 可知存在常数 c_{10} 使得

$$\|u^k\|_\infty \leqslant c_{10},\quad 0\leqslant k\leqslant n. \tag{6.37}$$

6.2.4 差分格式解的唯一性

定理 6.4 当 $\tau < 2/(1+4\sqrt{1+c_2^2}\ c_{10}^2)$ 时, 差分格式 (6.24)-(6.27) 的解是唯一的.

证明 由定理 6.2 的证明过程可知只要证明 (6.30) 存在唯一解. 设 (6.30) 另有解 $\{v_j\,|\,0\leqslant j\leqslant m\}$ 满足

$$\frac{2}{\tau}\left(v_j-u_j^k\right)=(1+\mathrm{i}c_1)\,\delta_x^2v_j+v_j-(1+\mathrm{i}c_2)|v_j|^2v_j,\quad 0\leqslant j\leqslant m. \tag{6.38}$$

由 (6.37) 知

$$\|w\|_\infty\leqslant c_{10},\quad \|v\|_\infty\leqslant c_{10}.$$

令

$$z_j=w_j-v_j,\quad 0\leqslant j\leqslant m,$$

将 (6.30) 和 (6.38) 相减, 得

$$\frac{2}{\tau}z_j=(1+\mathrm{i}c_1)\delta_x^2z_j+z_j-(1+\mathrm{i}c_2)\left(|w_j|^2w_j-|v_j|^2v_j\right),\quad 0\leqslant j\leqslant m.$$

用 z 与上式作内积, 得

$$\frac{2}{\tau}\|z\|^2=(1+\mathrm{i}c_1)\left(\delta_x^2z,z\right)+(z,z)-(1+\mathrm{i}c_2)\left(|w|^2w-|v|^2v,z\right). \tag{6.39}$$

注意到

$$\left||w_j|^2w_j-|v_j|^2v_j\right|\leqslant(|w_j|+|v_j|)^2\,|w_j-v_j|\leqslant 4c_{10}^2|z_j|,$$

可得

$$\left|\left(|w|^2w-|v|^2v,z\right)\right|\leqslant 4c_{10}^2\|z\|^2. \tag{6.40}$$

另外

$$(\delta_x^2z,z)=-|z|_1^2. \tag{6.41}$$

在 (6.39) 两边取实部, 并利用 (6.40) 和 (6.41), 得到

$$\begin{aligned}\frac{2}{\tau}\|z\|^2&=-|z|_1^2+\|z\|^2+\sqrt{1+c_2^2}\cdot 4c_{10}^2\|z\|^2\\&\leqslant\left(1+4\sqrt{1+c_2^2}\ c_{10}^2\right)\|z\|^2,\end{aligned}$$

当 $\tau<2/\left(1+4\sqrt{1+c_2^2}\ c_{10}^2\right)$ 时, $\|z\|^2=0$. $\square$

6.2.5 差分格式解的收敛性

定理 6.5 设 $\{U_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为问题 (6.1)–(6.3) 的解, $\{u_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (6.24)–(6.27) 的解. 记

$$e_j^k = U_j^k - u_j^k, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n,$$

则存在常数 c_{11} 使得

$$\left|e^k\right|_1 \leqslant c_{11}\left(\tau^2 + h^2\right), \quad 0 \leqslant k \leqslant n. \tag{6.42}$$

证明 将 (6.19), (6.12), (6.20), (6.23) 和 (6.24) –(6.27) 相减, 可得误差方程组

$$\begin{aligned}\delta_t e_j^{k+\frac{1}{2}} = {} & (1+\mathrm{i}c_1)\delta_x^2 e_j^{k+\frac{1}{2}} + e_j^{k+\frac{1}{2}} - (1+\mathrm{i}c_2)\Big(|U_j^{k+\frac{1}{2}}|^2 U_j^{k+\frac{1}{2}} - |u_j^{k+\frac{1}{2}}|^2 u_j^{k+\frac{1}{2}}\Big)\\ & + R_j^{k+\frac{1}{2}}, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n-1,\end{aligned} \tag{6.43}$$

$$e_j^0 = 0, \quad 0 \leqslant j \leqslant m. \tag{6.44}$$

由 (6.10) 存在常数 c_5 使得

$$\left|U_j^k\right| \leqslant c_5, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n. \tag{6.45}$$

由 (6.37) 和 (6.45), 知

$$\begin{aligned}&\Big||U_j^{k+\frac{1}{2}}|^2 U_j^{k+\frac{1}{2}} - |u_j^{k+\frac{1}{2}}|^2 u_j^{k+\frac{1}{2}}\Big|\\ \leqslant {} & \big(|U_j^{k+\frac{1}{2}}| + |u_j^{k+\frac{1}{2}}|\big)^2 \big|U_j^{k+\frac{1}{2}} - u_j^{k+\frac{1}{2}}\big|\\ \leqslant {} & (c_5 + c_{10})^2 \big|e_j^{k+\frac{1}{2}}\big|, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n-1.\end{aligned} \tag{6.46}$$

(I) 用 $e^{k+\frac{1}{2}}$ 与 (6.43) 作内积, 并取实部, 得

$$\begin{aligned}&\frac{1}{2\tau}\left(\|e^{k+1}\|^2 - \|e^k\|^2\right)\\ \leqslant {} & -|e^{k+\frac{1}{2}}|_1^2 + \|e^{k+\frac{1}{2}}\|^2 + \sqrt{1+c_2^2}\,(c_5+c_{10})^2\,\|e^{k+\frac{1}{2}}\|^2 + \|R^{k+\frac{1}{2}}\|\cdot\|e^{k+\frac{1}{2}}\|\\ \leqslant {} & \big[1 + \sqrt{1+c_2^2}(c_5+c_{10})^2\big]\|e^{k+\frac{1}{2}}\|^2 + \|R^{k+\frac{1}{2}}\|\cdot\|e^{k+\frac{1}{2}}\|, \quad 0 \leqslant k \leqslant n-1.\end{aligned}$$

由上式可得

$$\begin{aligned}\frac{1}{\tau}\left(\|e^{k+1}\| - \|e^k\|\right) \leqslant \Big[1 + \sqrt{1+c_2^2}(c_5+c_{10})^2\Big]\left\|e^{k+\frac{1}{2}}\right\| + \left\|R^{k+\frac{1}{2}}\right\|,&\\ 0 \leqslant k \leqslant n-1.&\end{aligned} \tag{6.47}$$

由 (6.13), (6.21), (6.22) 知

$$\|R^{k+\frac{1}{2}}\| \leqslant \sqrt{L}\max\{c_6, c_7\}\left(\tau^2+h^2\right), \quad 0 \leqslant k \leqslant n-1. \tag{6.48}$$

由 (6.47) 和 (6.48) 知

$$\begin{aligned}&\left\{1-\frac{1}{2}[1+\sqrt{1+c_2^2}(c_5+c_{10})^2]\tau\right\}\|e^{k+1}\|^2\\ \leqslant&\left\{1+\frac{1}{2}[1+\sqrt{1+c_2^2}\,(c_5+c_{10})^2\,]\tau\right\}\|e^k\|^2\\ &+\sqrt{L}\max\{c_6, c_7\}\,\tau(\tau^2+h^2), \quad 0 \leqslant k \leqslant n-1.\end{aligned}$$

当 $\frac{1}{2}[\sqrt{1+c_2^2}\,(c_5+c_{10})^2]\tau \leqslant \frac{1}{3}$ 时,

$$\begin{aligned}\|e^{k+1}\| \leqslant&\left\{1+\frac{3}{2}[1+\sqrt{1+c_2^2}\,(c_5+c_{10})^2]\tau\right\}\|e^k\|\\ &+\frac{3}{2}\sqrt{L}\max\{c_6, c_7\}\,\tau\left(\tau^2+h^2\right), \quad 0 \leqslant k \leqslant n-1.\end{aligned}$$

再由 Gronwall 不等式, 得到

$$\begin{aligned}\|e^k\| \leqslant& e^{\frac{3}{2}\left[1+\sqrt{1+c_2^2}(c_5+c_{10})^2\right]T} \cdot \frac{\sqrt{L}\max\{c_6, c_7\}}{1+\sqrt{1+c_2^2}\,(c_5+c_{10})^2}\left(\tau^2+h^2\right)\\ \equiv& c_{12}\left(\tau^2+h^2\right), \quad 0 \leqslant k \leqslant n.\end{aligned} \tag{6.49}$$

(II) 用 $\frac{1}{1-\mathrm{i}c_1}\delta_t e^{k+\frac{1}{2}}$ 与 (6.43) 作内积, 得

$$\begin{aligned}&\frac{1}{1+\mathrm{i}c_1}\|\delta_t e^{k+\frac{1}{2}}\|^2-\left(\delta_x^2 e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}\right)\\ =&\frac{1}{1+\mathrm{i}c_1}\left(e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}\right)-\frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\left(|U^{k+\frac{1}{2}}|^2 U^{k+\frac{1}{2}}-|u^{k+\frac{1}{2}}|^2 u^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}\right)\\ &+\frac{1}{1+\mathrm{i}c_1}\left(R^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}\right).\end{aligned}$$

取实部, 得到

$$\begin{aligned}&\frac{1}{1+c_1^2}\|\delta_t e^{k+\frac{1}{2}}\|^2+\frac{1}{2\tau}\left(|e^{k+1}|_1^2-|e^k|_1^2\right)\\ =&\mathrm{Re}\left\{\frac{1}{1+\mathrm{i}c_1}\left(e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{Re}\left\{-\frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\left(|U^{k+\frac{1}{2}}|^2U^{k+\frac{1}{2}}-|u^{k+\frac{1}{2}}|u^{k+\frac{1}{2}},\delta_te^{k+\frac{1}{2}}\right)\right\}\\
&+\mathrm{Re}\left\{\frac{1}{1+\mathrm{i}c_1}\left(R^{k+\frac{1}{2}},\delta_te^{k+\frac{1}{2}}\right)\right\}\\
\leqslant&\frac{1}{\sqrt{1+c_1^2}}\|e^{k+\frac{1}{2}}\|\cdot\|\delta_te^{k+\frac{1}{2}}\|+\frac{\sqrt{1+c_2^2}}{\sqrt{1+c_1^2}}\left(c_5+c_{10}\right)^2\|e^{k+\frac{1}{2}}\|\cdot\|\delta_te^{k+\frac{1}{2}}\|\\
&+\frac{1}{\sqrt{1+c_1^2}}\|R^{k+\frac{1}{2}}\|\cdot\|\delta_te^{k+\frac{1}{2}}\|\\
\leqslant&\frac{1}{4(1+c_1^2)}\|\delta_te^{k+\frac{1}{2}}\|^2+\|e^{k+\frac{1}{2}}\|^2+\frac{1}{4(1+c_1^2)}\|\delta_te^{k+\frac{1}{2}}\|^2\\
&+(1+c_2^2)(c_5+c_{10})^4\|e^{k+\frac{1}{2}}\|^2\\
&+\frac{1}{2(1+c_1^2)}\|\delta_te^{k+\frac{1}{2}}\|^2+\frac{1}{2}\|R^{k+\frac{1}{2}}\|^2,\quad 0\leqslant k\leqslant n-1,
\end{aligned}
$$

即

$$
\frac{1}{2\tau}\left(|e^{k+1}|_1^2-|e^k|_1^2\right)\leqslant\left[1+(1+c_2^2)(c_5+c_{10})^4\right]\|e^{k+\frac{1}{2}}\|^2+\frac{1}{2}\|R^{k+\frac{1}{2}}\|^2,\quad 1\leqslant k\leqslant n-1.
$$

由 (6.48) 和 (6.49), 得

$$
\begin{aligned}
\frac{1}{2\tau}\left(|e^{k+1}|_1^2-|e^k|_1^2\right)\leqslant&\left[1+(1+c_2^2)(c_5+c_{10})^4\right]c_{12}^2(\tau^2+h^2)^2\\
&+\frac{L}{2}\max\left\{c_6^2,c_7^2\right\}(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant n-1,
\end{aligned}
$$

即

$$
|e^{k+1}|_1^2\leqslant|e^k|_1^2+\left\{2\left[1+(1+c_2^2)(c_5+c_{10})^4\right]c_{12}^2+L\max\{c_6^2,c_7^2\}\right\}\tau\left(\tau^2+h^2\right)^2,\quad 0\leqslant k\leqslant n-1.
$$

递推得到

$$
|e^k|_1^2\leqslant\left\{2\left[1+(1+c_2^2)(c_5+c_{10})^4\right]c_{12}^2+L\max\{c_6^2,c_7^2\}\right\}T(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n.
$$

□

6.3 三层线性化差分格式

6.3.1 差分格式的建立

在结点 (x_j,t_k) 处考虑方程 (6.1) 有

$$u_t(x_j,t_k)=(1+\mathrm{i}c_1)u_{xx}(x_j,t_k)+u(x_j,t_k)-(1+\mathrm{i}c_2)|u(x_j,t_k)|^2u(x_j,t_k),$$
$$0\leqslant j\leqslant m,\ 1\leqslant k\leqslant n-1. \tag{6.50}$$

应用数值微分公式, 易知

$$\Delta_t U_j^k=(1+\mathrm{i}c_1)\delta_x^2U_j^{\bar{k}}+U_j^{\bar{k}}-(1+\mathrm{i}c_2)|U_j^k|^2U_j^{\bar{k}}+P_j^k,$$
$$1\leqslant j\leqslant m-1,\quad 1\leqslant k\leqslant n-1, \tag{6.51}$$

且存在常数 c_{13} 使得

$$|P_j^k|\leqslant c_{13}\left(\tau^2+h^2\right),\quad 1\leqslant j\leqslant m-1,\ 1\leqslant k\leqslant n-1. \tag{6.52}$$

由 (6.15) 和 (6.16) 可得

$$\begin{aligned}u_{xx}(x_0,t_k)&=\frac{1}{2}\left[u_{xx}(x_0,t_{k-1})+u_{xx}(x_0,t_{k+1})\right]+O(\tau^2)\\&=\frac{1}{2}\left[\frac{2}{h}\delta_xU_{\frac{1}{2}}^{k-1}+O(h^2)+\frac{2}{h}\delta_xU_{\frac{1}{2}}^{k+1}+O(h^2)\right]+O(\tau^2)\\&=\frac{2}{h}\delta_xU_{\frac{1}{2}}^{\bar{k}}+O(\tau^2+h^2),\end{aligned} \tag{6.53}$$

$$\begin{aligned}u_{xx}(x_m,t_k)&=\frac{1}{2}\left[u_{xx}(x_m,t_{k-1})+u_{xx}(x_m,t_{k+1})\right]+O(\tau^2)\\&=\frac{1}{2}\left[\left(-\frac{2}{h}\delta_xU_{m-\frac{1}{2}}^{k-1}+O(h^2)\right)+\left(-\frac{2}{h}\delta_xU_{m-\frac{1}{2}}^{k+1}+O(h^2)\right)\right]+O(\tau^2)\\&=-\frac{2}{h}\delta_xU_{m-\frac{1}{2}}^{\bar{k}}+O(\tau^2+h^2).\end{aligned} \tag{6.54}$$

对 (6.50) 中 $j=0$ 的方程应用 (6.53), $j=m$ 的方程应用 (6.54) 可得

$$\Delta_tU_0^k=(1+\mathrm{i}c_1)\frac{2}{h}\delta_xU_{\frac{1}{2}}^{\bar{k}}+U_0^{\bar{k}}-(1+\mathrm{i}c_2)|U_0^k|^2U_0^{\bar{k}}+P_0^k,$$
$$1\leqslant k\leqslant n-1, \tag{6.55}$$

$$\Delta_tU_m^k=(1+\mathrm{i}c_1)\left(-\frac{2}{h}\delta_xU_{m-\frac{1}{2}}^{\bar{k}}\right)+U_m^{\bar{k}}-(1+\mathrm{i}c_2)|U_m^k|^2U_m^{\bar{k}}+P_m^k,$$
$$1\leqslant k\leqslant n-1, \tag{6.56}$$

存在常数 c_{14}, 使得

$$\left|P_0^k\right| \leqslant c_{14}(\tau^2+h^2), \quad \left|P_m^k\right| \leqslant c_{14}(\tau^2+h^2), \quad 1 \leqslant k \leqslant n-1. \tag{6.57}$$

由 (6.1) 和 (6.2) 可得

$$u_t(x,0) = (1+\mathrm{i}c_1)\varphi_{xx}(x) + \varphi(x) - (1+\mathrm{i}c_2)|\varphi(x)|^2\varphi(x).$$

因而

$$U_j^1 = \varphi(x_j) + \tau u_t(x_j,0) + P_j^0, \quad 0 \leqslant j \leqslant m, \tag{6.58}$$

存在常数 c_{15} 使得

$$\left|P_j^0\right| \leqslant c_{15}\tau^2, \quad 0 \leqslant j \leqslant m, \tag{6.59}$$

$$\left|\delta_x P_{j+\frac{1}{2}}^0\right| \leqslant c_{15}\tau^2, \quad 0 \leqslant j \leqslant m-1. \tag{6.60}$$

注意到初值条件

$$U_j^0 = \varphi(x_j), \quad 0 \leqslant j \leqslant m. \tag{6.61}$$

在 (6.51), (6.55), (6.56), (6.58) 中略去小量项, 对问题 (6.1)–(6.3) 建立如下三层线性化差分格式

$$\Delta_t u_0^k = (1+\mathrm{i}c_1)\frac{2}{h}\delta_x u_{\frac{1}{2}}^{\bar{k}} + u_0^{\bar{k}} - (1+\mathrm{i}c_2)|u_0^k|^2 u_0^{\bar{k}}, \quad 1 \leqslant k \leqslant n-1, \tag{6.62}$$

$$\Delta_t u_j^k = (1+\mathrm{i}c_1)\delta_x^2 u_j^{\bar{k}} + u_j^{\bar{k}} - (1+\mathrm{i}c_2)|u_j^k|^2 u_j^{\bar{k}}, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{6.63}$$

$$\Delta_t u_m^k = (1+\mathrm{i}c_1)\left(-\frac{2}{h}\delta_x u_{m-\frac{1}{2}}^{\bar{k}}\right) + u_m^{\bar{k}} - (1+\mathrm{i}c_2)|u_m^k|^2 u_m^{\bar{k}}, \quad 1 \leqslant k \leqslant n-1, \tag{6.64}$$

$$u_j^0 = \varphi(x_j), \quad 0 \leqslant j \leqslant m, \tag{6.65}$$

$$u_j^1 = \varphi(x_j) + \tau u_t(x_j,0), \quad 0 \leqslant j \leqslant m. \tag{6.66}$$

6.3.2 差分格式解的有界性

定理 6.6 设 $\{u_j^k \mid 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (6.62)–(6.66) 的解, 则存在常数 c_{16} 使得

$$\|u^k\|_\infty \leqslant c_{16}, \quad 0 \leqslant k \leqslant n. \tag{6.67}$$

证明 由 (6.65)–(6.66) 知存在常数 c_{17} 使得

$$\|u^0\| \leqslant c_{17}, \quad |u^0|_1 \leqslant c_{17}, \tag{6.68}$$

$$\|u^1\| \leqslant c_{17}, \quad |u^1|_1 \leqslant c_{17}. \tag{6.69}$$

可将 (6.62)–(6.64) 统一写为

$$\Delta_t u_j^k=(1+\mathrm{i}c_1)\delta_x^2 u_j^{\bar{k}}+u_j^{\bar{k}}-(1+\mathrm{i}c_2)|u_j^k|^2 u_j^{\bar{k}},\quad 0\leqslant j\leqslant m,\ 1\leqslant k\leqslant n-1.\tag{6.70}$$

(I) 用 $u_j^{\bar{k}}$ 与 (6.70) 作内积, 然后取实部, 得

$$\begin{aligned}&\frac{1}{4\tau}\left(\|u^{k+1}\|^2-\|u^{k-1}\|^2\right)\\=&-|u^{\bar{k}}|_1^2+\|u^{\bar{k}}\|^2-\left(|u^k|^2u^{\bar{k}},u^{\bar{k}}\right)\\\leqslant&\|u^{\bar{k}}\|^2\\\leqslant&\left(\frac{\|u^{k+1}\|+\|u^{k-1}\|}{2}\right)^2,\quad 1\leqslant k\leqslant n-1.\end{aligned}$$

于是

$$\|u^{k+1}\|-\|u^{k-1}\|\leqslant\tau\left(\|u^{k+1}\|+\|u^{k-1}\|\right),\quad 1\leqslant k\leqslant n-1.$$

当 $\tau\leqslant\dfrac{1}{3}$ 时, 可得

$$\|u^{k+1}\|\leqslant(1+3\tau)\|u^{k-1}\|,\quad 1\leqslant k\leqslant n-1.$$

递推, 可得

$$\|u^k\|\leqslant(1+3\tau)^{\left[\frac{n}{2}\right]}\max\left\{\|u^1\|,\|u^0\|\right\}\leqslant e^{\frac{3T}{2}}\max\left\{\|u^1\|,\|u^0\|\right\},\quad 0\leqslant k\leqslant n-1.$$

注意到 (6.68)-(6.69), 可得

$$\left\|u^k\right\|\leqslant e^{\frac{3T}{2}}\cdot\max\left\{\|u^0\|,\|u^1\|\right\}\leqslant c_{17}e^{3T},\quad 1\leqslant k\leqslant n.\tag{6.71}$$

(II) 用 $\dfrac{1}{1-\mathrm{i}c_1}\Delta_t u^k$ 与 (6.70) 作内积, 得到

$$\frac{1}{1+\mathrm{i}c_1}\|\Delta_t u^k\|^2-\left(\delta_x^2u^{\bar{k}},\Delta_t u^k\right)=\frac{1}{1+\mathrm{i}c_1}\left(u^{\bar{k}},\Delta_t u^k\right)-\frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\left(|u^k|^2u^{\bar{k}},\Delta_t u^k\right).$$

取上式的实部, 得

$$\begin{aligned}&\frac{1}{1+c_1^2}\|\Delta_t u^k\|^2+\frac{1}{4\tau}\left(|u^{k+1}|_1^2-|u^{k-1}|_1^2\right)\\=&\mathrm{Re}\left\{\frac{1}{1+\mathrm{i}c_1}\left(u^{\bar{k}},\Delta_t u^k\right)\right\}+\mathrm{Re}\left\{-\frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\left(|u^k|^2u^{\bar{k}},\Delta_t u^k\right)\right\}\\\leqslant&\frac{1}{\sqrt{1+c_1^2}}\|u^{\bar{k}}\|\cdot\|\Delta_t u^k\|+\frac{\sqrt{1+c_2^2}}{\sqrt{1+c_1^2}}\||u^k|^2u^{\bar{k}}\|\cdot\|\Delta_t u^k\|\\\leqslant&\frac{1}{2(1+c_1^2)}\|\Delta_t u^k\|^2+\frac{1}{2}\|u^{\bar{k}}\|^2+\frac{1}{2(1+c_1^2)}\|\Delta_t u^k\|^2+\frac{1+c_2^2}{2}\||u^k|^2u^{\bar{k}}\|^2,\end{aligned}$$

即

$$\frac{1}{4\tau}\left(|u^{k+1}|_1^2-|u^{k-1}|_1^2\right)\leqslant\frac{1}{2}\|u^{\bar{k}}\|^2+\frac{1+c_2^2}{2}\left\||u^k|^2u^{\bar{k}}\right\|^2,\quad 1\leqslant k\leqslant n-1.\tag{6.72}$$

由 Young 不等式可得

$$|u_j^k|^4|u_j^{\bar{k}}|^2\leqslant\frac{2}{3}|u_j^k|^6+\frac{1}{3}|u_j^{\bar{k}}|^6.$$

再利用引理 6.6 得

$$\begin{aligned}&\left\||u^k|^2u^{\bar{k}}\right\|^2\\ \leqslant&\frac{2}{3}\|u^k\|_6^6+\frac{1}{3}\|u^{\bar{k}}\|_6^6\\ \leqslant&\frac{2}{3}\kappa^6\left(\left|u^k\right|_1^{\frac{1}{3}}\cdot\left\|u^k\right\|^{\frac{2}{3}}+\left\|u^k\right\|\right)^6+\frac{1}{3}\kappa^6\left(\left|u^{\bar{k}}\right|_1^{\frac{1}{3}}\cdot\left\|u^{\bar{k}}\right\|^{\frac{2}{3}}+\left\|u^{\bar{k}}\right\|\right)^6\\ \leqslant&\frac{2}{3}\kappa^62^5\left(\left|u^k\right|_1^2\cdot\left\|u^k\right\|^4+\left\|u^k\right\|^6\right)+\frac{1}{3}\kappa^62^5\left(\left|u^{\bar{k}}\right|_1^2\cdot\left\|u^{\bar{k}}\right\|^4+\left\|u^{\bar{k}}\right\|^6\right).\end{aligned}\tag{6.73}$$

将 (6.73) 代入 (6.72), 并利用 (6.71) 知存在常数 c_{18} 使得

$$\begin{aligned}&\frac{1}{4\tau}\left(\left|u^{k+1}\right|_1^2-\left|u^{k-1}\right|_1^2\right)\\ \leqslant&c_{18}\left(\left|u^k\right|_1^2+\left|u^{\bar{k}}\right|_1^2\right)+c_{18}\\ \leqslant&c_{18}\left(\frac{|u^k|_1^2+|u^{k+1}|_1^2}{2}+\frac{|u^{k-1}|_1^2+|u^k|_1^2}{2}\right)+c_{18},\quad 1\leqslant k\leqslant n-1,\end{aligned}$$

即

$$(1-2c_{18}\tau)\frac{|u^k|_1^2+|u^{k+1}|_1^2}{2}\leqslant(1+2c_{18}\tau)\frac{|u^{k-1}|_1^2+|u^k|_1^2}{2}+2c_{18}\tau,\quad 1\leqslant k\leqslant n-1.$$

当 $2c_{18}\tau\leqslant\dfrac{1}{3}$ 时,

$$\frac{|u^k|_1^2+|u^{k+1}|_1^2}{2}\leqslant(1+6c_{18}\tau)\frac{|u^{k-1}|_1^2+|u^k|_1^2}{2}+3c_{18}\tau,\quad 1\leqslant k\leqslant n-1.$$

由 Gronwall 不等式并注意到 (6.68)—(6.69), 得到

$$\frac{|u^k|_1^2+|u^{k+1}|_1^2}{2}\leqslant e^{6c_{18}T}\left(\frac{|u^0|_1^2+|u^1|_1^2}{2}+\frac{1}{2}\right)\leqslant e^{6c_{18}T}\left(c_{17}^2+\frac{1}{2}\right),\quad 0\leqslant k\leqslant n-1.\tag{6.74}$$

(III) 由 (6.71), (6.35) 以及引理 1.1 知 (6.67) 成立. □

6.3.3 差分格式解的存在性和唯一性

定理 6.7 差分格式 (6.62)–(6.66) 是唯一可解的.

证明 由 (6.65)–(6.66) 知 u^0, u^1 唯一确定.

设 u^{k-1}, u^k 已确定. 则由 (6.62)–(6.64) 可得关于 u^{k+1} 的线性方程组. 考虑它的齐次线性方程组

$$\frac{1}{2\tau}u_0^{k+1} = (1+\mathrm{i}c_1)\frac{1}{h}\delta_x u_{\frac{1}{2}}^{k+1} + \frac{1}{2}u_0^{k+1} - \frac{1}{2}(1+\mathrm{i}c_2)\left|u_0^k\right|^2 u_0^{k+1}, \tag{6.75}$$

$$\frac{1}{2\tau}u_j^{k+1} = \frac{1}{2}(1+\mathrm{i}c_1)\delta_x^2 u_j^{k+1} + \frac{1}{2}u_j^{k+1} - \frac{1}{2}(1+\mathrm{i}c_2)\left|u_j^k\right|^2 u_j^{k+1},\ 1 \leqslant j \leqslant m-1, \tag{6.76}$$

$$\frac{1}{2\tau}u_m^{k+1} = (1+\mathrm{i}c_1)\Big(-\frac{1}{h}\delta_x u_{m-\frac{1}{2}}^{k+1}\Big) + \frac{1}{2}u_m^{k+1} - \frac{1}{2}(1+\mathrm{i}c_2)\left|u_m^k\right|^2 u_m^{k+1}. \tag{6.77}$$

用 $\frac{1}{2}h\bar{u}_0^{k+1}$ 乘以 (6.75) 的两边, 用 $h\bar{u}_j^{k+1}$ 乘以 (6.76) 的两边, 用 $\frac{1}{2}h\bar{u}_m^{k+1}$ 乘以 (6.77) 的两边, 然后将结果相加得

$$\begin{aligned}
&\frac{1}{2\tau}\left\|u^{k+1}\right\|^2 \\
=& -\frac{1}{2}(1+\mathrm{i}c_1)\left|u^{k+1}\right|_1^2 + \frac{1}{2}\left\|u^{k+1}\right\|^2 \\
&- \frac{1}{2}(1+\mathrm{i}c_2)h\left[\frac{1}{2}\left|u_0^k\right|^2\left|u_0^{k+1}\right|^2 + \sum_{j=1}^{m-1}\left|u_j^k\right|^2\left|u_j^{k+1}\right|^2 + \frac{1}{2}\left|u_m^k\right|^2\left|u_m^{k+1}\right|^2\right].
\end{aligned}$$

两边取实部, 得

$$\begin{aligned}
\frac{1}{2\tau}\left\|u^{k+1}\right\|^2 =& -\frac{1}{2}\left|u^{k+1}\right|_1^2 + \frac{1}{2}\left\|u^{k+1}\right\|^2 \\
&- \frac{1}{2}h\left[\frac{1}{2}\left|u_0^k\right|^2\left|u_0^{k+1}\right|^2 + \sum_{j=1}^{m-1}\left|u_j^k\right|^2\left|u_j^{k+1}\right|^2 + \frac{1}{2}\left|u_m^k\right|^2\right] \\
\leqslant& \frac{1}{2}\left\|u^{k+1}\right\|^2.
\end{aligned}$$

当 $\tau < 1$ 时,

$$\left\|u^{k+1}\right\| = 0,$$

即

$$u^{k+1} = 0.$$

因而 (6.62)–(6.64) 关于 u^{k+1} 是唯一可解的. □

6.3.4 差分格式解的收敛性

定理 6.8 设 $\{U_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 是问题 (6.1)–(6.3) 的解, $\{u_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 是差分格式 (6.62)–(6.66) 的解. 记

$$e_j^k = U_j^k - u_j^k, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n,$$

则存在常数 c_{19} 使得

$$|e^k|_1 \leqslant c_{19}\left(\tau^2 + h^2\right), \quad 0 \leqslant k \leqslant n. \tag{6.78}$$

证明 将 (6.55), (6.51), (6.56), (6.61), (6.58) 依次与 (6.62)–(6.66) 相减, 得误差方程组

$$\Delta_t e_0^k = (1+\mathrm{i}c_1)\frac{2}{h}\delta_x e_{\frac{1}{2}}^{\bar{k}} + e_0^{\bar{k}} - (1+\mathrm{i}c_2)\left(\left|U_0^k\right|^2 U_0^{\bar{k}} - \left|u_0^k\right|^2 u_0^{\bar{k}}\right) + P_0^k,$$
$$1 \leqslant k \leqslant n-1, \tag{6.79}$$

$$\Delta_t e_j^k = (1+\mathrm{i}c_1)\delta_x^2 e_j^{\bar{k}} + e_j^{\bar{k}} - (1+\mathrm{i}c_2)\left(\left|U_j^k\right|^2 U_j^{\bar{k}} - \left|u_j^k\right|^2 u_j^{\bar{k}}\right) + P_j^k,$$
$$1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{6.80}$$

$$\Delta_t e_m^k = (1+\mathrm{i}c_1)\left(-\frac{2}{h}\delta_x e_{m-\frac{1}{2}}^{\bar{k}}\right) + e_m^{\bar{k}}$$
$$-(1+\mathrm{i}c_2)\left(\left|U_m^k\right|^2 U_m^{\bar{k}} - \left|u_m^k\right|^2 u_m^{\bar{k}}\right) + P_m^k, \quad 1 \leqslant k \leqslant n-1, \tag{6.81}$$

$$e_j^0 = 0, \quad 0 \leqslant j \leqslant m, \tag{6.82}$$

$$e_j^1 = P_j^0, \quad 0 \leqslant j \leqslant m. \tag{6.83}$$

可将 (6.79)–(6.81) 统一写为

$$\Delta_t e_j^k = (1+\mathrm{i}c_1)\delta_x^2 e_j^{\bar{k}} + e_j^{\bar{k}} - (1+\mathrm{i}c_2)\left(\left|U_j^k\right|^2 U_j^{\bar{k}} - \left|u_j^k\right|^2 u_j^{\bar{k}}\right) + P_j^k,$$
$$0 \leqslant j \leqslant m, \quad 1 \leqslant k \leqslant n-1, \tag{6.84}$$

由 (6.82)–(6.83) 和 (6.59)–(6.60) 得

$$\|e^0\| = 0, \quad \|e^1\| \leqslant \sqrt{L}c_{15}\left(\tau^2 + h^2\right), \tag{6.85}$$

$$\|e^0\|_1 = 0, \quad |e^1|_1 \leqslant \sqrt{L}c_{15}\left(\tau^2 + h^2\right). \tag{6.86}$$

由 (6.10) 和 (6.67) 知

$$|U_j^k| \leqslant c_5, \quad |u_j^k| \leqslant c_{16}, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n. \tag{6.87}$$

此外有

$$\left|U_j^k\right|^2 U_j^{\bar{k}} - \left|u_j^k\right|^2 u_j^{\bar{k}} = \left|u_j^k\right|^2 e_j^{\bar{k}} + \left(u_j^k \bar{e}_j^k + e_j^k \bar{U}_j^k\right) U_j^{\bar{k}}.$$

(I) 用 $e^{\bar{k}}$ 与 (6.84) 作内积, 得

$$\begin{aligned}\left(\Delta_t e^k, e^{\bar{k}}\right) =& (1+\mathrm{i}c_1)\left(\delta_x^2 e^{\bar{k}}, e^{\bar{k}}\right) + \left(e^{\bar{k}}, e^{\bar{k}}\right)\\ &- (1+\mathrm{i}c_2)\left(|U^k|^2 U^{\bar{k}} - |u^k|^2 u^{\bar{k}}, e^{\bar{k}}\right) + \left(P^k, e^{\bar{k}}\right)\\ =& - (1+\mathrm{i}c_1)\left|e^{\bar{k}}\right|_1^2 + \left\|e^{\bar{k}}\right\|^2\\ &- (1+\mathrm{i}c_2)\left(\left|u^k\right|^2 e^{\bar{k}} + \left(u^k \bar{e}^k + e^k \bar{U}^k\right) U^{\bar{k}}, e^{\bar{k}}\right) + \left(P^k, e^{\bar{k}}\right).\end{aligned}$$

上式两边取实部, 并利用 (6.87), 得

$$\begin{aligned}&\frac{1}{2\tau}\left(\left\|e^{k+1}\right\|^2 - \left\|e^{k-1}\right\|^2\right)\\ \leqslant& \left\|e^{\bar{k}}\right\|^2 + \mathrm{Re}\Big\{-(1+\mathrm{i}c_2)\big((u^k\bar{e}^k + e^k\bar{U}^k)U^{\bar{k}}, e^{\bar{k}}\big)\Big\} + \mathrm{Re}\Big\{\big(P^k, e^{\bar{k}}\big)\Big\}\\ \leqslant& \|e^{\bar{k}}\|^2 + \sqrt{1+c_2^2}\,(c_5+c_{16})\,c_5\left\|e^k\right\|\cdot\left\|e^{\bar{k}}\right\| + \left\|P^k\right\|\cdot\left\|e^{\bar{k}}\right\|\\ \leqslant& \left[\frac{\left\|e^{k+1}\right\| + \left\|e^{k-1}\right\|}{2} + \sqrt{1+c_2^2}\,(c_5+c_{16})\,c_5\left\|e^k\right\| + \left\|P^k\right\|\right]\cdot\frac{\left\|e^{k+1}\right\| + \left\|e^{k-1}\right\|}{2},\\ &\qquad 1\leqslant k\leqslant n-1.\end{aligned}$$

由上式可得

$$\begin{aligned}\frac{1}{\tau}\left(\left\|e^{k+1}\right\| - \left\|e^{k-1}\right\|\right) \leqslant \frac{\left\|e^{k+1}\right\| + \left\|e^{k-1}\right\|}{2} + \sqrt{1+c_2^2}\,(c_5+c_{16})\,c_5\left\|e^k\right\| + \left\|P^k\right\|,\\ 1\leqslant k\leqslant n-1.\end{aligned}$$

注意到 (6.52), 得

$$\begin{aligned}\left(1-\frac{1}{2}\tau\right)\left\|e^{k+1}\right\| \leqslant \left(1+\frac{1}{2}\tau\right)\left\|e^{k-1}\right\| + \sqrt{1+c_2^2}\,(c_5+c_{16})\,c_5\tau\left\|e^k\right\|\\ + \tau\sqrt{L}c_{13}\left(\tau^2+h^2\right), \quad 1\leqslant k\leqslant n-1.\end{aligned}$$

当 $\dfrac{1}{2}\tau \leqslant \dfrac{1}{3}$ 时,

$$\begin{aligned}\left\|e^{k+1}\right\| \leqslant& \left(1+\frac{3}{2}\tau\right)\left\|e^{k-1}\right\| + \frac{3}{2}\sqrt{1+c_2^2}\,(c_5+c_{16})\,c_5\tau\left\|e^k\right\| + \frac{3}{2}\sqrt{L}c_{13}\tau\left(\tau^2+h^2\right)\\ \leqslant& \left\{1+\frac{3}{2}\left[1+\sqrt{1+c_2^2}\,(c_5+c_{16})\,c_5\right]\tau\right\}\max\left\{\left\|e^{k-1}\right\|, \left\|e^k\right\|\right\}\\ &+ \frac{3}{2}\sqrt{L}c_{13}\tau\left(\tau^2+h^2\right), \quad 1\leqslant k\leqslant n-1,\end{aligned}$$

或

$$\max\left\{\|e^k\|,\|e^{k+1}\|\right\}\leqslant\left\{1+\frac{3}{2}\left[1+\sqrt{1+c_2^2}\,(c_5+c_{16})\,c_5\right]\tau\right\}\max\left\{\|e^{k-1}\|,\|e^k\|\right\}$$
$$+\frac{3}{2}\sqrt{L}c_{13}\tau\left(\tau^2+h^2\right),\quad 1\leqslant k\leqslant n-1.$$

由 Gronwall 不等式, 并注意到 (6.85), 得

$$\begin{aligned}&\max\left\{\|e^k\|,\|e^{k+1}\|\right\}\\ \leqslant& e^{\frac{3}{2}[1+\sqrt{1+c_2^2}(c_5+c_{16})c_5]T}\left[\max\left\{\|e^0\|,\|e^1\|\right\}+\frac{\sqrt{L}c_{13}}{1+\sqrt{1+c_2^2}(c_5+c_{16})c_5}\left(\tau^2+h^2\right)\right]\\ \leqslant& e^{\frac{3}{2}[1+\sqrt{1+c_2^2}(c_5+c_{16})c_5]T}\sqrt{L}\left[c_{15}+\frac{\sqrt{L}c_{13}}{1+\sqrt{1+c_2^2}(c_5+c_{16})c_5}\right]\left(\tau^2+h^2\right)\\ \equiv& c_{20}\left(\tau^2+h^2\right),\quad 0\leqslant k\leqslant n-1.\end{aligned}\tag{6.88}$$

(II) 用 $\dfrac{1}{1-\mathrm{i}c_1}\Delta_t e^k$ 与 (6.84) 作内积, 得

$$\frac{1}{1+\mathrm{i}c_1}\|\Delta_t e^k\|=\left(\delta_x^2 e^{\bar{k}},\Delta_t e^k\right)+\frac{1}{1+\mathrm{i}c_1}\left(e^{\bar{k}},\Delta_t e^k\right)$$
$$-\frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\left(\left|U^k\right|^2U^{\bar{k}}-\left|u^k\right|^2u^{\bar{k}},\Delta_t e^k\right)+\frac{1}{1+\mathrm{i}c_1}\left(P^k,\Delta_t e^k\right).$$

两边取实部, 得到

$$\begin{aligned}&\frac{1}{1+c_1^2}\|\Delta_t e^k\|^2+\frac{1}{4\tau}\left(\left|e^{k+1}\right|_1^2-\left|e^{k-1}\right|_1^2\right)\\ =&\mathrm{Re}\left\{\frac{1}{1+\mathrm{i}c_1}\left(e^{\bar{k}},\Delta_t e^k\right)\right\}+\mathrm{Re}\left\{-\frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\left(\left|u^k\right|^2e^{\bar{k}}+\left(u^k\bar{e}^k+e^k\bar{U}^k\right)U^{\bar{k}},\Delta_t e^k\right)\right\}\\ &+\mathrm{Re}\left\{\frac{1}{1+\mathrm{i}c_1}\left(P^k,\Delta_t e^k\right)\right\}\\ \leqslant&\frac{1}{\sqrt{1+c_1^2}}\left\|e^{\bar{k}}\right\|\cdot\|\Delta_t e^k\|+\frac{\sqrt{1+c_2^2}}{\sqrt{1+c_1^2}}\left(c_{16}^2\left\|e^{\bar{k}}\right\|+(c_5+c_{16})c_5\left\|e^k\right\|\right)\|\Delta_t e^k\|\\ &+\frac{1}{\sqrt{1+c_1^2}}\|P^k\|\cdot\|\Delta_t e^k\|\\ \leqslant&\frac{1}{4(1+c_1^2)}\left\|\Delta_t e^k\right\|^2+\left\|e^{\bar{k}}\right\|^2+\frac{1}{4(1+c_1^2)}\left\|\Delta_t e^k\right\|^2+(1+c_2^2)c_{16}^4\left\|e^{\bar{k}}\right\|^2\\ &+\frac{1}{4(1+c_1^2)}\left\|\Delta_t e^k\right\|^2+(1+c_2^2)(c_5+c_{16})^2c_5^2\left\|e^k\right\|^2\\ &+\frac{1}{4(1+c_1^2)}\left\|\Delta_t e^k\right\|^2+\left\|P^k\right\|^2,\quad 1\leqslant k\leqslant n-1,\end{aligned}$$

即

$$\frac{1}{4\tau}\left(\left|e^{k+1}\right|_1^2-\left|e^{k-1}\right|_1^2\right)$$
$$\leqslant\left[1+\left(1+c_2^2\right)c_{16}^4\right]\left\|e^{\bar{k}}\right\|^2+\left(1+c_2^2\right)\left(c_5+c_{16}\right)^2c_5^2\left\|e^k\right\|^2+\left\|P^k\right\|^2,\quad 1\leqslant k\leqslant n-1.$$

由 (6.88), (6.52), 得

$$\begin{aligned}&\frac{1}{4\tau}\left(\left|e^{k+1}\right|_1^2-\left|e^{k-1}\right|_1^2\right)\\ \leqslant&\left[1+\left(1+c_2^2\right)c_{16}^4\right]c_{20}^2\left(\tau^2+h^2\right)^2\\ &+\left(1+c_2^2\right)\left(c_5+c_{16}\right)^2c_5^2c_{20}^2\left(\tau^2+h^2\right)^2+Lc_{13}^2(\tau^2+h^2)^2\\ =&\left\{\left[1+\left(1+c_2^2\right)c_{16}^2\right]c_{20}^2+\left(1+c_2^2\right)\left(c_5+c_{16}\right)^2c_5^2c_{20}^2+Lc_{13}^2\right\}\left(\tau^2+h^2\right)^2\\ \equiv&c_{21}\left(\tau^2+h^2\right)^2,\quad 1\leqslant k\leqslant n-1.\end{aligned}$$

由上式可得

$$\left|e^{k+1}\right|_1^2\leqslant\left|e^{k-1}\right|_1^2+4c_{21}\tau\left(\tau^2+h^2\right)^2,\quad 1\leqslant k\leqslant n-1.$$

递推可得

$$\left|e^k\right|_1^2\leqslant\max\left\{\left|e^1\right|_1^2,\left|e^0\right|_1^2\right\}+2c_{21}k\tau\left(\tau^2+h^2\right)^2,\quad 1\leqslant k\leqslant n.$$

注意到 (6.86), 有

$$\left|e^k\right|_1^2\leqslant\left(Lc_{15}^2+2c_{21}T\right)\left(\tau^2+h^2\right)^2,\quad 1\leqslant k\leqslant n.$$ □

6.4 小结与延拓

本章对 Kuramoto-Tsuzuki 方程初边值问题建立了二层非线性差分格式和三层线性化差分格式. 证明了差分格式解的存在性、有界性和收敛性. 本章结果取自 [27]–[29]. Kuramoto-Tsuzuki 方程和 Schrödinger 方程都是复的半线性发展方程. 边界条件的提法不同. 胡秀玲 [17] 对 Kuramoto-Tsuzuki 方程建立了紧差分格式, 证明了差分格式解的收敛性.

第 7 章　Zakharov 方程的差分方法

7.1　引　　言

由于在超小电子设备、稠密天体物理、等离子系统和激光等离子体中，量子具有重要的作用. Vladimir Zakharov (1972 年) 最早建立了描述高频 Langmuir 波和低频率离子声波相互作用的非线性耦合波动方程组. 自那以后 Zakharov 系统引起人们的广泛研究.

本章研究 Zakharov 方程初边值问题

$$\mathrm{i}u_t + u_{xx} - uv = 0, \quad 0 < x < L, \quad 0 < t \leqslant T, \tag{7.1}$$

$$v_{tt} - v_{xx} - (|u|^2)_{xx} = 0, \quad 0 < x < L, \quad 0 < t \leqslant T, \tag{7.2}$$

$$u(x,0) = \varphi(x), \quad v(x,0) = \psi(x), \quad v_t(x,0) = \psi_1(x), \quad 0 < x < L, \tag{7.3}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad v(0,t) = 0, \quad v(L,t) = 0, \quad 0 \leqslant t \leqslant T \tag{7.4}$$

的有限差分方法，其中 $\varphi(0) = \varphi(L) = \psi(0) = \psi(L) = \psi_1(0) = \psi_1(L) = 0, u(x,t)$ 和 $\varphi(x)$ 为复值函数, $v(x,t)$, $\psi(x)$, $\psi_1(x)$ 为实值函数.

对于任意固定的 t, 令 $w(x,t)$ 为下列问题

$$w_{xx}(x,t) = v_t(x,t), \quad 0 < x < L, \tag{7.5}$$

$$w(0,t) = 0, \quad w(L,t) = 0 \tag{7.6}$$

的解. 则由常微分方程两点边值问题的理论知 $w(x,t)$ 是唯一存在的, 且易得估计式

$$|w(\cdot,t)|_1 \leqslant \frac{\sqrt{6}L}{6}\|v_t(\cdot,t)\|, \tag{7.7}$$

$$\|w(\cdot,t)\|_\infty \leqslant \frac{L\sqrt{6L}}{12}\|v_t(\cdot,t)\|. \tag{7.8}$$

特别地, 当 $t = 0$ 时有

$$w_{xx}(x,0) = \psi_1(x), \quad 0 < x < L, \tag{7.9}$$

$$w(0,0) = 0, \quad w(L,0) = 0. \tag{7.10}$$

可以解得

$$w(x,0) = \frac{x}{L}\int_0^L (L-s)\psi_1(s)\mathrm{d}s - \int_0^x (x-s)\psi_1(s)\mathrm{d}s. \tag{7.11}$$

记

$$F(\psi_1)(x) = \frac{x}{L}\int_0^L (L-s)\psi_1(s)\mathrm{d}s - \int_0^x (x-s)\psi_1(s)\mathrm{d}s. \tag{7.12}$$

则有

$$w(x,0) = F(\psi_1)(x), \quad 0 \leqslant x \leqslant L.$$

将 (7.5) 代入 (7.2), 可得

$$(w_t - v - |u|^2)_{xx} = 0, \quad 0 < x < L.$$

由于

$$(w_t - v - |u|^2)(0,t) = 0, \quad (w_t - v - |u|^2)(L,t) = 0.$$

所以

$$w_t - v - |u|^2 = 0, \quad 0 < x < L.$$

于是问题 (7.1)–(7.4) 等价于如下问题

$$\mathrm{i}u_t + u_{xx} - uv = 0, \quad 0 < x < L,\ 0 < t \leqslant T, \tag{7.13}$$

$$v_t - w_{xx} = 0, \qquad 0 < x < L,\ 0 < t \leqslant T, \tag{7.14}$$

$$w_t - v - |u|^2 = 0, \quad 0 < x < L,\ 0 < t \leqslant T, \tag{7.15}$$

$$u(x,0) = \varphi(x), \quad v(x,0) = \psi(x), \quad w(x,0) = F(\psi_1)(x), \quad 0 < x < L, \tag{7.16}$$

$$u(0,t) = 0, \quad u(L,t) = 0, \quad w(0,t) = 0, \quad w(L,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{7.17}$$

其中 $w(x,0) = F(\psi_1)(x)$ 由 (7.12) 确定.

定理 7.1 设 $\{u(x,t), v(x,t), w(x,t)\}$ 为 (7.13)–(7.17) 的解. 记

$$Q(t) = \int_0^L |u(x,t)|^2 \mathrm{d}x,$$

$$E(t) = \int_0^L \left[|u_x(x,t)|^2 + \frac{1}{2}w_x^2(x,t) + \frac{1}{2}v^2(x,t) + |u(x,t)|^2 v(x,t)\right]\mathrm{d}x,$$

则有

$$Q(t) = Q(0), \quad 0 \leqslant t \leqslant T, \tag{7.18}$$

$$E(t) = E(0), \quad 0 \leqslant t \leqslant T. \tag{7.19}$$

证明 (I) 用 $\bar{u}(x,t)$ 乘以 (7.13) 的两边, 并对 x 从 0 到 L 积分, 得

$$\mathrm{i}\int_0^L u_t(x,t)\bar{u}(x,t)\mathrm{d}x + \int_0^L u_{xx}(x,t)\bar{u}(x,t)\mathrm{d}x - \int_0^L |u(x,t)|^2 v(x,t)\mathrm{d}x = 0.$$

两边取虚部, 得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u(x,t)|^2\mathrm{d}x = 0. \tag{7.20}$$

因而 (7.18) 成立.

(II) 用 $-\bar{u}_t(x,t)$ 乘以 (7.13) 的两边, 并对 x 从 0 到 L 积分, 得

$$-\mathrm{i}\int_0^L |u_t(x,t)|^2\mathrm{d}x - \int_0^L u_{xx}(x,t)\bar{u}_t(x,t)\mathrm{d}x + \int_0^L v(x,t)u(x,t)\bar{u}_t(x,t)\mathrm{d}x = 0,$$

上式两边取实部, 得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u_x(x,t)|^2\mathrm{d}x + \int_0^L v(x,t)\frac{\mathrm{d}}{\mathrm{d}t}(|u(x,t)|^2)\mathrm{d}x = 0. \tag{7.21}$$

(III) 用 $w_t(x,t)$ 乘以 (7.14) 的两边, 并对 x 从 0 到 L 求积分, 得

$$\int_0^L v_t(x,t)w_t(x,t)\mathrm{d}x - \int_0^L w_t(x,t)w_{xx}(x,t)\mathrm{d}x = 0. \tag{7.22}$$

用 $-v_t(x,t)$ 乘以 (7.15) 的两边, 并对 x 从 0 到 L 求积分, 得

$$-\int_0^L w_t(x,t)v_t(x,t)\mathrm{d}x + \int_0^L v(x,t)v_t(x,t)\mathrm{d}x + \int_0^L |u(x,t)|^2 v_t(x,t)\mathrm{d}x = 0. \tag{7.23}$$

(IV) 将 (7.22) 和 (7.23) 相加, 得

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L w_x^2(x,t)\mathrm{d}x + \frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L v^2(x,t)\mathrm{d}x + \int_0^L |u(x,t)|^2 v_t(x,t)\mathrm{d}x = 0. \tag{7.24}$$

将 (7.21) 和 (7.24) 相加, 得

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left[|u_x(x,t)|^2 + \frac{1}{2}w_x^2(x,t) + \frac{1}{2}v^2(x,t) + |u(x,t)|^2 v(x,t)\right]\mathrm{d}x = 0.$$

因而 (7.19) 成立. □

由上述定理可得

$$\begin{aligned}&\int_0^L \left[|u_x(x,t)|^2 + \frac{1}{2}v^2(x,t)\right]\mathrm{d}x\\ \leqslant& -\int_0^L |u(x,t)|^2 v(x,t)\mathrm{d}x + E(0)\end{aligned}$$

$$\leqslant \frac{1}{4}\int_0^L v^2(x,t)\mathrm{d}x + \int_0^L |u(x,t)|^4 dx + E(0)$$
$$\leqslant \frac{1}{4}\int_0^L v^2(x,t)\mathrm{d}x + \|u(\cdot,t)\|_\infty^2 \|u(\cdot,t)\|^2 + E(0)$$
$$\leqslant \frac{1}{4}\int_0^L v^2(x,t)\mathrm{d}x + \|u(\cdot,t)\|_\infty^2 Q(0) + E(0)$$
$$\leqslant \frac{1}{4}\int_0^L v^2(x,t)\mathrm{d}x + \left(\varepsilon|u(\cdot,t)|_1^2 + \frac{1}{4\varepsilon}\|u(\cdot,t)\|^2\right) Q(0) + E(0),$$

取 $\varepsilon = \dfrac{1}{2Q(0)}$, 得到

$$|u(\cdot,t)|_1^2 + \frac{1}{2}\|v(\cdot,t)\|^2 \leqslant Q^3(0) + 2E(0), \quad 0 \leqslant t \leqslant T.$$

进一步可得

$$\frac{4}{L}\|u(\cdot,t)\|_\infty^2 + \frac{1}{2}\|v(\cdot,t)\|^2 \leqslant Q^3(0) + 2E(0), \quad 0 \leqslant t \leqslant T.$$

因而存在常数 c_1 使得

$$\|u(\cdot,t)\|_\infty \leqslant c_1, \quad \|v(\cdot,t)\| \leqslant c_1, \quad 0 \leqslant t \leqslant T.$$

7.2 二层非线性差分格式

7.2.1 差分格式的建立

在点 $(x_j, t_{k+\frac{1}{2}})$ 处考虑方程 (7.13)-(7.15), 利用数值微分公式, 可得

$$\mathrm{i}\delta_t U_j^{k+\frac{1}{2}} + \delta_x^2 U_j^{k+\frac{1}{2}} - U_j^{k+\frac{1}{2}} V_j^{k+\frac{1}{2}} = P_j^{k+\frac{1}{2}},$$
$$1 \leqslant j \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{7.25}$$
$$\delta_t V_j^{k+\frac{1}{2}} - \delta_x^2 W_j^{k+\frac{1}{2}} = Q_j^{k+\frac{1}{2}},$$
$$1 \leqslant j \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{7.26}$$
$$\delta_t W_j^{k+\frac{1}{2}} - V_j^{k+\frac{1}{2}} - \frac{|U_j^k|^2 + |U_j^{k+1}|^2}{2} = R_j^{k+\frac{1}{2}},$$
$$0 \leqslant j \leqslant m, \quad 0 \leqslant k \leqslant n-1, \tag{7.27}$$

存在常数 c_2 使得

$$|P_j^{k+\frac{1}{2}}| \leqslant c_2(\tau^2+h^2), \quad 1 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{7.28}$$

$$|Q_j^{k+\frac{1}{2}}| \leqslant c_2(\tau^2+h^2), \quad 1 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{7.29}$$

$$|R_j^{k+\frac{1}{2}}| \leqslant c_2\tau^2, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n-1, \tag{7.30}$$

$$\left|\frac{P_j^{k+\frac{1}{2}}-P_j^{k-\frac{1}{2}}}{\tau}\right| \leqslant c_2(\tau^2+h^2), \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{7.31}$$

$$\left|\frac{Q_j^{k+\frac{1}{2}}-Q_j^{k-\frac{1}{2}}}{\tau}\right| \leqslant c_2(\tau^2+h^2), \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{7.32}$$

$$\left|\frac{R_j^{k+\frac{1}{2}}-R_j^{k-\frac{1}{2}}}{\tau}\right| \leqslant c_2\tau^2, \quad 0 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n-1. \tag{7.33}$$

由初边值条件 (7.16)–(7.17)，有

$$U_j^0=\varphi(x_j), \quad V_j^0=\psi(x_j), \quad W_j^0=F(\psi_1)(x_j), \quad 0 \leqslant j \leqslant m, \tag{7.34}$$

$$U_j^0=0, \quad U_m^k=0, \quad W_0^k=0, \quad W_m^k=0, \quad 1 \leqslant k \leqslant n. \tag{7.35}$$

在 (7.25)–(7.27) 中略去小量项, 对问题 (7.13)–(7.17) 建立如下差分格式

$$\mathrm{i}\delta_t u_j^{k+\frac{1}{2}}+\delta_x^2 u_j^{k+\frac{1}{2}}-u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}}=0, \quad 1 \leqslant j \leqslant m-1, \ 0 \leqslant k \leqslant n-1, \tag{7.36}$$

$$\delta_t v_j^{k+\frac{1}{2}}-\delta_x^2 w_j^{k+\frac{1}{2}}=0, \quad 1 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{7.37}$$

$$\delta_t w_j^{k+\frac{1}{2}}-v_j^{k+\frac{1}{2}}-\frac{|u_j^k|^2+|u_j^{k+1}|^2}{2}=0, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n-1, \tag{7.38}$$

$$u_j^0=\varphi(x_j), \quad v_j^0=\psi(x_j), \quad w_j^0=G(\psi_1)_j, \quad 0 \leqslant j \leqslant m, \tag{7.39}$$

$$u_0^k=0, \quad u_m^k=0, \quad w_0^k=0, \quad w_m^k=0, \quad 1 \leqslant k \leqslant n, \tag{7.40}$$

其中 $w_j^0=G(\psi_1)_j\,(0 \leqslant j \leqslant m)$ 由下式确定:

$$\delta_x^2 w_j^0=\psi_1(x_j), \quad 1 \leqslant j \leqslant m-1, \tag{7.41}$$

$$w_0^0=0, \quad w_m^0=0. \tag{7.42}$$

由初边值的相容性可知

$$u_0^0=u_m^0=w_0^0=w_m^0=v_0^0=v_m^0=0.$$

再由 (7.38) 中 $j=0$ 和 $j=m$ 的方程可知

$$v_0^k=0, \quad v_m^k=0, \quad 1 \leqslant k \leqslant n.$$

换句话说 (7.38) 等价于

$$\delta_t w_j^{k+\frac{1}{2}} - v_j^{k+\frac{1}{2}} - \frac{|u_j^k|^2 + |u_j^{k+1}|^2}{2} = 0, \quad 1 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n-1,$$
$$v_0^k = 0, \quad v_m^k = 0, \quad 1 \leqslant k \leqslant n.$$

由 (7.37)–(7.38), (7.41) 可得

$$\frac{2}{\tau}(\delta_t v_j^{\frac{1}{2}} - \psi_1(x_j)) - \delta_x^2 v_j^{\frac{1}{2}} - \delta_x^2 \frac{|u_j^0|^2 + |u_j^1|^2}{2} = 0, \quad 1 \leqslant j \leqslant m-1, \tag{7.43}$$
$$\delta_t^2 v_j^k - \delta_x^2 \frac{v_j^{k+1} + 2v_j^k + v_j^{k-1}}{4} - \delta_x^2 \frac{|u_j^{k+1}|^2 + 2|u_j^k|^2 + |u_j^{k-1}|^2}{4} = 0,$$
$$1 \leqslant j \leqslant m-1, \quad 1 \leqslant k \leqslant n-1. \tag{7.44}$$

于是我们得到差分格式 (7.36)–(7.40) 的解 $\{u_j^k, v_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 满足

$$\mathrm{i}\delta_t u_j^{k+\frac{1}{2}} + \delta_x^2 u_j^{k+\frac{1}{2}} - u_j^{k+\frac{1}{2}} v_j^{k+\frac{1}{2}} = 0,\ 1 \leqslant j \leqslant m-1, \quad 0 \leqslant k \leqslant n-1, \tag{7.45}$$
$$\frac{2}{\tau}(\delta_t v_j^{\frac{1}{2}} - \psi_1(x_j)) - \delta_x^2 v_j^{\frac{1}{2}} - \delta_x^2 \frac{|u_j^0|^2 + |u_j^1|^2}{2} = 0, \quad 1 \leqslant j \leqslant m-1, \tag{7.46}$$
$$\delta_t^2 v_j^k - \delta_x^2 \frac{v_j^{k+1} + 2v_j^k + v_j^{k-1}}{4} - \delta_x^2 \frac{|u_j^{k+1}|^2 + 2|u_j^k|^2 + |u_j^{k-1}|^2}{4} = 0,$$
$$1 \leqslant j \leqslant m-1, \quad 1 \leqslant k \leqslant n-1, \tag{7.47}$$
$$u_j^0 = \varphi(x_j), \quad v_j^0 = \psi(x_j), \quad 0 \leqslant j \leqslant m, \tag{7.48}$$
$$u_0^k = 0, \quad u_m^k = 0, \quad v_0^k = 0, \quad v_m^k = 0, \quad 1 \leqslant k \leqslant n. \tag{7.49}$$

对原问题 (7.1)–(7.4) 建立差分格式 (7.45)–(7.49).

7.2.2 差分格式解的存在性

当第 k 层的值 $\{u^k, v^k, w^k\}$ 已知时, 可以将平均值 $\{u^{k+\frac{1}{2}},\ v^{k+\frac{1}{2}},\ w^{k+\frac{1}{2}}\}$ 看成未知量. 这时由 (7.36)–(7.40) 可以得到

$$\frac{2}{\tau}(u_j^{k+\frac{1}{2}} - u_j^k) - \mathrm{i}\delta_x^2 u_j^{k+\frac{1}{2}} + \mathrm{i}u_j^{k+\frac{1}{2}} v_j^{k+\frac{1}{2}} = 0,$$
$$1 \leqslant j \leqslant m-1, \tag{7.50}$$
$$\frac{2}{\tau}(v_j^{k+\frac{1}{2}} - v_j^k) - \delta_x^2 w_j^{k+\frac{1}{2}} = 0, \quad 1 \leqslant j \leqslant m-1, \tag{7.51}$$
$$\frac{2}{\tau}(w_j^{k+\frac{1}{2}} - w_j^k) - v_j^{k+\frac{1}{2}} - \frac{1}{2}(|u_j^k|^2 + |2u_j^{k+\frac{1}{2}} - u_j^k|^2) = 0,$$
$$1 \leqslant j \leqslant m-1, \tag{7.52}$$
$$u_0^{k+\frac{1}{2}} = u_m^{k+\frac{1}{2}} = v_0^{k+\frac{1}{2}} = v_m^{k+\frac{1}{2}} = w_0^{k+\frac{1}{2}} = w_m^{k+\frac{1}{2}} = 0. \tag{7.53}$$

由 (7.51) 可得

$$v_j^{k+\frac{1}{2}} = v_j^k + \frac{\tau}{2}\delta_x^2 w_j^{k+\frac{1}{2}}, \quad 1 \leqslant j \leqslant m-1. \tag{7.54}$$

将 (7.54) 代入 (7.50) 和 (7.52), 可得方程组

$$\frac{2}{\tau}(u_j^{k+\frac{1}{2}} - u_j^k) - \mathrm{i}\delta_x^2 u_j^{k+\frac{1}{2}} + \mathrm{i}u_j^{k+\frac{1}{2}}(v_j^k + \frac{\tau}{2}\delta_x^2 w_j^{k+\frac{1}{2}}) = 0,$$
$$1 \leqslant j \leqslant m-1, \tag{7.55}$$

$$\frac{2}{\tau}(w_j^{k+\frac{1}{2}} - w_j^k) - \left(v_j^k + \frac{\tau}{2}\delta_x^2 w_j^{k+\frac{1}{2}}\right) - \frac{1}{2}(|u_j^k|^2 + |2u_j^{k+\frac{1}{2}} - u_j^k|^2) = 0,$$
$$1 \leqslant j \leqslant m-1, \tag{7.56}$$

$$u_0^{k+\frac{1}{2}} = u_m^{k+\frac{1}{2}} = w_0^{k+\frac{1}{2}} = w_m^{k+\frac{1}{2}} = 0. \tag{7.57}$$

记

$$u_j = u_j^{k+\frac{1}{2}}, \quad w_j = w_j^{k+\frac{1}{2}}, \quad 0 \leqslant j \leqslant m.$$

可将 (7.55)–(7.57) 写为

$$\frac{2}{\tau}(u_j - u_j^k) - \mathrm{i}\delta_x^2 u_j + \mathrm{i}u_j\left(v_j^k + \frac{\tau}{2}\delta_x^2 w_j\right) = 0, \quad 1 \leqslant j \leqslant m-1, \tag{7.58}$$

$$\frac{2}{\tau}(w_j - w_j^k) - \left(v_j^k + \frac{\tau}{2}\delta_x^2 w_j\right) - \frac{1}{2}(|u_j^k|^2 + |2u_j - u_j^k|^2) = 0,$$
$$1 \leqslant j \leqslant m-1, \tag{7.59}$$

$$u_0 = u_m = w_0 = w_m = 0. \tag{7.60}$$

观察 (7.59) 和 (7.60) 可知, 如果 $(u_1, u_2, \cdots, u_{m-1})$ 已知, 则可唯一确定 $(w_1, w_2, \cdots, w_{m-1})$. 因而

$$w_j = w_j(u_1, u_2, \cdots, u_{m-1}), \quad j = 1, 2, \cdots, m-1.$$

这样可将 (7.58) 看成是关于 $u_1, u_2, \cdots, u_{m-1}$ 的非线性方程组.

定义算子 $\Pi : \mathring{\mathcal{U}}_h \to \mathring{\mathcal{U}}_h$:

$$\Pi(u)_j = \frac{2}{\tau}(u_j - u_j^k) - \mathrm{i}\delta_x^2 u_j + \mathrm{i}u_j\left(v_j^k + \frac{\tau}{2}\delta_x^2 w_j\right), \quad 1 \leqslant j \leqslant m-1,$$

其中 $(w_1, w_2, \cdots, w_{m-1})$ 由

$$\frac{2}{\tau}(w_j - w_j^k) - \left(v_j^k + \frac{\tau}{2}\delta_x^2 w_j\right) - \frac{1}{2}(|u_j^k|^2 + |2u_j - u_j^k|^2) = 0, \quad 1 \leqslant j \leqslant m-1,$$
$$w_0 = 0, \quad w_m = 0$$

确定. 计算可知

$$\mathrm{Re}(\Pi(u), u) = \frac{2}{\tau}\mathrm{Re}(u - u^k, u) = \frac{2}{\tau}\left[\|u\|^2 - \mathrm{Re}(u^k, u)\right] \geqslant \frac{2}{\tau}\|u\|(\|u\| - \|u^k\|).$$

当 $\|u\| = \|u^k\|$ 时, $\mathrm{Re}(\Pi(u), u) \geqslant 0$.

由 Browder 定理 (定理 1.3) 知方程组 (7.58)–(7.60) 存在解.

于是我们得到如下定理.

定理 7.2 差分格式 (7.36)–(7.40) 存在解.

我们可以用如下迭代格式求解 (7.58)–(7.60).

当 $\{u_j^{(l-1)}\,|\,1 \leqslant j \leqslant m-1\}$ 已知时, 解三对角线性方程组

$$\frac{2}{\tau}(w_j^{(l)} - w_j^k) - \left(v_j^k + \frac{\tau}{2}\delta_x^2 w_j^{(l)}\right) - \frac{1}{2}(|u_j^k|^2 + |2u_j^{(l-1)} - u_j^k|^2) = 0,$$
$$1 \leqslant j \leqslant m-1,$$
$$w_0^{(l)} = 0, \quad w_m^{(l)} = 0,$$

得到 $\{w_j^{(l)}\,|\,1 \leqslant j \leqslant m-1\}$. 然后再解三对角线性方程

$$\frac{2}{\tau}(u_j^{(l)} - u_j^k) - \mathrm{i}\delta_x^2 u_j^{(l)} + \mathrm{i}u_j^{(l)}\left(v_j^k + \frac{\tau}{2}\delta_x^2 w_j^{(l)}\right) = 0, \quad 1 \leqslant j \leqslant m-1,$$
$$u_0^{(l)} = 0, \quad u_m^{(l)} = 0,$$

得到 $\{u_j^{(l)}\,|\,1 \leqslant j \leqslant m-1\}$. 直至

$$\|u^{(l)} - u^{(l-1)}\|_\infty \leqslant \varepsilon.$$

取

$$u_j = u_j^{(l)}, \quad w_j = w_j^{(l)}, \quad 1 \leqslant j \leqslant m-1.$$

由

$$u_j^{k+1} = 2u_j - u_j^k, \quad w_j^{k+1} = 2w_j - w_j^k, \quad 1 \leqslant j \leqslant m-1$$

得到 $\{u^{k+1}, w^{k+1}\}$.

最后根据 (7.37), 得到

$$v_j^{k+1} = v_j^k + \tau\delta_x^2 w_j, \quad 1 \leqslant j \leqslant m-1.$$

7.2.3 差分格式解的守恒性和有界性

定理 7.3 设 $\{u_j^k, v_j^k, w_j^k\,|\,0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (7.36)–(7.40) 的解. 记

$$Q^k = \|u^k\|^2, \quad 0 \leqslant k \leqslant n, \tag{7.61}$$

$$E^k = |u^k|_1^2 + \frac{1}{2}|w^k|_1^2 + \frac{1}{2}\|v^k\|^2 + h\sum_{j=1}^{m-1}|u_j^k|^2 v_j^k, \quad 0 \leqslant k \leqslant n, \tag{7.62}$$

则有

$$Q^k = Q^0, \quad 0 \leqslant k \leqslant n, \tag{7.63}$$

$$E^k = E^0, \quad 0 \leqslant k \leqslant n. \tag{7.64}$$

证明 (I) 用 $h\bar{u}_j^{k+\frac{1}{2}}$ 乘以 (7.36), 对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}(\delta_t u_j^{k+\frac{1}{2}})\bar{u}_j^{k+\frac{1}{2}} + h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})\bar{u}_j^{k+\frac{1}{2}} - h\sum_{j=1}^{m-1} v_j^{k+\frac{1}{2}}|u_j^{k+\frac{1}{2}}|^2 = 0.$$

两边取虚部, 得

$$\frac{1}{2\tau}\left(\|u^{k+1}\|^2 - \|u^k\|^2\right) = 0, \quad 0 \leqslant k \leqslant n-1,$$

于是

$$Q^{k+1} = Q^k, \quad 0 \leqslant k \leqslant n-1.$$

因而 (7.63) 成立.

(II) 用 $-h\delta_t\bar{u}_j^{k+\frac{1}{2}}$ 乘以 (7.36), 并对 j 从 1 到 $m-1$ 求和, 得

$$-\mathrm{i}h\sum_{j=1}^{m-1}|\delta_t u_j^{k+\frac{1}{2}}| - h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{k+\frac{1}{2}})(\delta_t\bar{u}_j^{k+\frac{1}{2}}) + h\sum_{j=1}^{m-1} v_j^{k+\frac{1}{2}} u_j^{k+\frac{1}{2}}\delta_t\bar{u}_j^{k+\frac{1}{2}} = 0,$$
$$0 \leqslant k \leqslant n-1.$$

两边取实部, 得

$$\frac{1}{\tau}(|u^{k+1}|_1^2 - |u^k|_1^2) + h\sum_{j=1}^{m-1} v_j^{k+\frac{1}{2}}\frac{|u_j^{k+1}|^2 - |u_j^k|^2}{\tau} = 0, \quad 0 \leqslant k \leqslant n-1. \tag{7.65}$$

(III) 用 $h\delta_t w_j^{k+\frac{1}{2}}$ 乘以 (7.37), 并对 j 从 1 到 $m-1$ 求和, 得

$$h\sum_{j=1}^{m-1}(\delta_t v_j^{k+\frac{1}{2}})(\delta_t w_j^{k+\frac{1}{2}}) - h\sum_{j=1}^{m-1}(\delta_x^2 w_j^{k+\frac{1}{2}})(\delta_t w_j^{k+\frac{1}{2}}) = 0. \tag{7.66}$$

用 $-h\delta_t v_j^{k+\frac{1}{2}}$ 乘以 (7.38), 并对 j 从 1 到 $m-1$ 求和, 得

$$-h\sum_{j=1}^{m-1}(\delta_t w_j^{k+\frac{1}{2}})\delta_t v_j^{k+\frac{1}{2}} + h\sum_{j=1}^{m-1} v_j^{k+\frac{1}{2}}\delta_t v_j^{k+\frac{1}{2}} + h\sum_{j=1}^{m-1}\frac{|u_j^k|^2 + |u_j^{k+1}|^2}{2}\delta_t v_j^{k+\frac{1}{2}} = 0. \tag{7.67}$$

将 (7.66) 和 (7.67) 相加, 得

$$\frac{1}{2\tau}(|w^{k+1}|_1^2-|w^k|_1^2)+\frac{1}{2\tau}(\|v^{k+1}\|^2-\|v^k\|^2)+h\sum_{j=1}^{m-1}\frac{|u_j^k|^2+|u_j^{k+1}|^2}{2}\delta_t v_j^{k+\frac{1}{2}}=0,$$
$$0\leqslant k\leqslant n-1. \tag{7.68}$$

将 (7.65) 和 (7.68) 相加, 得

$$\frac{1}{\tau}\Bigg[\left(|u^{k+1}|_1^2+\frac{1}{2}|w^{k+1}|_1^2+\frac{1}{2}\|v^{k+1}\|^2+h\sum_{j=1}^{m-1}|u_j^{k+1}|^2v_j^{k+1}\right)$$
$$-\left(|u^k|_1^2+\frac{1}{2}|w^k|_1^2+\frac{1}{2}\|v^k\|^2+h\sum_{j=1}^{m-1}|u_j^k|^2v_j^k\right)\Bigg]=0,\quad 0\leqslant k\leqslant n-1,$$

即

$$\frac{1}{\tau}(E^{k+1}-E^k)=0,\quad 0\leqslant k\leqslant n-1.$$

因而 (7.64) 成立. □

由上述定理可知

$$\begin{aligned}
&|u^k|_1^2+\frac{1}{2}\|v^k\|^2\\
\leqslant&-h\sum_{j=1}^{m-1}|u_j^k|^2v_j^k+E^0\\
\leqslant&\,h\sum_{j=1}^{m-1}\left(|u_j^k|^4+\frac{1}{4}|v_j^k|^2\right)+E^0\\
\leqslant&\,\|u^k\|_\infty^2\|u^k\|^2+\frac{1}{4}\|v^k\|^2+E^0\\
=&\,\|u^k\|_\infty^2Q^0+\frac{1}{4}\|v^k\|^2+E^0\\
\leqslant&\left(\varepsilon|u^k|_1^2+\frac{1}{4\varepsilon}\|u^k\|^2\right)Q^0+\frac{1}{4}\|v^k\|^2+E^0\\
\leqslant&\left(\varepsilon|u^k|_1^2+\frac{1}{4\varepsilon}Q^0\right)Q^0+\frac{1}{4}\|v^k\|^2+E^0.
\end{aligned}$$

取 $\varepsilon=\dfrac{1}{2Q_0}$, 得

$$\frac{1}{2}|u^k|_1^2+\frac{1}{4}\|v^k\|^2\leqslant\frac{1}{2}(Q^0)^3+E^0,\quad 0\leqslant k\leqslant n.$$

再由引理 1.1(b) 得到

$$\frac{2}{L}\|u^k\|_\infty^2+\frac{1}{4}\|v^k\|^2\leqslant\frac{1}{2}(Q^0)^3+E^0,\quad 0\leqslant k\leqslant n.$$

因而存在常数 c_3 使得

$$\|u^k\|_\infty \leqslant c_3, \quad \|v^k\| \leqslant c_3, \quad 0 \leqslant k \leqslant n. \tag{7.69}$$

7.2.4　差分格式解的收敛性

定理 7.4　设 $\{U_j^k, V_j^k, W_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为问题 (7.13)–(7.17) 的解, $\{u_j^k, v_j^k, w_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (7.36)–(7.40) 的解. 记

$$\begin{aligned}
e_j^k &= U_j^k - u_j^k, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n,\\
f_j^k &= V_j^k - v_j^k, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n,\\
g_j^k &= W_j^k - w_j^k, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n,
\end{aligned}$$

则存在常数 c_4 使得

$$\|e^k\| + |e^k|_1 + \|f^k\| + |g^k|_1 \leqslant c_4(\tau^2 + h^2), \quad 0 \leqslant k \leqslant n. \tag{7.70}$$

证明　将 (7.25)–(7.27), (7.34)–(7.35) 与 (7.36)–(7.40) 相减, 得到误差方程组

$$\mathrm{i}\delta_t e_j^{k+\frac12} + \delta_x^2 e_j^{k+\frac12} - (U_j^{k+\frac12} V_j^{k+\frac12} - u_j^{k+\frac12} v_j^{k+\frac12}) = P_j^{k+\frac12},$$
$$1 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{7.71}$$

$$\delta_t f_j^{k+\frac12} - \delta_x^2 g_j^{k+\frac12} = Q_j^{k+\frac12}, \quad 1 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n-1, \tag{7.72}$$

$$\delta_t g_j^{k+\frac12} - f_j^{k+\frac12} - \left(\frac{|U_j^k|^2 + |U_j^{k+1}|^2}{2} - \frac{|u_j^k|^2 + |u_j^{k+1}|^2}{2}\right) = R_j^{k+\frac12},$$
$$0 \leqslant j \leqslant m, \quad 0 \leqslant k \leqslant n-1, \tag{7.73}$$

$$e_j^0 = 0, \quad f_j^0 = 0, \quad g_j^0 = F(\psi_1)(x_j) - G(\psi_1)_j, \quad 0 \leqslant j \leqslant m, \tag{7.74}$$

$$e_0^k = 0, \quad e_m^k = 0, \quad g_0^k = 0, \quad g_m^k = 0, \quad 1 \leqslant k \leqslant n. \tag{7.75}$$

由常微分方程两点边值问题数值解的理论可知存在常数 c 使得

$$|g^0|_1 \leqslant ch^2, \quad \|g^0\|_\infty \leqslant \frac{\sqrt{L}}{2} ch^2. \tag{7.76}$$

(I) 用 $h\bar{e}_j^{k+\frac12}$ 乘以 (7.71) 的两边, 对 j 从 1 到 $m-1$ 求和, 得

$$\begin{aligned}
&\mathrm{i}h\sum_{j=1}^{m-1}(\delta_t e_j^{k+\frac12})\bar{e}_j^{k+\frac12} + h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{k+\frac12})\bar{e}_j^{k+\frac12}\\
={}& h\sum_{j=1}^{m-1}\left(U_j^{k+\frac12}V_j^{k+\frac12} - u_j^{k+\frac12}v_j^{k+\frac12}\right)\bar{e}_j^{k+\frac12} + h\sum_{j=1}^{m-1}P_j^{k+\frac12}\bar{e}_j^{k+\frac12}\\
={}& h\sum_{j=1}^{m-1}\left(U_j^{k+\frac12}f_j^{k+\frac12} + e_j^{k+\frac12}v_j^{k+\frac12}\right)\bar{e}_j^{k+\frac12} + h\sum_{j=1}^{m-1}P_j^{k+\frac12}\bar{e}_j^{k+\frac12},
\end{aligned}$$

取虚部, 得到

$$\begin{aligned}&\frac{1}{2\tau}(\|e^{k+1}\|^2-\|e^k\|^2)\\&=\mathrm{Im}\left\{h\sum_{j=1}^{m-1}U_j^{k+\frac{1}{2}}f_j^{k+\frac{1}{2}}\bar{e}_j^{k+\frac{1}{2}}+h\sum_{j=1}^{m-1}P_j^{k+\frac{1}{2}}\bar{e}_j^{k+\frac{1}{2}}\right\}\\&\leqslant c_1\|f^{k+\frac{1}{2}}\|\cdot\|e^{k+\frac{1}{2}}\|+\|P^{k+\frac{1}{2}}\|\cdot\|e^{k+\frac{1}{2}}\|,\quad 0\leqslant k\leqslant n-1.\end{aligned}\tag{7.77}$$

(II) 用 $-h\delta_t\bar{e}_j^{k+\frac{1}{2}}$ 乘以 (7.71) 的两边, 对 j 从 1 到 $m-1$ 求和, 得

$$\begin{aligned}&-\mathrm{i}h\sum_{j=1}^{m-1}|\delta_te_j^{k+\frac{1}{2}}|^2-h\sum_{j=1}^{m-1}(\delta_x^2e_j^{k+\frac{1}{2}})\delta_t\bar{e}_j^{k+\frac{1}{2}}\\=&-h\sum_{j=1}^{m-1}(U_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}-u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}})\delta_t\bar{e}_j^{k+\frac{1}{2}}-h\sum_{j=1}^{m-1}P_j^{k+\frac{1}{2}}\delta_t\bar{e}_j^{k+\frac{1}{2}},\quad 0\leqslant k\leqslant n-1.\end{aligned}$$

两边取实部, 得

$$\begin{aligned}\frac{1}{2\tau}(|e^{k+1}|_1^2-|e^k|_1^2)=&\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}(U_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}-u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}})\delta_t\bar{e}_j^{k+\frac{1}{2}}\right\}\\&+\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}P_j^{k+\frac{1}{2}}\delta_t\bar{e}_j^{k+\frac{1}{2}}\right\},\quad 0\leqslant k\leqslant n-1.\end{aligned}\tag{7.78}$$

(III) 用 $h\delta_tg_j^{k+\frac{1}{2}}$ 乘以 (7.72) 的两边, 对 j 从 1 到 $m-1$ 求和, 得

$$h\sum_{j=1}^{m-1}(\delta_tf_j^{k+\frac{1}{2}})(\delta_tg_j^{k+\frac{1}{2}})-h\sum_{j=1}^{m-1}(\delta_x^2g_j^{k+\frac{1}{2}})\delta_tg_j^{k+\frac{1}{2}}=h\sum_{j=1}^{m-1}Q_j^{k+\frac{1}{2}}\delta_tg_j^{k+\frac{1}{2}},$$
$$0\leqslant k\leqslant n-1.$$

用 $-h\delta_tf_j^{k+\frac{1}{2}}$ 乘以 (7.73) 的两边, 对 j 从 1 到 $m-1$ 求和, 得

$$\begin{aligned}&-h\sum_{j=1}^{m-1}(\delta_tg_j^{k+\frac{1}{2}})(\delta_tf_j^{k+\frac{1}{2}})+h\sum_{j=1}^{m-1}f_j^{k+\frac{1}{2}}\delta_tf_j^{k+\frac{1}{2}}\\=&-h\sum_{j=1}^{m-1}\frac{(|U_j^{k+1}|^2-|u_j^{k+1}|^2)+(|U_j^k|^2-|u_j^k|^2)}{2}\delta_tf_j^{k+\frac{1}{2}}-h\sum_{j=1}^{m-1}R_j^{k+\frac{1}{2}}\delta_tf_j^{k+\frac{1}{2}},\end{aligned}$$
$$0\leqslant k\leqslant n-1.$$

将以上两式相加, 得到

$$
\begin{aligned}
&\frac{1}{2\tau}(|g^{k+1}|_1^2-|g^k|_1^2)+\frac{1}{2\tau}(\|f^{k+1}\|^2-\|f^k\|^2)\\
&+h\sum_{j=1}^{m-1}\frac{(|U_j^{k+1}|^2-|u_j^{k+1}|^2)f_j^{k+1}-(|U_j^k|^2-|u_j^k|^2)f_j^k}{\tau}\\
=&h\sum_{j=1}^{m-1}\frac{(|U_j^{k+1}|^2-|u_j^{k+1}|^2)-(|U_j^k|^2-|u_j^k|^2)}{\tau}f_j^{k+\frac{1}{2}}\\
&+h\sum_{j=1}^{m-1}Q_j^{k+\frac{1}{2}}\delta_t g_j^{k+\frac{1}{2}}-h\sum_{j=1}^{m-1}R_j^{k+\frac{1}{2}}\delta_t f_j^{k+\frac{1}{2}},\quad 0\leqslant k\leqslant n-1.
\end{aligned}\tag{7.79}
$$

将 (7.78) 乘以 2, 并与 (7.79) 相加, 得

$$
\begin{aligned}
&\frac{1}{\tau}\left\{\left[|e^{k+1}|_1^2+\frac{1}{2}|g^{k+1}|_1^2+\frac{1}{2}\|f^{k+1}\|^2+h\sum_{j=1}^{m-1}(|U_j^{k+1}|^2-|u_j^{k+1}|^2)f_j^{k+1}\right]\right.\\
&\left.-\left[|e^k|_1^2+\frac{1}{2}|g^k|_1^2+\frac{1}{2}\|f^k\|^2+h\sum_{j=1}^{m-1}(|U_j^k|^2-|u_j^k|^2)f_j^k\right]\right\}\\
=&2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}(U_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}-u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}})\delta_t\bar{e}_j^{k+\frac{1}{2}}\right\}\\
&+h\sum_{j=1}^{m-1}\frac{(|U_j^{k+1}|^2-|u_j^{k+1}|^2)-(|U_j^k|^2-|u_j^k|^2)}{\tau}f_j^{k+\frac{1}{2}}\\
&+2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}P_j^{k+\frac{1}{2}}\delta_t\bar{e}_j^{k+\frac{1}{2}}\right\}+h\sum_{j=1}^{m-1}Q_j^{k+\frac{1}{2}}\delta_t g_j^{k+\frac{1}{2}}\\
&-h\sum_{j=1}^{m-1}R_j^{k+\frac{1}{2}}\delta_t f_j^{k+\frac{1}{2}},\quad 0\leqslant k\leqslant n-1.
\end{aligned}\tag{7.80}
$$

记

$$
A^k=2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}(U_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}-u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}})\delta_t\bar{e}_j^{k+\frac{1}{2}}\right\},
$$

$$
B^k=h\sum_{j=1}^{m-1}\frac{(|U_j^{k+1}|^2-|u_j^{k+1}|^2)-(|U_j^k|^2-|u_j^k|^2)}{\tau}f_j^{k+\frac{1}{2}}.
$$

则

$$A^k = 2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}(U_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}} - u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}})\delta_t(\bar{U}_j^{k+\frac{1}{2}} - \bar{u}_j^{k+\frac{1}{2}})\right\}$$

$$= -2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(U_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}\delta_t\bar{U}_j^{k+\frac{1}{2}} - U_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}\delta_t\bar{u}_j^{k+\frac{1}{2}}\right.$$

$$\left. - u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}}\delta_t\bar{U}_j^{k+\frac{1}{2}} + u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}}\delta_t\bar{u}_j^{k+\frac{1}{2}})\right\}$$

$$= -2h\sum_{j=1}^{m-1}\left(V_j^{k+\frac{1}{2}}\frac{|U_j^{k+1}|^2 - |U_j^k|^2}{2\tau} + v_j^{k+\frac{1}{2}}\frac{|u_j^{k+1}|^2 - |u_j^k|^2}{2\tau}\right)$$

$$+ 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(U_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}\delta_t\bar{u}_j^{k+\frac{1}{2}} + u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}}\delta_t\bar{U}_j^{k+\frac{1}{2}})\right\},$$

$$B^k = h\sum_{j=1}^{m-1}\left(\frac{|U_j^{k+1}|^2 - |U_j^k|^2}{\tau} - \frac{|u_j^{k+1}|^2 - |u_j^k|^2}{\tau}\right)(V_j^{k+\frac{1}{2}} - v_j^{k+\frac{1}{2}}).$$

将以上两式相加, 得

$$A^k + B^k$$

$$= 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(U_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}\delta_t\bar{u}_j^{k+\frac{1}{2}} + u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}}\delta_t\bar{U}_j^{k+\frac{1}{2}})\right\}$$

$$+h\sum_{j=1}^{m-1}\left(-\frac{|U_j^{k+1}|^2 - |U_j^k|^2}{\tau}v_j^{k+\frac{1}{2}} - \frac{|u_j^{k+1}|^2 - |u_j^k|^2}{\tau}V_j^{k+\frac{1}{2}}\right)$$

$$= 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(U_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}\delta_t\bar{u}_j^{k+\frac{1}{2}} + u_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}}\delta_t\bar{U}_j^{k+\frac{1}{2}})\right\}$$

$$-2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}U_j^{k+\frac{1}{2}}(\delta_t\bar{U}_j^{k+\frac{1}{2}})v_j^{k+\frac{1}{2}} + u_j^{k+\frac{1}{2}}(\delta_t\bar{u}_j^{k+\frac{1}{2}})V_j^{k+\frac{1}{2}}\right\}$$

$$= 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(-e_j^{k+\frac{1}{2}}v_j^{k+\frac{1}{2}}\delta_t\bar{U}_j^{k+\frac{1}{2}} + e_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}\delta_t\bar{u}_j^{k+\frac{1}{2}})\right\}$$

$$= 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(V_j^{k+\frac{1}{2}}\delta_t\bar{u}_j^{k+\frac{1}{2}} - v_j^{k+\frac{1}{2}}\delta_t\bar{U}_j^{k+\frac{1}{2}})e_j^{k+\frac{1}{2}}\right\}$$

$$
\begin{aligned}
&= 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}[(V_j^{k+\frac{1}{2}}-v_j^{k+\frac{1}{2}})\delta_t\bar{U}_j^{k+\frac{1}{2}}+V_j^{k+\frac{1}{2}}(\delta_t\bar{u}_j^{k+\frac{1}{2}}-\delta_t\bar{U}_j^{k+\frac{1}{2}})]e_j^{k+\frac{1}{2}}\right\}\\
&= 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}f_j^{k+\frac{1}{2}}(\delta_t\bar{U}_j^{k+\frac{1}{2}})e_j^{k+\frac{1}{2}}\right\}+2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}V_j^{k+\frac{1}{2}}\bar{e}_j^{k+\frac{1}{2}}\delta_t e_j^{k+\frac{1}{2}}\right\}. \qquad (7.81)
\end{aligned}
$$

由 (7.71) 可知

$$
\delta_t e_j^{k+\frac{1}{2}} = \mathrm{i}\delta_x^2 e_j^{k+\frac{1}{2}} - \mathrm{i}(e_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}+u_j^{k+\frac{1}{2}}f_j^{k+\frac{1}{2}}) - \mathrm{i}P_j^{k+\frac{1}{2}}.
$$

于是

$$
\begin{aligned}
&-h\sum_{j=1}^{m-1}V_j^{k+\frac{1}{2}}\bar{e}_j^{k+\frac{1}{2}}\delta_t e_j^{k+\frac{1}{2}}\\
=&-h\sum_{j=1}^{m-1}V_j^{k+\frac{1}{2}}\bar{e}_j^{k+\frac{1}{2}}[\mathrm{i}\delta_x^2 e_j^{k+\frac{1}{2}} - \mathrm{i}(e_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}+u_j^{k+\frac{1}{2}}f_j^{k+\frac{1}{2}}) - \mathrm{i}P_j^{k+\frac{1}{2}}]\\
=&\,\mathrm{i}h\sum_{j=0}^{m-1}[\delta_x(V^{k+\frac{1}{2}}\bar{e}^{k+\frac{1}{2}})_{j+\frac{1}{2}}]\delta_x e_{j+\frac{1}{2}}^{k+\frac{1}{2}} + \mathrm{i}h\sum_{j=1}^{m-1}V_j^{k+\frac{1}{2}}\bar{e}_j^{k+\frac{1}{2}}(e_j^{k+\frac{1}{2}}V_j^{k+\frac{1}{2}}+u_j^{k+\frac{1}{2}}f_j^{k+\frac{1}{2}})\\
&+\mathrm{i}h\sum_{j=1}^{m-1}V_j^{k+\frac{1}{2}}\bar{e}_j^{k+\frac{1}{2}}P_j^{k+\frac{1}{2}}. \qquad (7.82)
\end{aligned}
$$

将 (7.82) 代入 (7.81), 并利用 Cauchy-Schwarz 不等式, 可得存在常数 c_5 使得

$$
A^k + B^k \leqslant c_5(\|e^{k+\frac{1}{2}}\|^2 + |e^{k+\frac{1}{2}}|_1^2 + \|f^{k+\frac{1}{2}}\|^2 + \|P^{k+\frac{1}{2}}\|^2). \qquad (7.83)
$$

记

$$
\hat{E}^k = 2(c_1+c_3)^2\|e^k\|^2 + |e^k|_1^2 + \frac{1}{2}\|f^k\|^2 + \frac{1}{2}|g^k|_1^2 + h\sum_{j=1}^{m-1}(|U_j^k|^2-|u_j^k|^2)f_j^k.
$$

可知

$$
\hat{E}^k \geqslant (c_1+c_3)^2\|e^k\|^2 + |e^k|_1^2 + \frac{1}{4}\|f^k\|^2 + \frac{1}{2}|g^k|_1^2. \qquad (7.84)
$$

将 (7.77) 乘以 $4(c_1+c_3)^2$, 并将结果和 (7.80) 相加, 然后利用 (7.83), 知存在常数 c_6 使得

$$
\begin{aligned}
\frac{1}{2\tau}(\hat{E}^{k+1}-\hat{E}^k) \leqslant{}& c_6(\hat{E}^{k+1}+\hat{E}^k) + c_6\|P^{k+\frac{1}{2}}\|^2 + 2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}P_j^{k+\frac{1}{2}}\delta_t\bar{e}_j^{k+\frac{1}{2}}\right\}\\
&+h\sum_{j=1}^{m-1}Q_j^{k+\frac{1}{2}}\delta_t g_j^{k+\frac{1}{2}} - h\sum_{j=1}^{m-1}R_j^{k+\frac{1}{2}}\delta_t f_j^{k+\frac{1}{2}}, \quad 0\leqslant k\leqslant n-1.
\end{aligned}
$$

在上式中将 k 换为 l, 并对 l 从 0 到 k 求和, 得

$$\begin{aligned}\frac{1}{2\tau}(\hat{E}^{k+1}-\hat{E}^0)\leqslant & c_6\sum_{l=0}^{k}(\hat{E}^{l+1}+\hat{E}^l)+c_6\sum_{l=0}^{k}\|P^{l+\frac{1}{2}}\|^2\\ &+2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}\sum_{l=0}^{k}P_j^{l+\frac{1}{2}}\delta_t\bar{e}_j^{l+\frac{1}{2}}\right\}+h\sum_{j=1}^{m-1}\sum_{l=0}^{k}Q_j^{l+\frac{1}{2}}\delta_t g_j^{l+\frac{1}{2}}\\ &-h\sum_{j=1}^{m-1}\sum_{l=0}^{k}R_j^{l+\frac{1}{2}}\delta_t f_j^{l+\frac{1}{2}},\quad 0\leqslant k\leqslant n-1.\end{aligned}\tag{7.85}$$

注意到

$$\begin{aligned}\sum_{l=0}^{k}P_j^{l+\frac{1}{2}}\delta_t\bar{e}_j^{l+\frac{1}{2}}&=\frac{1}{\tau}(P_j^{k+\frac{1}{2}}\bar{e}_j^{k+1}-P_j^{\frac{1}{2}}\bar{e}_j^0)-\sum_{l=0}^{k}\frac{P_j^{l+\frac{1}{2}}-P_j^{l-\frac{1}{2}}}{\tau}\bar{e}_j^l,\\ \sum_{l=0}^{k}Q_j^{l+\frac{1}{2}}\delta_t g_j^{l+\frac{1}{2}}&=\frac{1}{\tau}(Q_j^{k+\frac{1}{2}}g_j^{k+1}-Q_j^{\frac{1}{2}}g_j^0)-\sum_{l=0}^{k}\frac{Q_j^{l+\frac{1}{2}}-Q_j^{l-\frac{1}{2}}}{\tau}g_j^l,\\ \sum_{l=0}^{k}R_j^{l+\frac{1}{2}}\delta_t f_j^{l+\frac{1}{2}}&=\frac{1}{\tau}(R_j^{k+\frac{1}{2}}f_j^{k+1}-R_j^{\frac{1}{2}}f_j^0)-\sum_{l=0}^{k}\frac{R_j^{l+\frac{1}{2}}-R_j^{l-\frac{1}{2}}}{\tau}f_j^l,\end{aligned}$$

以及 (7.74), 由 (7.85) 可得

$$\begin{aligned}&\frac{1}{2\tau}(\hat{E}^{k+1}-\hat{E}^0)\\ \leqslant\ & c_6\hat{E}^{k+1}+2c_6\sum_{l=1}^{k}\hat{E}^l+c_6\hat{E}^0+c_6\sum_{l=0}^{k}\|P^{l+\frac{1}{2}}\|^2\\ &+2h\sum_{j=1}^{m-1}\left\{\frac{1}{\tau}\left|P_j^{k+\frac{1}{2}}\bar{e}_j^{k+1}-P_j^{\frac{1}{2}}\bar{e}_j^0\right|+\sum_{l=1}^{k}\left|\frac{P_j^{l+\frac{1}{2}}-P_j^{l-\frac{1}{2}}}{\tau}\right|\cdot|e_j^l|\right\}\\ &+h\sum_{j=1}^{m-1}\left\{\frac{1}{\tau}\left|Q_j^{k+\frac{1}{2}}g_j^{k+1}-Q_j^{\frac{1}{2}}g_j^0\right|+\sum_{l=1}^{k}\left|\frac{Q_j^{l+\frac{1}{2}}-Q_j^{l-\frac{1}{2}}}{\tau}\right|\cdot|g_j^l|\right\}\\ &+h\sum_{j=1}^{m-1}\left\{\frac{1}{\tau}\left|R_j^{k+\frac{1}{2}}f_j^{k+1}-R_j^{\frac{1}{2}}f_j^0\right|+\sum_{l=1}^{k}\left|\frac{R_j^{l+\frac{1}{2}}-R_j^{l-\frac{1}{2}}}{\tau}\right|\cdot|f_j^l|\right\}\\ \leqslant\ & c_6\hat{E}^{k+1}+2c_6\sum_{l=0}^{k}\hat{E}^l+c_6\hat{E}_0+c_6\sum_{l=0}^{k}\|P^{l+\frac{1}{2}}\|^2\\ &+\frac{2}{\tau}\|P^{k+\frac{1}{2}}\|\cdot\|e^{k+1}\|+\frac{1}{\tau}\|Q^{k+\frac{1}{2}}\|\cdot\|g^{k+1}\|+\frac{1}{\tau}\|Q^{\frac{1}{2}}\|\cdot\|g^0\|\end{aligned}$$

$$+\frac{1}{\tau}\|R^{k+\frac{1}{2}}\|\cdot\|f^{k+1}\|+2\sum_{l=1}^{k}\left\|\frac{P^{l+\frac{1}{2}}-P^{l-\frac{1}{2}}}{\tau}\right\|\cdot\|e^l\|$$
$$+\sum_{l=1}^{k}\left\|\frac{Q^{l+\frac{1}{2}}-Q^{l-\frac{1}{2}}}{\tau}\right\|\cdot\|g^l\|+\sum_{l=1}^{k}\left\|\frac{R^{l+\frac{1}{2}}-R^{l-\frac{1}{2}}}{\tau}\right\|\cdot\|f^l\|.$$

进一步可以得到

$$\begin{aligned}
&\frac{1}{2\tau}(\hat{E}^{k+1}-\hat{E}^0)\\
\leqslant\ & c_6\hat{E}^{k+1}+2c_6\sum_{l=1}^{k}\hat{E}^l+c_6\hat{E}_0+c_6\sum_{l=0}^{k}\|P^{l+\frac{1}{2}}\|\\
&+\frac{1}{\tau}\left(\frac{1}{4}\cdot\frac{6}{L^2}\|e^{k+1}\|^2+4\cdot\frac{L^2}{6}\|P^{k+\frac{1}{2}}\|^2\right)\\
&+\frac{1}{\tau}\left(\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{6}{L^2}\|g^{k+1}\|^2+\frac{L^2}{3}\|Q^{k+\frac{1}{2}}\|^2\right)\\
&+\frac{1}{\tau}\left(\frac{1}{2}\cdot\|Q^{\frac{1}{2}}\|^2+\frac{1}{2}\|g^0\|^2\right)+\frac{1}{\tau}\left(\frac{1}{4}\cdot\frac{1}{2}\|f^{k+1}\|^2+2\|R^{k+\frac{1}{2}}\|^2\right)\\
&+\sum_{l=1}^{k}\left(\frac{6}{L^2}\|e^l\|^2+\frac{L^2}{6}\left\|\frac{P^{l+\frac{1}{2}}-P^{l-\frac{1}{2}}}{\tau}\right\|^2\right)\\
&+\sum_{l=1}^{k}\left(\frac{1}{2}\cdot\frac{6}{L^2}\|g^l\|^2+\frac{1}{2}\cdot\frac{L^2}{6}\left\|\frac{Q^{l+\frac{1}{2}}-Q^{l-\frac{1}{2}}}{\tau}\right\|^2\right)\\
&+\sum_{l=1}^{k}\left(\frac{1}{2}\|f^l\|^2+\frac{1}{2}\left\|\frac{R^{l+\frac{1}{2}}-R^{l-\frac{1}{2}}}{\tau}\right\|^2\right)\\
\leqslant\ & c_6\hat{E}^{k+1}+2c_6\sum_{l=1}^{k}\hat{E}^l+c_6\hat{E}_0+c_6\sum_{l=1}^{k}\|P^{l+\frac{1}{2}}\|^2\\
&+\frac{1}{4\tau}\left(|e^{k+1}|_1^2+\frac{1}{2}|g^{k+1}|_1^2+\frac{1}{2}\|f^{k+1}\|^2\right)\\
&+\frac{1}{\tau}\left(\frac{2}{3}L^2\|P^{k+\frac{1}{2}}\|^2+\frac{L^2}{3}\|Q^{k+\frac{1}{2}}\|^2+2\|R^{k+\frac{1}{2}}\|^2+\frac{1}{2}\|Q^{\frac{1}{2}}\|^2+\frac{1}{2}\|g^0\|^2\right)\\
&+\sum_{l=1}^{k}\left(|e^l|_1^2+\frac{1}{2}|g^l|_1^2+\frac{1}{2}\|f^l\|^2\right)\\
&+\sum_{l=1}^{k}\left(\frac{L^2}{6}\left\|\frac{P^{l+\frac{1}{2}}-P^{l-\frac{1}{2}}}{\tau}\right\|^2+\frac{L^2}{12}\left\|\frac{Q^{l+\frac{1}{2}}-Q^{l-\frac{1}{2}}}{\tau}\right\|^2\right.\\
&\left.+\frac{1}{2}\left\|\frac{R^{l+\frac{1}{2}}-R^{l-\frac{1}{2}}}{\tau}\right\|^2\right),\quad 0\leqslant k\leqslant n-1.
\end{aligned}$$

将上式乘以 4τ, 并移项得

$$
\begin{aligned}
&(1-4c_6\tau)\hat{E}^{k+1}\\
\leqslant &4(1+2c_6)\tau\sum_{l=1}^{k}\hat{E}^l+4c_6\hat{E}_0+4\Big(\frac{2}{3}L^2\|P^{k+\frac{1}{2}}\|^2\\
&+\frac{L^2}{3}\|Q^{k+\frac{1}{2}}\|^2+2\|R^{k+\frac{1}{2}}\|^2+2\|R^{k+\frac{1}{2}}\|^2+\frac{1}{2}\|Q^{\frac{1}{2}}\|+\frac{1}{2}\|g^0\|^2\Big)\\
&+4\tau\sum_{l=1}^{k}\Big(\frac{L^2}{6}\Big\|\frac{P^{l+\frac{1}{2}}-P^{l-\frac{1}{2}}}{\tau}\Big\|^2+\frac{L^2}{12}\Big\|\frac{Q^{l+\frac{1}{2}}-Q^{l-\frac{1}{2}}}{\tau}\Big\|^2\\
&+\frac{1}{2}\Big\|\frac{R^{l+\frac{1}{2}}-R^{l-\frac{1}{2}}}{\tau}\Big\|^2\Big),\quad 0\leqslant k\leqslant n-1.
\end{aligned}
$$

注意到 (7.28)–(7.33), 当 $4c_6\tau\leqslant\dfrac{1}{3}$ 时存在常数 c_7 使得

$$
\hat{E}^{k+1}\leqslant c_7\tau\sum_{l=1}^{k}\hat{E}^l+c_7(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant n-1.
$$

由 Gronwall 不等式得到

$$
\hat{E}^{k+1}\leqslant c_7e^{c_7\tau}(\tau^2+h^2)^2,\quad 0\leqslant k\leqslant n-1.
$$

再注意到 (7.84) 知 (7.70) 成立. □

7.3 三层线性化局部解耦差分格式

7.3.1 差分格式的建立

在点 (x_j,t_k) 处考虑方程 (7.13)–(7.15), 利用数值微分公式, 可得

$$
\mathrm{i}\Delta_tU_j^k+\delta_x^2U_j^{\bar{k}}-U_j^{\bar{k}}V_j^k=\hat{P}_j^k,\quad 1\leqslant j\leqslant m-1,\ 1\leqslant k\leqslant n-1, \tag{7.86}
$$

$$
\Delta_tV_j^k-\delta_x^2W_j^{\bar{k}}=\hat{Q}_j^k,\quad 1\leqslant j\leqslant m-1,\ 1\leqslant k\leqslant n-1, \tag{7.87}
$$

$$
\Delta_tW_j^k-V_j^{\bar{k}}-|U_j^k|^2=\hat{R}_j^k,\quad 0\leqslant j\leqslant m,\ 1\leqslant k\leqslant n-1, \tag{7.88}
$$

存在常数 c_8 使得

$$
|\hat{P}_j^k|\leqslant c_8(\tau^2+h^2),\quad 1\leqslant j\leqslant m-1,\ 1\leqslant k\leqslant n-1, \tag{7.89}
$$

$$
|\hat{Q}_j^k|\leqslant c_8(\tau^2+h^2),\quad 1\leqslant j\leqslant m-1,\ 1\leqslant k\leqslant n-1, \tag{7.90}
$$

$$|\hat{R}_j^k| \leqslant c_8\tau^2, \quad 0 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n-1, \tag{7.91}$$

$$\left|\frac{\hat{P}_j^{k+1} - \hat{P}_j^{k-1}}{2\tau}\right| \leqslant c_8(\tau^2 + h^2), \quad 1 \leqslant j \leqslant m-1,\ 2 \leqslant k \leqslant n-2, \tag{7.92}$$

$$\left|\frac{\hat{Q}_j^{k+1} - \hat{Q}_j^{k-1}}{2\tau}\right| \leqslant c_8(\tau^2 + h^2), \quad 1 \leqslant j \leqslant m-1,\ 2 \leqslant k \leqslant n-2, \tag{7.93}$$

$$\left|\frac{\hat{R}_j^{k+1} - \hat{R}_j^{k-1}}{2\tau}\right| \leqslant c_8\tau^2, \quad 0 \leqslant j \leqslant m,\ 2 \leqslant k \leqslant n-2. \tag{7.94}$$

易知

$$\hat{R}_0^k = 0, \quad \hat{R}_m^k = 0, \quad 1 \leqslant k \leqslant n-1.$$

由方程 (7.13)–(7.15) 及初值条件 (7.16) 可求得 $u_t(x,0), v_t(x,0), w_t(x,0)$.

在 $(x_j, t_{\frac{1}{2}})$ 处考虑方程 (7.13)–(7.15), 利用数值微分公式, 可得

$$\mathrm{i}\delta_t U_j^{\frac{1}{2}} + \delta_x^2 U_j^{\frac{1}{2}} - U_j^{\frac{1}{2}} \hat{v}_j^{\frac{1}{2}} = \hat{P}_j^0, \quad 1 \leqslant j \leqslant m-1, \tag{7.95}$$

$$\delta_t V_j^{\frac{1}{2}} - \delta_x^2 W_j^{\frac{1}{2}} = \hat{Q}_j^0, \quad 1 \leqslant j \leqslant m-1, \tag{7.96}$$

$$\delta_t W_j^{\frac{1}{2}} - V_j^{\frac{1}{2}} - |\hat{u}_j^{\frac{1}{2}}|^2 = \hat{R}_j^0, \quad 0 \leqslant j \leqslant m, \tag{7.97}$$

其中

$$\hat{u}_j^{\frac{1}{2}} = u(x_j, 0) + \frac{\tau}{2} u_t(x_j, 0), \quad \hat{v}_j^{\frac{1}{2}} = v(x_j, 0) + \frac{\tau}{2} v_t(x_j, 0), \quad 1 \leqslant j \leqslant m-1, \tag{7.98}$$

且存在常数 c_9 使得

$$|\hat{P}_j^0| \leqslant c_9(\tau^2 + h^2), \quad 1 \leqslant j \leqslant m-1, \tag{7.99}$$

$$|\hat{Q}_j^0| \leqslant c_9(\tau^2 + h^2), \quad 1 \leqslant j \leqslant m-1, \tag{7.100}$$

$$|\hat{R}_j^0| \leqslant c_9\tau^2, \quad 0 \leqslant j \leqslant m. \tag{7.101}$$

注意到初边值条件 (7.16)—(7.17) 有

$$U_j^0 = \varphi(x_j), \quad V_j^0 = \psi(x_j), \quad W_j^0 = F(\psi_1)(x_j), \quad 0 \leqslant j \leqslant m, \tag{7.102}$$

$$U_0^k = 0, \quad U_m^k = 0, \quad W_0^k = 0, \quad W_m^k = 0, \quad 1 \leqslant k \leqslant n, \tag{7.103}$$

其中 $U_j^0 = F(\psi_1)(x)$ 由 (7.12) 给定.

在 (7.86)–(7.88), (7.95)–(7.97) 中略去小量项, 对问题 (7.13)–(7.17) 建立如下差分格式

$$\mathrm{i}\delta_t u_j^{\frac{1}{2}} + \delta_x^2 u_j^{\frac{1}{2}} - u_j^{\frac{1}{2}} \hat{v}_j^{\frac{1}{2}} = 0, \quad 1 \leqslant j \leqslant m-1, \tag{7.104}$$

$$\delta_t v_j^{\frac{1}{2}} - \delta_x^2 w_j^{\frac{1}{2}} = 0, \quad 1 \leqslant j \leqslant m-1, \tag{7.105}$$

$$\delta_t w_j^{\frac{1}{2}} - v_j^{\frac{1}{2}} - |\hat{u}_j^{\frac{1}{2}}|^2 = 0, \quad 0 \leqslant j \leqslant m, \tag{7.106}$$

$$\mathrm{i}\Delta_t u_j^k + \delta_x^2 u_j^{\bar{k}} - u_j^{\bar{k}} v_j^k = 0, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{7.107}$$

$$\Delta_t v_j^k - \delta_x^2 w_j^{\bar{k}} = 0, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{7.108}$$

$$\Delta_t w_j^k - v_j^{\bar{k}} - |u_j^k|^2 = 0, \quad 0 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n-1, \tag{7.109}$$

$$u_j^0 = \varphi(x_j), \quad v_j^0 = \psi(x_j), \quad w_j^0 = G(\psi_1)_j, \quad 0 \leqslant j \leqslant m, \tag{7.110}$$

$$u_0^k = 0, \quad u_m^k = 0, \quad w_0^k = 0, \quad w_m^k = 0, \quad 1 \leqslant k \leqslant n, \tag{7.111}$$

其中 $\hat{u}_j^{\frac{1}{2}}$ 和 $\hat{v}_j^{\frac{1}{2}}$ 由 (7.98) 确定, $w_j^0 = G(\psi_1)_j (1 \leqslant j \leqslant m-1)$ 由下式确定

$$\delta_x^2 w_j^0 = \psi_1(x_j), \quad 1 \leqslant j \leqslant m-1, \tag{7.112}$$

$$w_0^0 = 0, \quad w_m^0 = 0. \tag{7.113}$$

由 (7.109)–(7.111) 知

$$v_0^k = 0, \quad v_m^k = 0, \quad 0 \leqslant k \leqslant n.$$

7.3.2 差分格式解的存在性

由 (7.106) 可得

$$\delta_x^2 \frac{w_j^{\frac{1}{2}} - w_j^0}{\tau/2} - \delta_x^2 v_j^{\frac{1}{2}} - \delta_x^2 (|\hat{u}_j^{\frac{1}{2}}|^2) = 0, \quad 1 \leqslant j \leqslant m-1.$$

将 (7.112) 代入并注意到 (7.105), 得到

$$\frac{2}{\tau}\left[\delta_t v_j^{\frac{1}{2}} - \psi_1(x_j)\right] - \delta_x^2 v_j^{\frac{1}{2}} - \delta_x^2 (|\hat{u}_j^{\frac{1}{2}}|^2) = 0, \quad 1 \leqslant j \leqslant m-1. \tag{7.114}$$

由 (7.109) 可得

$$\delta_x^2 \frac{w_j^{\bar{k}} - w_j^{k-1}}{\tau} - \delta_x^2 v_j^{\bar{k}} - \delta_x^2 (|u_j^k|^2) = 0, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1.$$

利用 (7.108), 得

$$\frac{1}{\tau}(\Delta_t v_j^k - \delta_x^2 w_j^{k-1}) - \delta_x^2 v_j^{\bar{k}} - \delta_x^2 (|u_j^k|^2) = 0, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1. \tag{7.115}$$

我们可以按如下步骤求解差分格式 (7.104)–(7.113).

(1) 由 (7.112)–(7.113) 解三对角方程组得到 $\{w_j^0\,|\,0\leqslant j\leqslant m\}$, 由 (7.98) 确定 $\{\hat{u}_j^{\frac{1}{2}}\,|\,0\leqslant j\leqslant m\}$ 和 $\{\hat{v}_j^{\frac{1}{2}}\,|\,0\leqslant j\leqslant m\}$.

(2) 解三对角方程组 (7.104) 及边界条件 $u_0^1=0, u_m^1=0$, 得到 $\{u_j^1\,|\,0\leqslant j\leqslant m\}$; 解三对角方程组 (7.114) 及边界条件 $v_0^1=0, v_m^1=0$, 得到 $\{v_j^1\,|\,0\leqslant j\leqslant m\}$; 由 (7.106) 得到 $\{w_j^1\,|\,0\leqslant j\leqslant m\}$.

(3) 当 $\{u_j^{k-1}, u_j^k, v_j^{k-1}, v_j^k, w_j^{k-1}, w_j^k\,|\,0\leqslant j\leqslant m\}$ 已知时, 解三对角方程组 (7.107) 及边界条件 $u_0^{k+1}=0, u_m^{k+1}=0$ 得到 $\{u_j^{k+1}\,|\,0\leqslant j\leqslant m\}$; 解三对角方程组 (7.115) 及边界条件 $v_0^{k+1}=0, v_m^{k+1}=0$ 得到 $\{v_j^{k+1}\,|\,0\leqslant j\leqslant m\}$; 由 (7.109) 得到 $\{w_j^{k+1}\,|\,0\leqslant j\leqslant m\}$.

由以上过程可知, 每一层求解, 只需独立地解两个三对角线方程组. 每一个三对角方程组的系数矩阵都是严格对角占优的. 因而差分格式的解是唯一存在的.

7.3.3 差分格式解的守恒性和有界性

定理 7.5 设 $\{u_j^k, v_j^k, w_j^k\,|\,0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ 为差分格式 (7.104)–(7.113) 的解. 记

$$\begin{aligned}
&Q^k=\|u^k\|^2,\quad 0\leqslant k\leqslant n,\\
&E^k=\frac{1}{2}(|u^{k+1}|_1^2+|u^k|_1^2)+\frac{1}{4}(|w^{k+1}|_1^2+|w^k|_1^2)+\frac{1}{4}(\|v^{k+1}\|^2+\|v^k\|^2)\\
&\qquad\quad+\frac{1}{2}h\sum_{j=1}^{m-1}(v_j^k|u_j^{k+1}|^2+v_1^{k+1}|u_j^k|^2),\quad 0\leqslant k\leqslant n-1,
\end{aligned}$$

则有

$$Q^k=Q^0,\quad 0\leqslant k\leqslant n; \tag{7.116}$$

$$\begin{aligned}
&\frac{1}{2}(|u^1|_1^2+|u^0|_1^2)+\frac{1}{4}(|w^1|_1^2+|w^0|_1^2)+\frac{1}{4}(\|v^1\|^2+\|v^0\|^2)\\
&\quad+\frac{1}{2}\left(h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}|u_j^1|^2+h\sum_{j=1}^{m-1}|\hat{u}_j^{\frac{1}{2}}|^2v_j^1\right)\\
&=|u^0|_1^2+\frac{1}{2}|w^0|_1^2+\frac{1}{2}\|v^0\|^2+\frac{1}{2}\left(h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}|u_j^0|^2+h\sum_{j=1}^{m-1}|\hat{u}_j^{\frac{1}{2}}|^2v_j^0\right);
\end{aligned} \tag{7.117}$$

$$E^k=E^0,\quad 1\leqslant k\leqslant n-1. \tag{7.118}$$

证明 (I) 用 $h\bar{u}_j^{\frac{1}{2}}$ 乘以 (7.104), 并对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}(\delta_t u_j^{\frac{1}{2}})\bar{u}_j^{\frac{1}{2}} + h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\frac{1}{2}})\bar{u}_j^{\frac{1}{2}} - h\sum_{j=1}^{m-1}|u_j^{\frac{1}{2}}|^2\hat{v}_j^{\frac{1}{2}} = 0.$$

两边取虚部, 得到

$$\frac{1}{2\tau}\left(\|u^1\|^2 - \|u^0\|^2\right) = 0.$$

于是

$$Q^1 = Q^0. \tag{7.119}$$

用 $h\bar{u}_j^{\bar{k}}$ 乘以 (7.107), 并对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}(\Delta_t u_j^k)\bar{u}_j^{\bar{k}} + h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\bar{k}})\bar{u}_j^{\bar{k}} - h\sum_{j=1}^{m-1}|u_j^{\bar{k}}|^2 v_j^k = 0, \quad 1 \leqslant k \leqslant n-1.$$

两边取虚部, 得到

$$\frac{1}{2\tau}(\|u^{k+1}\|^2 - \|u^{k-1}\|^2) = 0, \quad 1 \leqslant k \leqslant n-1,$$

即

$$\|u^{k+1}\|^2 = \|u^{k-1}\|^2, \quad 1 \leqslant k \leqslant n-1,$$

或

$$Q^{k+1} = Q^{k-1}, \quad 1 \leqslant k \leqslant n-1, \tag{7.120}$$

综合 (7.119) 和 (7.120) 可得 (7.116).

(II) 用 $-h\delta_t\bar{u}_j^{\frac{1}{2}}$ 乘以 (7.104), 并对 j 从 1 到 $m-1$ 求和, 得

$$-\mathrm{i}h\sum_{j=1}^{m-1}|\delta_t u_j^{\frac{1}{2}}|^2 - h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\frac{1}{2}})\delta_t\bar{u}_j^{\frac{1}{2}} + h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}u_j^{\frac{1}{2}}\delta_t\bar{u}_j^{\frac{1}{2}} = 0.$$

两边取实部, 得到

$$\frac{1}{2\tau}(|u^1|_1^2 - |u^0|_1^2) + h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}\frac{|u_j^1|^2 - |u_j^0|^2}{2\tau} = 0.$$

即

$$|u^1|_1^2 + h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}|u_j^1|^2 = |u^0|_1^2 + h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}|u_j^0|^2. \tag{7.121}$$

用 $h\delta_t w_j^{\frac{1}{2}}$ 乘以 (7.105), 并对 j 从 1 到 $m-1$ 求和, 得

$$h\sum_{j=1}^{m-1}(\delta_t v_j^{\frac{1}{2}})\delta_t w_j^{\frac{1}{2}} - h\sum_{j=1}^{m-1}(\delta_x^2 w_j^{\frac{1}{2}})\delta_t w_j^{\frac{1}{2}} = 0.$$

用 $-h\delta_t v_j^{\frac{1}{2}}$ 乘以 (7.106), 并对 j 从 1 到 $m-1$ 求和, 得

$$-h\sum_{j=1}^{m-1}(\delta_t w_j^{\frac{1}{2}})\delta_t v_j^{\frac{1}{2}} + h\sum_{j=1}^{m-1} v_j^{\frac{1}{2}}\delta_t v_j^{\frac{1}{2}} + h\sum_{j=1}^{m-1}|\hat{u}_j^{\frac{1}{2}}|^2\delta_t v_j^{\frac{1}{2}} = 0.$$

将以上两式相加, 得

$$\frac{1}{2\tau}(|w^1|_1^2 - |w^0|_1^2) + \frac{1}{2\tau}(\|v^1\|^2 - \|v^0\|^2) + \frac{1}{\tau}|\hat{u}_j^{\frac{1}{2}}|^2(v_j^1 - v_j^0) = 0,$$

即

$$\frac{1}{2}|w^1|_1^2 + \frac{1}{2}\|v^1\|^2 + h\sum_{j=1}^{m-1}|\hat{u}_j^{\frac{1}{2}}|^2 v_j^1 = \frac{1}{2}|w^0|_1^2 + \frac{1}{2}\|v^0\|^2 + h\sum_{j=1}^{m-1}|\hat{u}_j^{\frac{1}{2}}|^2 v_j^0. \tag{7.122}$$

将 (7.121) 和 (7.122) 相加, 得

$$\begin{aligned}&|u^1|_1^2 + \frac{1}{2}|w^1|_1^2 + \frac{1}{2}\|v^1\|^2 + h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}|u_j^1|^2 + h\sum_{j=1}^{m-1}|\hat{u}_j^{\frac{1}{2}}|^2 v_j^1\\ =&|u^0|_1^2 + \frac{1}{2}|w^0|_1^2 + \frac{1}{2}\|v^0\|^2 + h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}|u_j^0|^2 + h\sum_{j=1}^{m-1}|\hat{u}_j^{\frac{1}{2}}|^2 v_j^0,\end{aligned}$$

即

$$\begin{aligned}&\frac{1}{2}(|u^1|_1^2 + |u^0|_1^2) + \frac{1}{4}(|w^1|_1^2 + |w^0|_1^2) + \frac{1}{4}(\|v^1\|^2 + \|v^0\|^2)\\ &+\frac{1}{2}\left(h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}|u_j^1|^2 + h\sum_{j=1}^{m-1}|\hat{u}_j^{\frac{1}{2}}|^2 v_j^1\right)\\ =&|u^0|_1^2 + \frac{1}{2}|w^0|_1^2 + \frac{1}{2}\|v^0\|^2 + \frac{1}{2}\left(h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}|u_j^0|^2 + h\sum_{j=1}^{m-1}|\hat{u}_j^{\frac{1}{2}}|^2 v_j^0\right).\end{aligned}$$

因而 (7.117) 成立.

(III) 用 $h\Delta_t\bar{u}_j^{\bar{k}}$ 乘以 (7.107), 并对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}|\Delta_t u_j^k|^2 + h\sum_{j=1}^{m-1}(\delta_x^2 u_j^{\bar{k}})\Delta_t\bar{u}_j^k - h\sum_{j=1}^{m-1}v_j^k u_j^{\bar{k}}\Delta_t\bar{u}_j^k = 0.$$

两边取实部得

$$-\frac{1}{4\tau}(|u^{k+1}|_1^2-|u^{k-1}|_1^2)-h\sum_{j=1}^{m-1}v_j^k\frac{|u_j^{k+1}|^2-|u_j^{k-1}|^2}{4\tau}=0,\quad 1\leqslant k\leqslant n-1,$$

即

$$\frac{1}{2\tau}(|u^{k+1}|_1^2-|u^{k-1}|_1^2)+h\sum_{j=1}^{m-1}v_j^k\frac{|u_j^{k+1}|^2-|u_j^{k-1}|^2}{2\tau}=0,\quad 1\leqslant k\leqslant n-1. \tag{7.123}$$

用 $h\Delta_t w_j^k$ 乘以 (7.108) 的两边, 并对 j 从 1 到 $m-1$ 求和, 得

$$h\sum_{j=1}^{m-1}(\Delta_t v_j^k)\Delta_t w_j^k-h\sum_{j=1}^{m-1}(\delta_x^2 w_j^{\bar{k}})\Delta_t w_j^k=0,\quad 1\leqslant k\leqslant n-1. \tag{7.124}$$

用 $-h\Delta_t v_j^k$ 乘以 (7.109) 的两边, 并对 j 从 1 到 $m-1$ 求和, 得

$$-h\sum_{j=1}^{m-1}(\Delta_t w_j^k)(\Delta_t v_j^k)+h\sum_{j=1}^{m-1}v_j^{\bar{k}}\Delta_t v_j^k+h\sum_{j=1}^{m-1}|u_j^k|^2\Delta_t v_j^k=0,\quad 1\leqslant k\leqslant n-1. \tag{7.125}$$

将 (7.124) 和 (7.125) 相加, 得

$$\frac{1}{4\tau}(|w^{k+1}|_1^2-|w^{k-1}|_1^2)+\frac{1}{4\tau}(\|v^{k+1}\|^2-\|v^{k-1}\|^2)+h\sum_{j=1}^{m-1}|u_j^k|^2\Delta_t v_j^k=0,\quad 1\leqslant k\leqslant n-1. \tag{7.126}$$

将 (7.123) 和 (7.126) 相加, 得

$$\begin{aligned}&\frac{1}{2\tau}(|u^{k+1}|_1^2-|u^{k-1}|_1^2)+\frac{1}{4\tau}(|w^{k+1}|_1^2-|w^{k-1}|_1^2)+\frac{1}{4\tau}(\|v^{k+1}\|^2-\|v^{k-1}\|^2)\\&+h\sum_{j=1}^{m-1}\frac{1}{2\tau}\left[v_j^k(|u_j^{k+1}|^2-|u_j^{k-1}|^2)+|u_j^k|^2(v_j^{k+1}-v_j^{k-1})\right]=0,\quad 1\leqslant k\leqslant n-1,\end{aligned}$$

即

$$\begin{aligned}&\frac{1}{2}(|u^{k+1}|_1^2+|u^k|_1^2)+\frac{1}{4}(|w^{k+1}|_1^2+|w^k|_1^2)+\frac{1}{4}(\|v^{k+1}\|^2+|v^k\|^2)\\&+\frac{1}{2}h\sum_{j=1}^{m-1}(v_j^k|u_j^{k+1}|^2+v_j^{k+1}|u_j^k|^2)\\=&\frac{1}{2}(|u^k|_1^2+|u^{k-1}|_1^2)+\frac{1}{4}(|w^k|_1^2+|w^{k-1}|_1^2)+\frac{1}{4}(\|v^k\|^2+|v^{k-1}\|^2)\\&+\frac{1}{2}h\sum_{j=1}^{m-1}(v_j^{k-1}|u_j^k|^2+v_j^k|u_j^{k-1}|^2),\quad 1\leqslant k\leqslant n-1,\end{aligned}$$

或

$$E^k = E^{k-1}, \quad 1 \leqslant k \leqslant n-1.$$

因而 (7.118) 成立. □

由定理 7.5 可知存在常数 c_{10} 使得

$$\|u^k\|_\infty \leqslant c_{10}, \quad \|v^k\| \leqslant c_{10}, \quad \|w^k\|_\infty \leqslant c_{10}, \quad 0 \leqslant k \leqslant n.$$

7.3.4 差分格式解的收敛性

定理 7.6 设 $\{U_j^k, V_j^k, W_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为问题 (7.13)–(7.17) 的解, $\{u_j^k, v_j^k, w_j^k \,|\, 0 \leqslant j \leqslant m, 0 \leqslant k \leqslant n\}$ 为差分格式 (7.104)–(7.113) 的解. 记

$$e_j^k = U_j^k - u_j^k, \quad f_j^k = V_j^k - v_j^k, \quad g_j^k = W_j^k - w_j^k, \quad 0 \leqslant j \leqslant m,\ 0 \leqslant k \leqslant n.$$

则存在常数 c_{11} 使得

$$\|e^k\| + |e^k|_1 + \|f^k\| + |g^k|_1 \leqslant c_{11}(\tau^2 + h^2), \quad 0 \leqslant k \leqslant n. \tag{7.127}$$

证明 将 (7.95)–(7.97), (7.86)–(7.88), (7.102)–(7.103) 和 (7.104)–(7.111) 依次相减, 得误差方程组

$$\mathrm{i}\delta_t e_j^{\frac{1}{2}} + \delta_x^2 e_j^{\frac{1}{2}} - e_j^{\frac{1}{2}} \hat{v}_j^{\frac{1}{2}} = P_j^0, \quad 1 \leqslant j \leqslant m-1, \tag{7.128}$$

$$\delta_t f_j^{\frac{1}{2}} - \delta_x^2 g_j^{\frac{1}{2}} = \hat{Q}_j^0, \quad 1 \leqslant j \leqslant m-1, \tag{7.129}$$

$$\delta_t g_j^{\frac{1}{2}} - f_j^{\frac{1}{2}} = \hat{R}_j^0, \quad 0 \leqslant j \leqslant m, \tag{7.130}$$

$$\mathrm{i}\Delta_t e_j^k + \delta_x^2 e_j^{\bar{k}} - (U_j^{\bar{k}} V_j^k - u_j^{\bar{k}} v_j^k) = \hat{P}_j^k, \qquad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{7.131}$$

$$\Delta_t f_j^k - \delta_x^2 g_j^{\bar{k}} = \hat{Q}_j^k, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, \tag{7.132}$$

$$\Delta_t g_j^k - f_j^{\bar{k}} - (|U_j^k|^2 - |u_j^k|^2) = \hat{R}_j^k, \quad 0 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant n-1, \tag{7.133}$$

$$e_j^0 = 0,\ f_j^0 = 0,\ g_j^0 = F(\psi_1)(x_j) - G(\psi_1)_j, \quad 0 \leqslant j \leqslant m, \tag{7.134}$$

$$e_0^k = 0,\ e_m^k = 0,\ g_0^k = 0,\ g_m^k = 0,\ 1 \leqslant k \leqslant n. \tag{7.135}$$

由 (7.134)–(7.135) 知

$$\|e^0\| = 0, \quad |e^0|_1 = 0, \quad \|f^0\| = 0. \tag{7.136}$$

由 (7.76) 知存在常数 c 使得

$$|g^0|_1 \leqslant ch^2, \quad \|g^0\|_\infty \leqslant \frac{\sqrt{L}}{2} ch^2. \tag{7.137}$$

此外存在常数 c_{12} 使得

$$|\hat{u}_j^{\frac{1}{2}}| \leqslant c_{12}, \quad |\hat{v}_j^{\frac{1}{2}}| \leqslant c_{12}, \quad 1 \leqslant j \leqslant m-1. \tag{7.138}$$

(I) 用 $h\bar{e}_j^{\frac{1}{2}}$ 与 (7.128) 相乘, 并对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}(\delta_t e_j^{\frac{1}{2}})\bar{e}_j^{\frac{1}{2}} + h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\frac{1}{2}})\bar{e}_j^{\frac{1}{2}} - h\sum_{j=1}^{m-1}|e_j^{\frac{1}{2}}|^2\hat{v}_j^{\frac{1}{2}} = h\sum_{j=1}^{m-1}P_j^0 e_j^{\frac{1}{2}}.$$

两边取虚部, 得

$$\frac{1}{2\tau}(\|e^1\|^2 - \|e^0\|^2) = \mathrm{Im}\left\{h\sum_{j=1}^{m-1}P_j^0 e_j^{\frac{1}{2}}\right\} \leqslant \|P^0\|\cdot\|e^{\frac{1}{2}}\|.$$

注意到 $e^0=0$, 得

$$\frac{1}{2\tau}\|e^1\|^2 \leqslant \|P^0\|\cdot\frac{1}{2}\|e^1\|.$$

再利用 (7.99), 得到

$$\|e^1\| \leqslant \tau\|P^0\| \leqslant c_9\sqrt{L}\tau(\tau^2+h^2) \leqslant c_9\sqrt{L}(\tau^2+h^2). \tag{7.139}$$

(II) 用 $h\delta_t\bar{e}_j^{\frac{1}{2}}$ 与 (7.128) 相乘, 并对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}|\delta_t e_j^{\frac{1}{2}}|^2 + h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\frac{1}{2}})(\delta_t\bar{e}_j^{\frac{1}{2}}) - h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}e_j^{\frac{1}{2}}\delta_t\bar{e}_j^{\frac{1}{2}} = h\sum_{j=1}^{m-1}P_j^0\delta_t e_j^{\frac{1}{2}}.$$

两边取实部, 得

$$-\frac{1}{2\tau}(|e^1|_1^2 - |e^0|_1^2) - h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}\frac{|e_j^1|^2 - |e_j^0|^2}{2\tau} = \mathrm{Re}\Big\{h\sum_{j=1}^{m-1}P_j^0\delta_t\bar{e}_j^{\frac{1}{2}}\Big\}.$$

注意到 $e^0=0$, 有

$$|e^1|_1^2 + h\sum_{j=1}^{m-1}\hat{v}_j^{\frac{1}{2}}|e_j^1|^2 = -2\,\mathrm{Re}\left\{h\sum_{j=1}^{m-1}P_j^0\bar{e}_j^1\right\} \leqslant 2\|P^0\|\cdot\|e^1\| \leqslant \|P^0\|^2 + \|e^1\|^2.$$

利用 (7.138), 得

$$|e^1|_1^2 \leqslant (c_{12}+1)\|e^1\|^2 + \|P^0\|^2.$$

再由 (7.99) 和 (7.139), 得

$$|e^1|_1^2 \leqslant (c_{12}+1)c_9^2 L(\tau^2+h^2)^2 + c_9^2 L(\tau^2+h^2)^2,$$

即

$$|e^1|_1 \leqslant \sqrt{(c_{12}+2)L}c_9(\tau^2+h^2). \tag{7.140}$$

(III) 用 $h\delta_t g_j^{\frac{1}{2}}$ 与 (7.129) 相乘, 并对 j 从 1 到 $m-1$ 求和, 得

$$h\sum_{j=1}^{m-1}(\delta_t f_j^{\frac{1}{2}})\delta_t g_j^{\frac{1}{2}} - h\sum_{j=1}^{m-1}(\delta_x^2 g_j^{\frac{1}{2}})\delta_t g_j^{\frac{1}{2}} = h\sum_{j=1}^{m-1}\hat{Q}_j^0\delta_t g_j^{\frac{1}{2}}.$$

用 $-h\delta_t f_j^{\frac{1}{2}}$ 与 (7.130) 相乘, 并对 j 从 1 到 $m-1$ 求和, 得

$$-h\sum_{j=1}^{m-1}(\delta_t g_j^{\frac{1}{2}})\delta_t f_j^{\frac{1}{2}} + h\sum_{j=1}^{m-1} f_j^{\frac{1}{2}}\delta_t f_j^{\frac{1}{2}} = -h\sum_{j=1}^{m-1}\hat{R}_j^0\delta_t f_j^{\frac{1}{2}}.$$

将以上二式相加, 得

$$\frac{1}{2\tau}(|g^1|_1^2-|g^0|_1^2)+\frac{1}{2\tau}(\|f^1\|^2-\|f^0\|^2) = h\sum_{j=1}^{m-1}\hat{Q}_j^0\delta_t g_j^{\frac{1}{2}} - h\sum_{j=1}^{m-1}\hat{R}_j^0\delta_t f_j^{\frac{1}{2}}.$$

注意到 $f^0=0$, (7.137), 有

$$\begin{aligned}
&|g^1|_1^2+\|f^1\|^2\\
\leqslant{}&|g^0|_1^2+2h\sum_{j=1}^{m-1}\hat{Q}_j^0(g_j^1-g_j^0)-2h\sum_{j=1}^{m-1}\hat{R}_j^0 f_j^1\\
\leqslant{}&|g^0|_1^2+2\|\hat{Q}^0\|\cdot\|g^1\|+2\|\hat{Q}^0\|\cdot\|g^0\|+2\|\hat{R}^0\|\cdot\|f^1\|\\
\leqslant{}&|g^0|_1^2+\left(\frac{3}{L^2}\|g^1\|^2+\frac{L^2}{3}\|\hat{Q}^0\|^2\right)+\left(\|\hat{Q}^0\|^2+\|g^0\|^2\right)+\left(\frac{1}{2}\|f^1\|^2+2\|\hat{R}^0\|^2\right)\\
\leqslant{}&\frac{1}{2}|g^1|_1^2+\frac{1}{2}\|f^1\|^2+\left(1+\frac{L^2}{6}\right)|g^0|_1^2+\left(1+\frac{L^2}{3}\right)\|\hat{Q}^0\|^2+2\|\hat{R}^0\|^2.
\end{aligned}$$

注意到 (7.100)–(7.101), 可得

$$\begin{aligned}
|g^1|_1^2+\|f^1\|^2\leqslant{}&2\left(1+\frac{L^2}{6}\right)|g^0|_1^2+2\left(1+\frac{L^2}{3}\right)\|\hat{Q}^0\|^2+4\|\hat{R}^0\|^2\\
\leqslant{}&2\left(1+\frac{L^2}{6}\right)c^2h^4+2\left(4+\frac{L^2}{3}\right)c_9^2L(\tau^2+h^2)^2\\
\leqslant{}&\left[2\left(1+\frac{L^2}{6}\right)c^2+2\left(4+\frac{L^2}{3}\right)c_9^2L\right](\tau^2+h^2)^2.
\end{aligned} \tag{7.141}$$

由 (7.136), (7.137), (7.139)–(7.141) 知 (7.127) 对 $k=0$ 和 $k=1$ 成立.

(IV) 用 $h\bar{e}_j^{\bar{k}}$ 与 (7.131) 的两边, 并对 j 从 1 到 $m-1$ 求和, 得

$$\mathrm{i}h\sum_{j=1}^{m-1}(\Delta_t e_j^k)\bar{e}_j^{\bar{k}}+\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\bar{k}})\bar{e}_j^{\bar{k}}-h\sum_{j=1}^{m-1}(e_j^{\bar{k}}v_j^k+U_j^{\bar{k}}f_j^k)\bar{e}_j^{\bar{k}}=h\sum_{j=1}^{m-1}\hat{P}_j^k\bar{e}_j^{\bar{k}}.$$

上式取虚部, 得

$$\begin{aligned}\frac{1}{4\tau}(\|e^{k+1}\|^2-\|e^{k-1}\|^2)&\leqslant \mathrm{Im}\Big\{h\sum_{j=1}^{m-1}U_j^{\bar{k}}f_j^k\bar{e}_j^{\bar{k}}+h\sum_{j=1}^{m-1}\hat{P}_j^k\bar{e}_j^{\bar{k}}\Big\}\\&\leqslant c_1\|f^k\|\cdot\|e^{\bar{k}}\|+\|\hat{P}^k\|\cdot\|e^{\bar{k}}\|.\end{aligned}\tag{7.142}$$

(V) 用 $-h\Delta_t\bar{e}_j^k$ 与 (7.131) 相乘, 并对 j 从 1 到 $m-1$ 求和, 得

$$\begin{aligned}&-\mathrm{i}h\sum_{j=1}^{m-1}|\Delta_t e_j^k|^2-h\sum_{j=1}^{m-1}(\delta_x^2 e_j^{\bar{k}})\delta_t\bar{e}_j^{\bar{k}}+h\sum_{j=1}^{m-1}(U_j^{\bar{k}}V_j^k-u_j^{\bar{k}}v_j^k)\Delta_t\bar{e}_j^k\\&=-h\sum_{j=1}^{m-1}\hat{P}_j^k\Delta_t\bar{e}_j^k,\quad 1\leqslant k\leqslant n-1.\end{aligned}$$

两边取实部, 得

$$\begin{aligned}\frac{1}{4\tau}(|e^{k+1}|_1^2-|e^{k-1}|_1^2)=&-\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(U_j^{\bar{k}}V_j^k-u_j^{\bar{k}}v_j^k)\Delta_t\bar{e}_j^k\right\}\\&+\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}\hat{P}_j^k\Delta_t\bar{e}_j^k\right\},\quad 1\leqslant k\leqslant n-1.\end{aligned}\tag{7.143}$$

(VI) 用 $h\Delta_t g_j^k$ 与 (7.132) 相乘, 并对 j 从 1 到 $m-1$ 求和, 得

$$h\sum_{j=1}^{m-1}(\Delta_t f_j^k)\Delta_t g_j^k-h\sum_{j=1}^{m-1}(\delta_x^2 g_j^{\bar{k}})\Delta_t g_j^k=h\sum_{j=1}^{m-1}\hat{Q}_j^k\Delta_t g_j^k,\quad 1\leqslant k\leqslant n-1.\tag{7.144}$$

用 $-h\Delta_t f_j^k$ 与 (7.133) 相乘, 并对 j 从 1 到 $m-1$ 求和, 得

$$\begin{aligned}-h\sum_{j=1}^{m-1}(\Delta_t g_j^k)\Delta_t f_j^k+h\sum_{j=1}^{m-1}f_j^{\bar{k}}\Delta_t f_j^k+h\sum_{j=1}^{m-1}(|U_j^k|^2-|u_j^k|^2)\Delta_t f_j^k-h\sum_{j=1}^{m-1}\hat{R}_j^k\Delta_t f_j^k,\\1\leqslant k\leqslant n-1.\end{aligned}\tag{7.145}$$

将 (7.144) 和 (7.145) 相加, 得

$$
\begin{aligned}
&\frac{1}{4\tau}(|g^{k+1}|_1^2-|g^{k-1}|_1^2)+\frac{1}{4\tau}(\|f^{k+1}\|^2-\|f^{k-1}\|^2)\\
&+\frac{1}{\tau}\left[h\sum_{j=1}^{m-1}\frac{(|U_j^{k+1}|^2-|u_j^{k+1}|^2)f_j^k+(|U_j^k|^2-|u_j^k|^2)f_j^{k+1}}{2}\right.\\
&\left.-h\sum_{j=1}^{m-1}\frac{(|U_j^k|^2-|u_j^k|^2)f_j^{k-1}+(|U_j^{k-1}|^2-|u_j^{k-1}|^2)f_j^k}{2}\right]\\
=&h\sum_{j=1}^{m-1}\Big(\frac{|U_j^{k+1}|^2-|U_j^{k-1}|^2}{2\tau}-\frac{|u_j^{k+1}|^2-|u_j^{k-1}|^2}{2\tau}\Big)f_j^k\\
&+h\sum_{j=1}^{m-1}\hat{Q}_j^k\Delta_t g_j^k-h\sum_{j=1}^{m-1}\hat{R}_j^k\Delta_t f_j^k.
\end{aligned}
\tag{7.146}
$$

记

$$
C^k=-2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(U_j^{\bar{k}}V_j^k-u_j^{\bar{k}}v_j^k)\Delta_t\bar{e}_j^k\right\},
$$

$$
D^k=h\sum_{j=1}^{m-1}\Big(\frac{|U_j^{k+1}|^2-|U_j^{k-1}|^2}{2\tau}-\frac{|u_j^{k+1}|^2-|u_j^{k-1}|^2}{2\tau}\Big)f_j^k.
$$

计算得

$$
\begin{aligned}
C^k=&-2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(U_j^{\bar{k}}V_j^k-u_j^{\bar{k}}v_j^k)(\Delta_t\bar{U}_j^k-\Delta_t\bar{u}_j^k)\right\}\\
=&-2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(U_j^{\bar{k}}V_j^k\Delta_t\bar{U}_j^k+u_j^{\bar{k}}v_j^k\Delta_t\bar{u}_j^k-U_j^{\bar{k}}V_j^k\Delta_t\bar{u}_j^k-u_j^{\bar{k}}v_j^k\Delta_t\bar{U}_j^k)\right\};\\
D^k=&2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(U_j^{\bar{k}}\Delta_t\bar{U}_j^k-u_j^{\bar{k}}\Delta_t\bar{u}_j^k)(V_j^k-v_j^k)\right\}\\
=&2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(U_j^{\bar{k}}V_j^k\Delta_t\bar{U}_j^k+u_j^{\bar{k}}v_j^k\Delta_t\bar{u}_j^k-u_j^{\bar{k}}V_j^k\Delta_t\bar{u}_j^k-U_j^{\bar{k}}v_j^k\Delta_t\bar{U}_j^k)\right\}.
\end{aligned}
$$

将以上两式相加，得到

$$
\begin{aligned}
C^k + D^k &= 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}[(U_j^{\bar{k}} - u_j^{\bar{k}})V_j^k\Delta_t\bar{u}_j^k - (U_j^{\bar{k}} - u_j^{\bar{k}})v_j^k\Delta_t\bar{U}_j^k]\right\}\\
&= 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(V_j^k\Delta_t\bar{u}_j^k - v_j^k\Delta_t\bar{U}_j^k)e_j^{\bar{k}}\right\}\\
&= 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}[V_j^k(\Delta_t\bar{u}_j^k - \Delta_t\bar{U}_j^k) + (V_j^k - v_j^k)\Delta_t\bar{U}_j^k]e_j^{\bar{k}}\right\}\\
&= 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}(-V_j^k\Delta_t\bar{e}_j^k + f_j^k\Delta_t\bar{U}_j^k)e_j^{\bar{k}}\right\}\\
&= 2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}V_j^k\bar{e}_j^{\bar{k}}\Delta_t e_j^k\right\} + 2\mathrm{Re}\left\{h\sum_{j=1}^{m-1}f_j^k(\Delta_t\bar{U}_j^k)e_j^{\bar{k}}\right\}. \quad (7.147)
\end{aligned}
$$

由 (7.131) 可知

$$
\begin{aligned}
\Delta_t e_j^k &= \mathrm{i}\delta_x^2 e_j^{\bar{k}} - \mathrm{i}(U_j^{\bar{k}}V_j^k - u_j^{\bar{k}}v_j^k) - \mathrm{i}\hat{P}_j^k\\
&= \mathrm{i}\delta_x^2 e_j^{\bar{k}} - \mathrm{i}(U_j^{\bar{k}}f_j^k + e_j^{\bar{k}}v_j^k) - \mathrm{i}\hat{P}_j^k. \quad (7.148)
\end{aligned}
$$

将 (7.148) 代入 (7.147), 可得存在常数 c_{12} 使得

$$
C^k + D^k \leqslant c_{12}(\|e^{\bar{k}}\|^2 + |e^{\bar{k}}|_1^2 + \|f^k\|^2 + \|\hat{P}^k\|^2). \quad (7.149)
$$

记

$$
\begin{aligned}
F^k = {}& 2(c_1 + c_{10})^2\frac{\|e^{k+1}\|^2 + \|e^k\|^2}{2} + \frac{|e^{k+1}|_1^2 + |e^k|_1^2}{2} + \frac{\|f^{k+1}\|^2 + \|f^k\|^2}{4}\\
&+ \frac{|g^{k+1}|_1^2 + |g^k|_1^2}{4} + h\sum_{j=1}^{m-1}\frac{(|U_j^{k+1}|^2 - |u_j^{k+1}|^2)f_j^k + (|U_j^k|^2 - |u_j^k|^2)f_j^{k+1}}{2}.
\end{aligned}
$$

则

$$
\begin{aligned}
F^k \geqslant {}& (c_1 + c_{10})^2\frac{\|e^{k+1}\|^2 + \|e^k\|^2}{2} + \frac{|e^{k+1}|_1^2 + |e^k|_1^2}{2}\\
&+ \frac{\|f^{k+1}\|^2 + \|f^k\|^2}{8} + \frac{|g^{k+1}|_1^2 + |g^k|_1^2}{4}.
\end{aligned}
$$

将 (7.142) 乘以 $4(c_1 + c_{10})^2$, 将 (7.143) 乘以 2, 并将结果与 (7.146) 相加, 然后利用 (7.149), 得到存在常数 c_{13} 和 c_{14} 使得

$$
\begin{aligned}
\frac{1}{\tau}(F^k-F^{k-1})\leqslant\, & c_{13}(\|e^{\bar{k}}\|^2+|e^{\bar{k}}|_1^2+\|f^k\|^2+\|\hat{P}^k\|^2)\\
&+2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}\hat{P}_j^k\Delta_t\bar{e}_j^k\right\}+h\sum_{j=1}^{m-1}\hat{Q}_j^k\Delta_t g^k-h\sum_{j=1}^{m-1}\hat{R}_j^k\Delta_t f_j^k\\
\leqslant\, & c_{14}(F^k+F^{k-1})+2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}\hat{P}_j^k\Delta_t\bar{e}_j^k\right\}\\
&+h\sum_{j=1}^{m-1}\hat{Q}_j^k\Delta_t g_j^k-h\sum_{j=1}^{m-1}\hat{R}_j^k\Delta_t f_j^k,\quad 1\leqslant k\leqslant n-1.
\end{aligned}
$$

将上式中 k 换为 l, 并对 l 从 1 到 k 求和, 得

$$
\begin{aligned}
\frac{1}{\tau}(F^k-F^0)\leqslant\, & c_{14}(F^k+F^0)+2c_{14}\sum_{l=1}^{k-1}F^l\\
&+2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}\sum_{l=1}^{k}\hat{P}_j^l\Delta_t\bar{e}_j^l\right\}+h\sum_{j=1}^{m-1}\sum_{l=1}^{k}\hat{Q}_j^l\Delta_t g_j^l\\
&-h\sum_{j=1}^{m-1}\sum_{l=1}^{k}\hat{R}_j^l\Delta_t f_j^l,\quad 1\leqslant k\leqslant n-1.
\end{aligned}\tag{7.150}
$$

计算得

$$
\begin{aligned}
\sum_{l=1}^{k}\hat{P}_j^l\Delta_t\bar{e}_j^l&=\frac{1}{2\tau}\sum_{l=1}^{k}\hat{P}_j^l(\bar{e}_j^{l+1}-\bar{e}_j^{l-1})=\frac{1}{2\tau}\left(\sum_{l=2}^{k+1}\hat{P}_j^{l-1}\bar{e}_j^l-\sum_{l=0}^{k-1}\hat{P}_j^{l+1}\bar{e}_j^l\right)\\
&=\frac{1}{2\tau}(\hat{P}_j^{k-1}\bar{e}_j^k+\hat{P}_j^k\bar{e}_j^{k+1}-\hat{P}_j^1\bar{e}_j^0-\hat{P}_j^2\bar{e}_j^1)-\sum_{l=2}^{k-1}\frac{\hat{P}_j^{l+1}-\hat{P}_j^{l-1}}{2\tau}\bar{e}_j^l.
\end{aligned}
$$

于是

$$
\begin{aligned}
&\left|2\mathrm{Re}\left\{-h\sum_{j=1}^{m-1}\sum_{l=1}^{k}\hat{P}_j^l\Delta_t\bar{e}_j^l\right\}\right|\\
\leqslant\, & \frac{1}{\tau}(\|\hat{P}^{k-1}\|\cdot\|e^k\|+\|\hat{P}^k\|\cdot\|e^{k+1}\|+\|\hat{P}^1\|\cdot\|e^0\|+\|\hat{P}^2\|\cdot\|e^1\|)\\
&+2\sum_{l=2}^{k-1}\left\|\frac{P^{l+1}-\hat{P}^{l-1}}{2\tau}\right\|\cdot\|e^l\|\\
\leqslant\, & \frac{1}{2\tau}\left[\left(\frac{3}{L^2}\|e^k\|^2+\frac{L^2}{3}\|\hat{P}^{k-1}\|^2\right)+\left(\frac{3}{L^2}\|e^{k+1}\|^2+\frac{L^2}{3}\|\hat{P}^k\|^2\right)+\frac{3}{L^2}\|e^1\|^2+\frac{L^2}{3}\|\hat{P}^2\|^2\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=2}^{k-1}\left(\frac{3}{L^2}\|e^l\|^2+\frac{L^2}{3}\left\|\frac{\hat{P}^{l+1}-\hat{P}^{l-1}}{2\tau}\right\|^2\right)\\
\leqslant&\frac{1}{2\tau}\left[\frac{|e^k|_1^2+|e^{k+1}|_1^2}{2}+\frac{|e^1|_1^2}{2}+\frac{L^2}{3}(\|\hat{P}^{k-1}\|^2+\|\hat{P}^k\|^2+\|\hat{P}^2\|^2)\right]\\
&+\frac{1}{2}\sum_{l=2}^{k-1}|e^l|_1^2+\frac{L^2}{3}\sum_{l=2}^{k-1}\left\|\frac{\hat{P}^{l+1}-\hat{P}^{l-1}}{2\tau}\right\|^2,\quad 1\leqslant k\leqslant n-1.
\end{aligned}\tag{7.151}
$$

同理, 由

$$
\sum_{l=1}^{k}\hat{Q}_j^l\Delta_t g_j^l=\frac{1}{2\tau}(\hat{Q}_j^{k-1}g_j^k+\hat{Q}_j^k g_j^{k+1}-\hat{Q}_j^1 g_j^0-\hat{Q}_j^2 g_j^1)-\sum_{l=2}^{k-1}\frac{\hat{Q}_j^{l+1}-\hat{Q}_j^{l-1}}{2\tau}g_j^l,
$$

得

$$
\begin{aligned}
&\left|h\sum_{j=1}^{m-1}\sum_{l=1}^{k}\hat{Q}_j^l\Delta_t g_j^l\right|\\
\leqslant&\frac{1}{2\tau}(\|\hat{Q}^{k-1}\|\cdot\|g^k\|+\|\hat{Q}^k\|\cdot\|g^{k+1}\|+\|\hat{Q}^1\|\cdot\|g^0\|+\|\hat{Q}^2\|\cdot\|g^1\|)\\
&+\sum_{l=2}^{k-1}\left\|\frac{\hat{Q}^{l+1}-\hat{Q}^{l-1}}{2\tau}\right\|\cdot\|g^l\|\\
\leqslant&\frac{1}{2\tau}\left[\left(\frac{1}{4}\cdot\frac{6}{L^2}\|g^k\|^2+\frac{L^2}{6}\|\hat{Q}^{k-1}\|^2\right)+\left(\frac{1}{4}\cdot\frac{6}{L^2}\|g^{k+1}\|^2+\frac{L^2}{6}\|\hat{Q}^k\|^2\right)\right.\\
&\left.+\left(\frac{1}{4}\cdot\frac{6}{L^2}\|g^0\|^2+\frac{L^2}{6}\|\hat{Q}^1\|^2\right)+\left(\frac{1}{4}\cdot\frac{6}{L^2}\|g^1\|^2+\frac{L^2}{6}\|\hat{Q}^2\|^2\right)\right]\\
&+\sum_{l=2}^{k-1}\left(\frac{1}{4}\cdot\frac{6}{L^2}\|g^l\|^2+\frac{L^2}{6}\left\|\frac{\hat{Q}^{l+1}-\hat{Q}^{l-1}}{2\tau}\right\|^2\right)\\
\leqslant&\frac{1}{2\tau}\left[\frac{1}{4}(|g^k|_1^2+|g^{k+1}|_1^2)+\frac{1}{4}(|g^0|_1^2+|g^1|_1^2)\right.\\
&\left.+\frac{L^2}{6}(\|\hat{Q}^{k-1}\|^2+\|\hat{Q}^k\|^2+\|\hat{Q}^1\|^2+\|\hat{Q}^2\|^2)\right]\\
&+\frac{1}{4}\sum_{l=2}^{k-1}|g^l|_1^2+\frac{L^2}{6}\sum_{l=2}^{k-1}\left\|\frac{\hat{Q}^{l+1}-\hat{Q}^{l-1}}{2\tau}\right\|^2,\quad 1\leqslant k\leqslant n-1.
\end{aligned}\tag{7.152}
$$

由

$$
\sum_{l=1}^{k}\hat{R}_j^l\Delta_t f_j^l=\frac{1}{2\tau}(\hat{R}_j^{k-1}f_j^k+\hat{R}_j^k f_j^{k+1}-\hat{R}_j^1 f_j^0-\hat{R}_j^2 f_j^1)-\sum_{l=2}^{k-1}\frac{\hat{R}_j^{l+1}-\hat{R}_j^{l-1}}{2\tau}f_j^l,
$$

得

$$
\begin{aligned}
&\left|-h\sum_{j=1}^{m-1}\sum_{l=1}^{k}\hat{R}_j^l\Delta_t f_j^l\right| \\
\leqslant&\frac{1}{2\tau}(\|\hat{R}^{k-1}\|\cdot\|f^k\|+\|\hat{R}^k\|\cdot\|f^{k+1}\|+\|\hat{R}^1\|\cdot\|f^0\|+\|\hat{R}^2\|\cdot\|f^1\|) \\
&+\sum_{l=2}^{k-1}\left\|\frac{\hat{R}^{l+1}-\hat{R}^{l-1}}{2\tau}\right\|\cdot\|f^l\| \\
\leqslant&\frac{1}{2\tau}\left[\left(\frac{1}{4}\|f^k\|^2+\|\hat{R}^{k-1}\|^2\right)+\left(\frac{1}{4}\|f^{k+1}\|^2+\|\hat{R}^k\|^2\right)+\left(\frac{1}{4}\|f^1\|^2+\|\hat{R}^2\|^2\right)\right] \\
&+\sum_{l=2}^{k-1}\left(\frac{1}{4}\|f^l\|^2+\left\|\frac{\hat{R}^{l+1}-\hat{R}^{l-1}}{2\tau}\right\|^2\right). \qquad (7.153)
\end{aligned}
$$

将 (7.151)–(7.153) 代入 (7.150)，并利用 (7.89)–(7.94), (7.136), (7.139), (7.140)–(7.141), 当 $2\tau c_{14}\leqslant\dfrac{1}{3}$ 时, 存在常数 c_{15} 使得

$$F^k\leqslant c_{15}\sum_{l=1}^{k-1}F^l+c_{15}(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n-1.$$

由 Gronwall 不等式, 得

$$F^k\leqslant e^{c_{15}T}c_{15}(\tau^2+h^2)^2,\quad 1\leqslant k\leqslant n-1.$$

因而 (7.127) 成立. □

7.4 小结与延拓

本章首先引进新变量 $w(x,t)$, 将初边值问题 (7.1)–(7.4) 写为等价的初边值问题 (7.13)–(7.17). 证明了初边值问题 (7.13)–(7.17) 的解满足两个守恒律. 接着在 7.2 节和 7.3 节研究了两个差分格式.

7.2 节给出的差分格式是二层非线性差分格式. 证明了差分格式解的存在性和唯一性, 证明了差分格式解满足两个能量守恒律, 证明了差分格式解的收敛性.

7.3 节建立了三层线性化局部解耦差分格式. 在每一时间层上只要解两个独立的三对角线性方程组. 证明了差分格式解的存在性和唯一性, 证明了差分格式解满足两个能量守恒律, 证明了差分格式解的收敛性.

国内学者郭柏灵、常谦顺较早地研究了 Zakharov 方程的有限差分方法 [2, 13, 15]. 本章是在他们的工作基础上发展而成的.

与 Zakharov 方程相近的一类非线性方程为 Klein-Gordon-Zakharov 方程:

$$
\begin{aligned}
&u_{tt} - u_{xx} + u + uv + |u|^2 u = 0, \\
&v_{tt} - u_{xx} = (|u|^2)_{xx}.
\end{aligned}
$$

Wang 在 [33] 对上述问题建立了三层非线性差分格式, 证明了差分格式的解满足离散的守恒律、差分格式解的存在性和收敛性. Wang 在 [34] 中对上述问题建立了三层线性化差分格式.

多维 Zakharov 方程形式如下

$$
\begin{aligned}
&\mathrm{i}u_t + \Delta u - uv = 0, \\
&v_{tt} - \Delta v - \Delta(|u|^2) = 0.
\end{aligned}
$$

第 8 章　Ginzburg-Landau 方程的有限差分方法

Ginzburg-Landau 方程是由物理学家 Ginzburg 和 Landau 在 20 世纪 50 年代作为低温超导模型提出的. 这个模型于 2003 年获 Nobel 物理学奖. 该模型在非平衡流体动力学系统和物理相变过程等领域也有广泛的应用.

8.1 引　　言

本章考虑二维 Ginzburg-Landau 方程初边值问题

$$u_t-(\nu+\mathrm{i}\alpha)\Delta u+(\kappa+\mathrm{i}\beta)|u|^2u-\gamma u=0,\quad (x,y)\in\Omega,\quad 0<t\leqslant T,\tag{8.1}$$

$$u(x,y,t)=0,\quad (x,y)\in\partial\Omega,\quad 0<t\leqslant T,\tag{8.2}$$

$$u(x,y,0)=\varphi(x,y),\quad (x,y)\in\bar{\Omega},\tag{8.3}$$

其中 $\Omega=(0,L_1)\times(0,L_2),\partial\Omega$ 为 Ω 的边界, Δ 为 Laplace 算子, $\nu>0,\kappa>0,\alpha,\beta,\gamma$ 为给定的实常数. 当 $x\in\partial\Omega$ 时 $\varphi(x,y)=0$.

在介绍差分格式之前, 我们先用能量方法给出问题 (8.1)–(8.3) 解的先验估计式.

定理 8.1　设 $u(x,y,t)$ 为问题 (8.1)–(8.3) 的解, 则有

$$\|u(\cdot,\cdot,t)\|\leqslant e^{\gamma t}\|u(\cdot,\cdot,0)\|,\quad 0\leqslant t\leqslant T.\tag{8.4}$$

证明　用 u 与 (8.1) 的两边作内积, 得

$$(u_t,u)-(\nu+\mathrm{i}\alpha)(\Delta u,u)+(\kappa+\mathrm{i}\beta)\|u\|_4^4-\gamma\|u\|^2=0,\quad 0<t\leqslant T.\tag{8.5}$$

注意到 (8.2), 由分部积分公式, 得

$$-(\Delta u,u)=\|\nabla u\|^2,\quad 0\leqslant t\leqslant T.\tag{8.6}$$

将 (8.6) 代入 (8.5), 并取实部, 得

$$\frac{1}{2}(\|u(\cdot,\cdot,t)\|^2)_t+\nu\|\nabla u(\cdot,\cdot,t)\|^2+\kappa\|u(\cdot,\cdot,t)\|_4^4-\gamma\|u(\cdot,\cdot,t)\|^2=0.$$

于是

$$(\|u(\cdot,\cdot,t)\|^2)_t\leqslant 2\gamma\|u(\cdot,\cdot,t)\|^2,\quad 0\leqslant t\leqslant T.$$

由 Gronwall 不等式, 得到

$$\|u(\cdot,\cdot,t)\|^2 \leqslant e^{2\gamma t}\|u(\cdot,\cdot,0)\|^2, \quad 0 \leqslant t \leqslant T.$$

两边开平方, 得

$$\|u(\cdot,\cdot,t)\| \leqslant e^{\gamma t}\|u(\cdot,\cdot,0)\|, \quad 0 \leqslant t \leqslant T.$$

□

记

$$c_0 = \max_{(x,y,t)\in\overline{\Omega}\times[0,T]} |u(x,y,t)|.$$

8.2 二层非线性差分格式

取正整数 m_1, m_2 和 n, 并令 $h_1 = \dfrac{L_1}{m_1}$, $h_2 = \dfrac{L_2}{m_2}$, $\tau = \dfrac{T}{n}$. 记

$$x_i = ih_1, \quad y_j = jh_2, \quad t_k = k\tau, \quad t_{k+\frac{1}{2}} = \frac{1}{2}(t_k + t_{k+1}),$$

$$\Omega_{h_1,h_2} = \{(x_i,y_j)\,|\,0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2\}, \quad \Omega_\tau = \{t_k\,|\,0 \leqslant k \leqslant n\},$$

$$\overline{\omega} = \{(i,j)\,|\,0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2\},$$

$$\omega = \{(i,j)\,|\,1 \leqslant i \leqslant m_1 - 1, 1 \leqslant j \leqslant m_2 - 1\}, \quad \partial\omega = \overline{\omega}\backslash\omega,$$

$$w_i = \begin{cases} 1, & 1 \leqslant i \leqslant m_1 - 1, \\ \dfrac{1}{2}, & i = 0, m_1, \end{cases} \qquad \overline{w}_j = \begin{cases} 1, & 1 \leqslant j \leqslant m_2 - 1, \\ \dfrac{1}{2}, & j = 0, m_2. \end{cases}$$

记

$$\mathcal{V}_h = \{v\,|\,v\text{为 }\Omega_{h_1,h_2}\text{ 上的网格函数}\},$$
$$\mathring{\mathcal{V}}_h = \{v\,|\,v \in \mathcal{V}_h, \text{当}(i,j) \in \partial\omega\text{时}v_{ij} = 0\}.$$

设 $u \in \mathcal{V}_h$, 引进如下记号:

$$\delta_x u_{i+\frac{1}{2},j} = \frac{1}{h_1}(v_{i+1,j} - v_{ij}), \quad \delta_x^2 v_{ij} = \frac{1}{h_1^2}(v_{i-1,j} - 2v_{ij} + v_{i+1,j}),$$
$$\delta_y v_{i,j+\frac{1}{2}} = \frac{1}{h_2}(v_{i,j+1} - v_{ij}), \quad \delta_y^2 v_{ij} = \frac{1}{h_2^2}(v_{i,j-1} - 2v_{i,j} + v_{i,j+1}),$$
$$\Delta_h v_{ij} = \delta_x^2 u_{ij} + \delta_y^2 u_{ij}.$$

易知

$$\delta_x^2 u_{ij} = \frac{1}{h_1}(\delta_x u_{i+\frac{1}{2},j} - \delta_x u_{i-\frac{1}{2},j}), \quad \delta_y^2 u_{ij} = \frac{1}{h_2}(\delta_y u_{i,j+\frac{1}{2}} - \delta_y u_{i,j-\frac{1}{2}}).$$

设 $u,v\in\mathcal{V}_h$, 引进如下内积和范数

$$(u,v)=h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}w_i\overline{w}_ju_{ij}\bar{v}_{ij},\quad \|u\|=\sqrt{(u,u)},$$

$$\|\delta_xu\|=\sqrt{h_1h_2\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2}\overline{w}_j|\delta_xu_{i+\frac{1}{2},j}|^2},\quad \|\delta_yu\|=\sqrt{h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2-1}w_i|\delta_yu_{i,j+\frac{1}{2}}|^2},$$

$$\|\nabla_hu\|=\sqrt{\|\delta_xu\|^2+\|\delta_yu\|^2},\quad \|\Delta_hu\|=\sqrt{h_1h_2\sum_{i=1}^{m_1-1}\sum_{j=1}^{m_2-1}(\Delta_hu_{ij})^2},$$

$$\|u\|_4=\sqrt[4]{h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}w_i\overline{w}_j|u_{ij}|^4},\quad \|u\|_\infty=\max_{0\leqslant i\leqslant m_1,0\leqslant j\leqslant m_2}|u_{ij}|,\quad |u|_1=\|\nabla_hu\|.$$

记

$$\mathcal{S}_\tau=\{w\,|\,w=(w_0,w_1,\cdots,w_n)\text{为}\Omega_\tau\text{上的网格函数}\}.$$

设 $w\in\mathcal{S}_\tau$, 引进如下记号:

$$w^{k+\frac{1}{2}}=\frac{1}{2}(w^k+w^{k+1}),\qquad w^{\bar{k}}=\frac{1}{2}(w^{k+1}+w^{k-1}),$$

$$\delta_tw^{k+\frac{1}{2}}=\frac{1}{\tau}(w^{k+1}-w^k),\quad \Delta_tw^k=\frac{1}{2\tau}(w^{k+1}-w^{k-1}).$$

易知

$$\Delta_tw^k=\frac{1}{2}(\delta_tw^{k-\frac{1}{2}}+\delta_tw^{k+\frac{1}{2}}).$$

设 $u=\{u_{ij}^k\,|\,0\leqslant i\leqslant m_1,1\leqslant j\leqslant m_2,0\leqslant k\leqslant n\}$ 为 $\Omega_{h_1,h_2}\times\Omega_\tau$ 上的网格函数, 则 $v=\{u_{ij}^k\,|\,0\leqslant i\leqslant m_1,0\leqslant j\leqslant m_2\}$ 为 Ω_{h_1,h_2} 上的网格函数, $w=\{u_{ij}^k\,|\,0\leqslant k\leqslant n\}$ 为 Ω_τ 上的网格函数.

引理 8.1 ([4]) 设 $u\in\mathring{\mathcal{V}}_h$, 则有

$$\|u\|\leqslant\frac{1}{\sqrt{\frac{6}{L_1^2}+\frac{6}{L_2^2}}}|u|_1.$$

证明 由引理 1.1(b) 可得

$$h_1\sum_{i=1}^{m_1-1}u_{ij}^2\leqslant\frac{L_1^2}{6}h_1\sum_{i=1}^{m_1}(\delta_xu_{i-\frac{1}{2},j})^2.$$

将上式乘以 h_2 并对 j 求和, 得到

$$h_1h_2\sum_{i=1}^{m_1-1}\sum_{j=1}^{m_2-1}u_{ij}^2\leqslant\frac{L_1^2}{6}h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2-1}(\delta_xu_{i-\frac{1}{2},j})^2,$$

即

$$\frac{6}{L_1^2}\|u\|^2 \leqslant \|\delta_x u\|^2.$$

同理, 可得

$$\frac{6}{L_2^2}\|u\|^2 \leqslant \|\delta_y u\|^2.$$

将以上两式相加, 得到

$$\left(\frac{6}{L_1^2}+\frac{6}{L_2^2}\right)\|u\|^2 \leqslant \|\delta_x u\|^2+\|\delta_y u\|^2=|u|_1^2.$$

因而

$$\|u\| \leqslant \frac{1}{\sqrt{\frac{6}{L_1^2}+\frac{6}{L_2^2}}}|u|_1.$$

□

引理 8.2 设 $u \in \mathring{\mathcal{V}}_h$, 则有

$$(\Delta_h u, u)=-|u|_1^2, \qquad |u|_1^2 \leqslant \|u\| \cdot \|\Delta_h u\|.$$

证明 由分部求和公式得到

$$\begin{aligned}
(\Delta_h u, u) &= h_1 h_2 \sum_{i=1}^{m_1-1} \sum_{j=1}^{m_2-1}\left(\delta_x^2 u_{ij}+\delta_y^2 u_{ij}\right) u_{ij} \\
&= h_1 h_2 \sum_{i=1}^{m_1-1} \sum_{j=1}^{m_2-1}\left(\delta_x^2 u_{ij}\right) u_{ij}+h_1 h_2 \sum_{i=1}^{m_1-1} \sum_{j=1}^{m_2-1}\left(\delta_y^2 u_{ij}\right) u_{ij} \\
&= h_2 \sum_{j=1}^{m_2-1}\left[h_1 \sum_{i=1}^{m_1-1}\left(\delta_x^2 u_{ij}\right) u_{ij}\right]+h_1 \sum_{i=1}^{m_1-1}\left[h_2 \sum_{j=1}^{m_2-1}\left(\delta_y^2 u_{ij}\right) u_{ij}\right] \\
&= h_2 \sum_{j=1}^{m_2-1}\left[-h_1 \sum_{i=1}^{m_1}\left(\delta_x u_{i-\frac{1}{2}, j}\right)^2\right]+h_1 \sum_{i=1}^{m_1-1}\left[-h_2 \sum_{j=1}^{m_2}\left(\delta_y^2 u_{i, j-\frac{1}{2}}\right)^2\right] \\
&= -|u|_1^2 .
\end{aligned}$$

再由 Cauchy-Schwarz 不等式得到

$$|u|_1^2=-(\Delta_h u, u) \leqslant \|u\| \cdot \|\Delta_h u\|.$$

□

引理 8.3 ([21]) 设 $u \in \mathring{\mathcal{V}}_h$, 则有

$$|u|_1 \leqslant \frac{1}{\sqrt{\dfrac{6}{L_1^2}+\dfrac{6}{L_2^2}}}\|\Delta_h u\|.$$

证明　综合应用引理 8.1 和引理 8.2 得到

$$|u|_1^2 \leqslant \|u\| \cdot \|\Delta_h u\| \leqslant \frac{1}{\sqrt{\dfrac{6}{L_1^2}+\dfrac{6}{L_2^2}}}|u|_1\|\Delta_h u\|.$$

两边约去 $|u|_1$, 得到

$$|u|_1 \leqslant \frac{1}{\sqrt{\dfrac{6}{L_1^2}+\dfrac{6}{L_2^2}}}\|\Delta_h u\|.$$

□

引理 8.4 ([21])　设 $u \in \mathring{\mathcal{V}}_h$, 则有

$$\|u\|_\infty \leqslant \frac{1}{8}\sqrt{2L_1L_2}\|\Delta_h u\|.$$

证明　注意到

$$\begin{aligned}
&\|\Delta_h u\|^2\\
=&h_1h_2\sum_{i=1}^{m_1-1}\sum_{j=1}^{m_2-1}\left(\delta_x^2u_{ij}+\delta_y^2u_{ij}\right)^2\\
=&h_1h_2\sum_{i=1}^{m_1-1}\sum_{j=1}^{m_2-1}\left(\delta_x^2u_{ij}\right)^2+2h_1h_2\sum_{i=1}^{m_1-1}\sum_{j=1}^{m_2-1}\left(\delta_x^2u_{ij}\right)\left(\delta_y^2u_{ij}\right)\\
&+h_1h_2\sum_{i=1}^{m_1-1}\sum_{j=1}^{m_2-1}\left(\delta_y^2u_{ij}\right)^2\\
=&h_1h_2\sum_{i=1}^{m_1-1}\sum_{j=1}^{m_2-1}\left(\delta_x^2u_{ij}\right)^2+2h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(\delta_x\delta_yu_{i-\frac{1}{2},j-\frac{1}{2}}\right)^2\\
&+h_1h_2\sum_{i=1}^{m_1-1}\sum_{j=1}^{m_2-1}\left(\delta_y^2u_{ij}\right)^2\\
=&\|\delta_x^2u\|^2+2\|\delta_x\delta_yu\|^2+\|\delta_y^2u\|^2,
\end{aligned}$$

可得

$$\|\delta_x\delta_yu\|^2 \leqslant \frac{1}{2}\|\Delta_h u\|^2. \tag{8.7}$$

对任意 $(i,j)\in\omega$, 有

$$\begin{aligned}
u_{ij}^2 &\leqslant \frac{L_1}{4}h_1\sum_{l=1}^{m_1}(\delta_xu_{l-\frac{1}{2},j})^2\\
&\leqslant \frac{L_1}{4}h_1\sum_{l=1}^{m_1}\left[\frac{L_2}{4}h_2\sum_{s=1}^{m_2}(\delta_y\delta_xu_{l-\frac{1}{2},s-\frac{1}{2}})^2\right]\\
&=\frac{L_1L_2}{16}\|\delta_y\delta_xu\|^2.
\end{aligned}$$

因而

$$\|u\|_\infty^2 \leqslant \frac{L_1L_2}{16}\|\delta_y\delta_x u\|^2.$$

应用 (8.7), 得到

$$\|u\|_\infty^2 \leqslant \frac{L_1L_2}{32}\|\Delta_h u\|^2.$$

两边开方, 得

$$\|u\|_\infty \leqslant \frac{1}{8}\sqrt{2L_1L_2}\|\Delta_h u\|.$$

□

8.2.1 差分格式的建立

定义 $\Omega_{h_1\times h_2}\times\Omega_\tau$ 上的网格函数 U :

$$U_{ij}^k = u(x_i, y_j, t_k), \quad (i,j)\in\overline{\omega}, \quad 0\leqslant k\leqslant n.$$

在点 $(x_i, y_j, t_{k+\frac{1}{2}})$ 处考虑方程 (8.1), 有

$$\begin{aligned}&u_t(x_i,y_j,t_{k+\frac{1}{2}}) - (\nu+\mathrm{i}\alpha)\Delta u(x_i,y_j,t_{k+\frac{1}{2}}) + (\kappa+\mathrm{i}\beta)(|u|^2u)(x_i,y_j,t_{k+\frac{1}{2}})\\&-\gamma u(x_i,y_j,t_{k+\frac{1}{2}}) = 0, \quad (i,j)\in\omega, \quad 0\leqslant k\leqslant n-1.\end{aligned}$$

应用数值微分公式可得

$$\begin{aligned}\delta_t U_{ij}^{k+\frac{1}{2}} - (\nu+\mathrm{i}\alpha)\Delta_h U_{ij}^{k+\frac{1}{2}} + (\kappa+\mathrm{i}\beta)|U_{ij}^{k+\frac{1}{2}}|^2 U_{ij}^{k+\frac{1}{2}} - \gamma U_{ij}^{k+\frac{1}{2}} = R_{ij}^{k+\frac{1}{2}},\\ (i,j)\in\omega, \quad 0\leqslant k\leqslant n-1,\end{aligned} \tag{8.8}$$

且存在常数 c_1 使得

$$|R_{ij}^{k+\frac{1}{2}}| \leqslant c_1(\tau^2+h_1^2+h_2^2), \quad (i,j)\in\omega,\ 0\leqslant k\leqslant n-1. \tag{8.9}$$

在 (8.8) 中略去小量项 $R_{ij}^{k+\frac{1}{2}}$, 并注意到初边值条件

$$U_{ij}^0 = \varphi(x_i, y_j), \quad (i,j)\in\omega, \tag{8.10}$$

$$U_{ij}^k = 0, \quad (i,j)\in\partial\omega,\ 0\leqslant k\leqslant n, \tag{8.11}$$

对 (8.1)–(8.3) 建立如下差分格式

$$\begin{aligned}\delta_t u_{ij}^{k+\frac{1}{2}} - (\nu+\mathrm{i}\alpha)\Delta_h u_{ij}^{k+\frac{1}{2}} + (\kappa+\mathrm{i}\beta)|u_{ij}^{k+\frac{1}{2}}|^2 u_{ij}^{k+\frac{1}{2}} - \gamma u_{ij}^{k+\frac{1}{2}} = 0,\\ (i,j)\in\omega, \quad 0\leqslant k\leqslant n-1,\end{aligned} \tag{8.12}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j)\in\omega, \tag{8.13}$$

$$u_{ij}^k = 0, \quad (i,j)\in\partial\omega,\ 0\leqslant k\leqslant n. \tag{8.14}$$

8.2.2 差分格式解的存在性

差分格式 (8.12)–(8.14) 是一个二层非线性差分格式. 称

$$u^k=\{u_{ij}^k\,|\,(i,j)\in\overline{\omega}\}$$

为第 k 层值. 由 (8.13)–(8.14) 知第 0 层值 u^0 已知. 现假设已求得第 k 层值 u^k. 可将 (8.12), (8.14) 看成是关于平均值 $u^{k+\frac{1}{2}}=\dfrac{1}{2}(u^k+u^{k+1})$ 的方程组:

$$\frac{2}{\tau}(u_{ij}^{k+\frac{1}{2}}-u_{ij}^k)-(\nu+\mathrm{i}\alpha)\Delta_h u_{ij}^{k+\frac{1}{2}}+(\kappa+\mathrm{i}\beta)|u_{ij}^{k+\frac{1}{2}}|^2u_{ij}^{k+\frac{1}{2}}-\gamma u_{ij}^{k+\frac{1}{2}}=0,\quad (i,j)\in\omega, \tag{8.15}$$

$$u_{ij}^{k+\frac{1}{2}}=0,\quad (i,j)\in\partial\omega. \tag{8.16}$$

若求得了 $\{u_{ij}^{k+\frac{1}{2}}\,|\,(i,j)\in\bar{\omega}\}$, 则有

$$u_{ij}^{k+1}=2u_{ij}^{k+\frac{1}{2}}-u_{ij}^k,\quad (i,j)\in\bar{\omega}.$$

记

$$w_{ij}=u_{ij}^{k+\frac{1}{2}},\quad (i,j)\in\overline{\omega}.$$

则 (8.15)–(8.16) 可写为

$$\frac{2}{\tau}(w_{ij}-u_{ij}^k)-(\nu+\mathrm{i}\alpha)\Delta_h w_{ij}+(\kappa+\mathrm{i}\beta)|w_{ij}|^2w_{ij}-\gamma w_{ij}=0,\quad (i,j)\in\omega, \tag{8.17}$$

$$w_{ij}=0,\quad (i,j)\in\partial\omega. \tag{8.18}$$

记

$$\Pi(w)_{ij}=\frac{2}{\tau}(w_{ij}-u_{ij}^k)-(\nu+\mathrm{i}\alpha)\Delta_h w_{ij}+(\kappa+\mathrm{i}\beta)|w_{ij}|^2w_{ij}-\gamma w_{ij},\quad (i,j)\in\omega.$$

则

$$\begin{aligned}\big(\Pi(w),w\big)=&\frac{2}{\tau}(\|w\|^2-(u^k,w))-(\nu+\mathrm{i}\alpha)(\Delta_h w,w)+(\kappa+\mathrm{i}\beta)\|w\|_4^4-\gamma\|w\|^2\\=&\frac{2}{\tau}(\|w\|^2-(u^k,w))+(\nu+\mathrm{i}\alpha)|w|_1^2+(\kappa+\mathrm{i}\beta)\|w\|_4^4-\gamma\|w\|^2.\end{aligned}$$

于是

$$\begin{aligned}\mathrm{Re}\big(\Pi(w),w\big)=&\frac{2}{\tau}(\|w\|^2-\mathrm{Re}(u^k,w))+\nu|w|_1^2+\kappa\|w\|_4^4-\gamma\|w\|^2\\\geqslant&\frac{2}{\tau}(\|w\|^2-\|u^k\|\cdot\|w\|)-\gamma\|w\|^2\\=&\frac{2}{\tau}\left[\left(1-\frac{1}{2}\gamma\tau\right)\|w\|-\|u^k\|\right]\cdot\|w\|.\end{aligned}$$

当 $\frac{1}{2}\gamma\tau < 1$ 且 $\|w\| = \dfrac{\|u^k\|}{1-\frac{1}{2}\gamma\tau}$ 时,

$$\mathrm{Re}(\Pi(w), w) \geqslant 0.$$

由 Browder 定理 (定理 1.3) 知存在 w^* 且 $\|w^*\| \leqslant \dfrac{\|u^k\|}{1-\frac{1}{2}\gamma\tau}$ 使得

$$\Pi(w^*) = 0.$$

即 (8.17)–(8.18) 存在解. 于是得到如下定理.

定理 8.2 差分格式 (8.12)–(8.14) 存在解.

8.2.3 差分格式解的有界性

定理 8.3 设 $\{u_{ij}^k \,|\, (i,j) \in \overline{\omega}, 0 \leqslant k \leqslant n\}$ 为差分格式 (8.12)–(8.14) 的解. 则当 $\gamma\tau \leqslant \frac{2}{3}$ 时, 有

$$\|u^k\| \leqslant e^{\frac{3}{2}T\max\{0,\gamma\}}\|u^0\|, \quad 0 \leqslant k \leqslant n.$$

证明 用 $u^{k+\frac{1}{2}}$ 与 (8.12) 的两边作内积, 得

$$\begin{aligned}&(\delta_t u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}) - (\nu + \mathrm{i}\alpha)(\Delta_h u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})\\&+(\kappa + \mathrm{i}\beta)(|u^{k+\frac{1}{2}}|^2 u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}) - \gamma(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}) = 0.\end{aligned} \tag{8.19}$$

注意到

$$-(\Delta_h u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}}) = \|\nabla_h u^{k+\frac{1}{2}}\|^2,$$

对 (8.19) 两边取实部, 得

$$\frac{1}{2\tau}(\|u^{k+1}\|^2 - \|u^k\|^2) + \nu\|\nabla_h u^{k+\frac{1}{2}}\|^2 + \kappa\|u^{k+\frac{1}{2}}\|_4^4 - \gamma\|u^{k+\frac{1}{2}}\|^2 = 0, \quad 0 \leqslant k \leqslant n-1. \tag{8.20}$$

当 $\gamma \leqslant 0$ 时, 有

$$\frac{1}{2\tau}(\|u^{k+1}\|^2 - \|u^k\|^2) \leqslant 0, \quad 0 \leqslant k \leqslant n-1.$$

于是

$$\|u^k\| \leqslant \|u^0\|, \quad 0 \leqslant k \leqslant n.$$

当 $\gamma > 0$ 时, 由 (8.20) 得到

$$\frac{1}{2\tau}(\|u^{k+1}\|^2 - \|u^k\|^2) \leqslant \gamma \cdot \|u^{k+\frac{1}{2}}\|^2 \leqslant \left(\frac{\|u^k\| + \|u^{k+1}\|}{2}\right)^2, \quad 0 \leqslant k \leqslant n-1.$$

将上式两边约去 $\frac{1}{2}(\|u^{k+1}\|+\|u^{k+1}\|)$, 得

$$\frac{1}{\tau}(\|u^{k+1}\|-\|u^k\|)\leqslant\frac{\gamma}{2}(\|u^k\|+\|u^{k+1}\|),\quad 0\leqslant k\leqslant n-1. \tag{8.21}$$

当 $\tau\leqslant\frac{2}{3}$ 时, 由上式可得

$$\|u^{k+1}\|\leqslant\frac{1+\frac{\gamma}{2}\tau}{1-\frac{\gamma}{2}\tau}\|u^k\|\leqslant\left(1+\frac{3\gamma\tau}{2}\right)\|u^k\|,\quad 0\leqslant k\leqslant n-1.$$

递推得到

$$\|u^k\|\leqslant e^{\frac{3\gamma}{2}k\tau}\|u^0\|\leqslant e^{\frac{3\gamma}{2}T}\|u^0\|,\quad 0\leqslant k\leqslant n.$$

□

注 8.1 由上述定理可知存在常数 c_2 使得

$$\|u^k\|\leqslant c_2,\quad 0\leqslant k\leqslant n. \tag{8.22}$$

注 8.2 将 (8.20) 中的 k 换为 l, 再对 l 从 0 到 k 求和, 并利用 (8.22) 可得

$$\|u^{k+1}\|^2+2\tau\nu\sum_{l=0}^{k}\|\nabla_h u^{k+\frac{1}{2}}\|^2+2\kappa\tau\sum_{l=0}^{k}\|u^{k+\frac{1}{2}}\|_4^4\leqslant(1+2T)c_2^2,$$
$$0\leqslant k\leqslant n-1. \tag{8.23}$$

8.2.4 差分格式解的收敛性

引理 8.5([42]) 设 $u\in\mathcal{V}_h$, 则有

$$\|u\|_4^4\leqslant\left[4\|\nabla_h u\|^2+\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\|u\|^2\right]\|u\|^2.$$

证明 当 $0\leqslant s\leqslant m\leqslant m_1$ 时

$$\begin{aligned}|u_{mj}|^2-|u_{sj}|^2=&\sum_{i=s}^{m-1}(|u_{i+1,j}|^2-|u_{ij}|^2)\\=&\sum_{i=s}^{m-1}(|u_{i+1,j}|-|u_{ij}|)(|u_{i+1,j}|+|u_{ij}|)\\\leqslant&h_1\sum_{i=s}^{m-1}\left|\frac{u_{i+1,j}-u_{ij}}{h_1}\right|(|u_{i+1,j}|+|u_{ij}|)\\\leqslant&\left(h_1\sum_{i=s}^{m-1}|\delta_x u_{i+\frac{1}{2},j}|^2\right)^{\frac{1}{2}}\left[h_1\sum_{j=s}^{m-1}(|u_{i+1,j}|+|u_{ij}|)^2\right]^{\frac{1}{2}}\\\leqslant&2\left(h_1\sum_{i=0}^{m_1-1}|\delta_x u_{i+\frac{1}{2},j}|^2\right)^{\frac{1}{2}}\left(h_1\sum_{i=0}^{m_1}w_i|u_{ij}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

上式对 $0 \leqslant m \leqslant s \leqslant m_1$ 也是成立的. 因而

$$|u_{mj}|^2 \leqslant 2\left(h_1 \sum_{i=0}^{m_1-1} |\delta_x u_{i+\frac{1}{2},j}|^2\right)^{\frac{1}{2}} \left(h_1 \sum_{i=0}^{m_1} w_i |u_{ij}|^2\right)^{\frac{1}{2}} + |u_{sj}|^2,$$
$$0 \leqslant m, s \leqslant m_1.$$

将上式乘以 $w_s h_1$, 并对 s 从 0 到 m_1 求和, 得

$$L_1|u_{mj}|^2 \leqslant 2L_1 \left(h_1 \sum_{i=0}^{m_1-1} |\delta_x u_{i+\frac{1}{2},j}|^2\right)^{\frac{1}{2}} \left(h_1 \sum_{i=0}^{m_1} w_i |u_{ij}|^2\right)^{\frac{1}{2}} + h_1 \sum_{s=0}^{m_1} w_s |u_{sj}|^2,$$
$$0 \leqslant m \leqslant m_1.$$

上式除以 L_1, 并对 m 取最大值, 得到

$$\max_{0\leqslant i\leqslant m_1} |u_{ij}|^2 \leqslant 2\left(h_1 \sum_{i=0}^{m_1-1} |\delta_x u_{i+\frac{1}{2},j}|^2\right)^{\frac{1}{2}} \left(h_1 \sum_{i=0}^{m_1} w_i |u_{ij}|^2\right)^{\frac{1}{2}} + \frac{1}{L_1} h_1 \sum_{s=0}^{m_1} w_s |u_{sj}|^2.$$

再将上式乘以 $h_2 \overline{w}_j$, 并对 j 从 0 到 m_2 求和, 得

$$\begin{aligned}
&h_2 \sum_{j=0}^{m_2} \overline{w}_j \max_{0\leqslant i\leqslant m_1} |u_{ij}|^2 \\
\leqslant &2h_2 \sum_{j=0}^{m_2} \overline{w}_j \left(h_1 \sum_{i=0}^{m_1-1} |\delta_x u_{i+\frac{1}{2},j}|^2\right)^{\frac{1}{2}} \left(h_1 \sum_{i=0}^{m_1} w_i |u_{ij}|^2\right)^{\frac{1}{2}} + \frac{1}{L_1}\|u\|^2 \\
\leqslant &2\left(h_2 \sum_{j=0}^{m_2} \overline{w}_j h_1 \sum_{i=0}^{m_1-1} |\delta_x u_{i+\frac{1}{2},j}|^2\right)^{\frac{1}{2}} \left(h_2 \sum_{j=0}^{m_2} \overline{w}_j h_1 \sum_{i=0}^{m_1} w_i |u_{ij}|^2\right)^{\frac{1}{2}} + \frac{1}{L_1}\|u\|^2 \\
= &2\|\delta_x u\|\ \|u\| + \frac{1}{L_1}\|u\|^2.
\end{aligned} \tag{8.24}$$

类似地可得

$$h_1 \sum_{i=0}^{m_1} w_i \max_{0\leqslant j\leqslant m_2} |u_{ij}|^2 \leqslant 2\|\delta_y u\|\ \|u\| + \frac{1}{L_2}\|u\|^2. \tag{8.25}$$

现在来估计 $\|u\|_4^4$.

$$\|u\|_4^4 = h_1 h_2 \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} w_i \overline{w}_j |u_{ij}|^4 = h_1 \sum_{i=0}^{m_1} w_i \left[h_2 \sum_{j=0}^{m_2} \overline{w}_j |u_{ij}|^4\right]$$

$$\leqslant h_1 \sum_{i=0}^{m_1} w_i \left[\left(\max_{0 \leqslant j \leqslant m_2} |u_{ij}|^2 \right) \left(h_2 \sum_{j=0}^{m_2} \overline{w}_j |u_{ij}|^2 \right) \right]$$

$$\leqslant \left(h_1 \sum_{i=0}^{m_1} w_i \max_{0 \leqslant j \leqslant m_2} |u_{ij}|^2 \right) \left(h_2 \sum_{j=0}^{m_2} \overline{w}_j \max_{0 \leqslant i \leqslant m_1} |u_{ij}|^2 \right).$$

利用 (8.24) 和 (8.25) 可得

$$\begin{aligned}
\|u\|_4^4 &\leqslant \left(2\|\delta_y u\| \, \|u\| + \frac{1}{L_2}\|u\|^2 \right) \left(2\|\delta_x u\| \, \|u\| + \frac{1}{L_1}\|u\|^2 \right) \\
&= \left(2\|\delta_y u\| + \frac{1}{L_2}\|u\| \right) \left(2\|\delta_x u\| + \frac{1}{L_1}\|u\| \right) \|u\|^2 \\
&\leqslant \left(4\|\delta_y u\|^2 + \frac{1}{L_2^2}\|u\|^2 + 4\|\delta_x u\|^2 + \frac{1}{L_1^2}\|u\|^2 \right) \|u\|^2 \\
&= \left[4\|\nabla_h u\|^2 + \left(\frac{1}{L_1^2} + \frac{1}{L_2^2} \right) \|u\|^2 \right] \|u\|^2.
\end{aligned}$$

□

定理 8.4　设 $\{U_{ij}^k \,|\, (i,j) \in \overline{\omega}, \, 0 \leqslant k \leqslant n\}$ 为 (8.1)–(8.3) 的解, $\{u_{ij}^k \,|\, (i,j) \in \overline{\omega}, \, 0 \leqslant k \leqslant n\}$ 为差分格式 (8.12)–(8.14) 的解. 记

$$e_{ij}^k = U_{ij}^k - u_{ij}^k, \quad (i,j) \in \overline{\omega}, \quad 0 \leqslant k \leqslant n.$$

则存在常数 c_3 使得

$$\|e^k\| \leqslant c_3(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant k \leqslant n. \tag{8.26}$$

证明　将 (8.8), (8.10)–(8.11) 分别和 (8.12)–(8.14) 相减, 可得误差方程组

$$\delta_t e_{ij}^{k+\frac{1}{2}} - (\nu + \mathrm{i}\alpha)\Delta_h e_{ij}^{k+\frac{1}{2}} + (\kappa + \mathrm{i}\beta)(|U_{ij}^{k+\frac{1}{2}}|^2 u_{ij}^{k+\frac{1}{2}} - |u_{ij}^{k+\frac{1}{2}}|^2 u_{ij}^{k+\frac{1}{2}})$$

$$-\gamma e_{ij}^{k+\frac{1}{2}} = R_{ij}^{k+\frac{1}{2}}, \quad (i,j) \in \overline{\omega}, \, 0 \leqslant k \leqslant n-1, \tag{8.27}$$

$$e_{ij}^0 = 0, \quad (i,j) \in \omega, \tag{8.28}$$

$$e_{ij}^k = 0, \quad (i,j) \in \partial\omega, \, 0 \leqslant k \leqslant n. \tag{8.29}$$

用 $e^{k+\frac{1}{2}}$ 与 (8.27) 作内积, 并取实部, 可得

$$\frac{1}{2\tau}(\|e^{k+1}\|^2 - \|e^k\|^2) + \nu\|\nabla_h e^{k+\frac{1}{2}}\|^2 - \gamma\|e^{k+\frac{1}{2}}\|^2$$

$$= -\mathrm{Re}\left\{(k + \mathrm{i}\beta)(|U^{k+\frac{1}{2}}|^2 U^{k+\frac{1}{2}} - |u^{k+\frac{1}{2}}|^2 u^{k+\frac{1}{2}}, e^{k+\frac{1}{2}})\right\} + \mathrm{Re}\left\{(R^{k+\frac{1}{2}}, e^{k+\frac{1}{2}})\right\},$$

$$0 \leqslant k \leqslant n-1. \tag{8.30}$$

注意到

$$
\begin{aligned}
&|U_{ij}^{k+\frac{1}{2}}|^2U_{ij}^{k+\frac{1}{2}}-|u_{ij}^{k+\frac{1}{2}}|^2u_{ij}^{k+\frac{1}{2}}\\
=&|U_{ij}^{k+\frac{1}{2}}|^2U_{ij}^{k+\frac{1}{2}}-|U_{ij}^{k+\frac{1}{2}}-e_{ij}^{k+\frac{1}{2}}|^2\cdot(U_{ij}^{k+\frac{1}{2}}-e_{ij}^{k+\frac{1}{2}})\\
=&2|U_{ij}^{k+\frac{1}{2}}|^2e_{ij}^{k+\frac{1}{2}}+(U_{ij}^{k+\frac{1}{2}})^2\overline{e}_{ij}^{k+\frac{1}{2}}-\overline{U}_{ij}^{k+\frac{1}{2}}(e_{ij}^{k+\frac{1}{2}})^2-2U_{ij}^{k+\frac{1}{2}}|e_{ij}^{k+\frac{1}{2}}|^2+|e_{ij}^{k+\frac{1}{2}}|^2e_{ij}^{k+\frac{1}{2}},
\end{aligned}
$$

有

$$
\begin{aligned}
&|-\mathrm{Re}\{(\kappa+\mathrm{i}\beta)(|U^{k+\frac{1}{2}}|^2U^{k+\frac{1}{2}}-|u^{k+\frac{1}{2}}|^2u^{k+\frac{1}{2}},e^{k+\frac{1}{2}})\}|\\
\leqslant&\sqrt{\kappa^2+\beta^2}h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\omega_i\overline{\omega}_j(3c_0^2|e_{ij}^{k+\frac{1}{2}}|^2+3c_0|e_{ij}^{k+\frac{1}{2}}|^3)\\
=&\sqrt{\kappa^2+\beta^2}(3c_0^2\|e^{k+\frac{1}{2}}\|^2+3c_0\|e^{k+\frac{1}{2}}\|_4^2\cdot\|e^{k+\frac{1}{2}}\|)\\
\leqslant&\sqrt{\kappa^2+\beta^2}\left\{3c_0^2\|e^{k+\frac{1}{2}}\|^2+3c_0\left[4\|\nabla_he^{k+\frac{1}{2}}\|^2+\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\|e^{k+\frac{1}{2}}\|^2\right]^{1/2}\|e^{k+\frac{1}{2}}\|^2\right\}\\
\leqslant&\sqrt{\kappa^2+\beta^2}\left[3c_0^2\|e^{k+\frac{1}{2}}\|^2+\varepsilon\left(4\|\nabla_he^{k+\frac{1}{2}}\|^2+(\frac{1}{L_1^2}+\frac{1}{L_2^2})\|e^{k+\frac{1}{2}}\|^2\right)+\frac{9c_0^2}{4\varepsilon}\|e^{k+\frac{1}{2}}\|^4\right].
\end{aligned}
$$

取 $4\sqrt{\kappa^2+\beta^2}\varepsilon=\nu$, 并注意到 $\|e^{k+\frac{1}{2}}\|\leqslant\|U^{k+\frac{1}{2}}-u^{k+\frac{1}{2}}\|\leqslant c_0\sqrt{L_1L_2}+c_2$, 得

$$
\begin{aligned}
&|-\mathrm{Re}\{(\kappa+\mathrm{i}\beta)(|U^{k+\frac{1}{2}}|^2U^{k+\frac{1}{2}}-|u^{k+\frac{1}{2}}|^2u^{k+\frac{1}{2}},e^{k+\frac{1}{2}})\}|\\
\leqslant&\sqrt{\kappa^2+\beta^2}\left[3c_0^2+\frac{\nu}{4\sqrt{\kappa^2+\beta^2}}\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\right.\\
&\left.+\frac{9c_0^2}{4}\cdot\frac{4\sqrt{\kappa^2+\beta^2}}{\nu}(c_0\sqrt{L_1L_2}+c_2)^2\right]\|e^{k+\frac{1}{2}}\|^2+\nu\|\nabla_he^{k+\frac{1}{2}}\|^2\\
=&\left[3c_0^2\sqrt{\kappa^2+\beta^2}+\frac{\nu}{4}\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)+\frac{9c_2^2(\kappa^2+\beta^2)}{\nu}(c_0\sqrt{L_1L_2}+c_2)^2\right]\|e^{k+\frac{1}{2}}\|^2\\
&+\nu\|\nabla_he^{k+\frac{1}{2}}\|^2.
\end{aligned}
$$

将上式代入到 (8.30), 得

$$
\begin{aligned}
&\frac{1}{2\tau}(\|e^{k+1}\|^2-\|e^k\|^2)\\
\leqslant&\left[3c_0^2\sqrt{\kappa^2+\beta^2}+\frac{\nu}{4}\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)+\frac{9c_2^2(\kappa^2+\beta^2)}{\nu}(c_0\sqrt{L_1L_2}+c_2)^2|\gamma|\right]\|e^{k+\frac{1}{2}}\|^2\\
&+\|e^{k+\frac{1}{2}}\|\cdot\|R^{k+\frac{1}{2}}\|,\quad 0\leqslant k\leqslant n-1.
\end{aligned}
$$

注意到 $\|e^{k+\frac{1}{2}}\| \leqslant \dfrac{\|e^{k+1}\| + \|e^k\|}{2}$ 及 $\|R^{k+\frac{1}{2}}\| \leqslant c_1\sqrt{L_1L_2}(\tau^2 + h_1^2 + h_2^2)$, 得到

$$\begin{aligned}&\frac{1}{\tau}(\|e^{k+1}\| - \|e^k\|)\\ \leqslant&\left[3c_0^2\sqrt{\kappa^2+\beta^2} + \frac{\nu}{4}\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right) + \frac{9c_2^2(\kappa^2+\beta^2)}{\nu}(c_0\sqrt{L_1L_2}+c_2)^2|\gamma|\right]\frac{\|e^{k+1}\| + \|e^k\|}{2}\\ &+ c_1\sqrt{L_1L_2}(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant k \leqslant n-1.\end{aligned}$$

记

$$c_4 = \frac{1}{2}\left[3c_0^2\sqrt{\kappa^2+\beta^2} + \frac{\nu}{4}\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right) + \frac{9c_2^2(\kappa^2+\beta^2)}{\nu}(c_0\sqrt{L_1L_2}+c_2)^2|\gamma|\right],$$

可知

$$(1-c_4\tau)\|e^{k+1}\| \leqslant (1+c_4\tau)\|e^k\| + c_1\sqrt{L_1L_2}\tau(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant k \leqslant n-1.$$

当 $c_4\tau \leqslant \dfrac{1}{3}$ 时

$$\|e^{k+1}\| \leqslant (1+3c_4\tau)\|e^k\| + \frac{3}{2}c_1\sqrt{L_1L_2}\tau(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant k \leqslant n-1.$$

由 Gronwall 不等式, 得

$$\begin{aligned}\|e^{k+1}\| \leqslant& e^{3c_4k\tau}\left[\|e^0\| + \frac{c_1\sqrt{L_1L_2}}{2c_4}(\tau^2 + h_1^2 + h_2^2)\right]\\ \leqslant& e^{3c_4T}\cdot\frac{c_1\sqrt{L_1L_2}}{2c_4}(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant k \leqslant n-1.\end{aligned}$$

□

8.3　三层线性化差分格式

8.3.1　差分格式的建立

在点 (x_i, y_j, t_k) 处考虑方程 (8.1), 有

$$\begin{aligned}&u_t(x_i,y_j,t_k) - (\nu + \mathrm{i}\alpha)\Delta u(x_i,y_j,t_k) + (\kappa + \mathrm{i}\beta)(|u|^2u)(x_i,y_j,t_k)\\ &-\gamma u(x_i,y_j,t_k) = 0, \qquad (i,j)\in\omega,\ 1\leqslant k\leqslant n.\end{aligned} \tag{8.31}$$

定义

$$\nabla_\tau U_{ij}^k = \frac{1}{\tau}(U_{ij}^k - U_{ij}^{k-1}), \quad \nabla_{2\tau}U_{ij}^k = \frac{1}{2\tau}(U_{ij}^k - U_{ij}^{k-2}).$$

应用数值微分公式, 有

$$\begin{aligned} u_t(x_i, y_j, t_k) &= 2\nabla_\tau U_{ij}^k - \nabla_{2\tau} U_{ij}^k + O(\tau^2) \\ &= \frac{1}{2\tau}(3U_{ij}^k - 4U_{ij}^{k-1} + U_{ij}^{k-2}) + O(\tau^2), \end{aligned} \tag{8.32}$$

$$\begin{aligned} u(x_i, y_j, t_k) &= 2u(x_i, y_j, t_{k-1}) - u(x_i, y_j, t_{k-2}) + O(\tau^2) \\ &= 2U_{ij}^{k-1} - U_{ij}^{k-2} + O(\tau^2). \end{aligned} \tag{8.33}$$

将以上两式代入 (8.31), 得

$$\begin{aligned} &2\nabla_\tau U_{ij}^k - \nabla_{2\tau} U_{ij}^k - (\nu + \mathrm{i}\alpha)\Delta_h U_{ij}^k + (\kappa + \mathrm{i}\beta)|2U_{ij}^{k-1} - U_{ij}^{k-2}|^2 U_{ij}^k - \gamma U_{ij}^k = R_{ij}^k, \\ &\qquad (i,j) \in \omega,\ 2 \leqslant k \leqslant n, \end{aligned} \tag{8.34}$$

存在常数 c_6 使得

$$|R_{ij}^k| \leqslant c_6(\tau^2 + h_1^2 + h_2^2), \quad (i,j) \in \omega, \quad 2 \leqslant k \leqslant n. \tag{8.35}$$

由 (8.31) 还可得到

$$\nabla_\tau U_{ij}^1 - (\nu + \mathrm{i}\alpha)\Delta_h U_{ij}^1 + (\kappa + \mathrm{i}\beta)|U_{ij}^0|^2 U_{ij}^1 - \gamma U_{ij}^1 = R_{ij}^1, \quad (i,j) \in \omega, \tag{8.36}$$

存在常数 c_7 使得

$$|R_{ij}^1| \leqslant c_7(\tau + h_1^2 + h_2^2), \quad (i,j) \in \omega. \tag{8.37}$$

在 (8.34) 和 (8.36) 中略去小量项, 并注意到初边值条件

$$U_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, \tag{8.38}$$

$$U_{ij}^k = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant k \leqslant n, \tag{8.39}$$

对 (8.1)–(8.3) 建立如下线性化差分格式

$$\begin{aligned} &2\nabla_\tau u_{ij}^k - \nabla_{2\tau} u_{ij}^k - (\nu + \mathrm{i}\alpha)\Delta_h u_{ij}^k + (\kappa + \mathrm{i}\beta)|2u_{ij}^{k-1} - u_{ij}^{k-2}|^2 u_{ij}^k - \gamma u_{ij}^k = 0, \\ &\qquad (i,j) \in \omega, \quad 2 \leqslant k \leqslant n, \end{aligned} \tag{8.40}$$

$$\nabla_\tau u_{ij}^1 - (\nu + \mathrm{i}\alpha)\Delta_h u_{ij}^1 + (\kappa + \mathrm{i}\beta)|u_{ij}^0|^2 u_{ij}^1 - \gamma u_{ij}^1 = 0, \quad (i,j) \in \omega, \tag{8.41}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega, \tag{8.42}$$

$$u_{ij}^k = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant k \leqslant n. \tag{8.43}$$

8.3.2 差分格式解的存在性

定理 8.5 当 $\gamma \leqslant 0$, 或当 $\gamma > 0$ 且 $\tau < 1/\gamma$ 时，差分格式 (8.40)–(8.43) 解是存在唯一的.

证明 由 (8.42)–(8.43) 知 u^0 已给定.

由 (8.41) 和 (8.43) 可得关于 u^1 的线性方程组. 考虑其齐次线性方程组:

$$\frac{1}{\tau}u_{ij}^1 - (\nu + \mathrm{i}\alpha)\Delta_h u_{ij}^1 + (\kappa + \mathrm{i}\beta)|u_{ij}^0|^2 u_{ij}^1 - \gamma u_{ij}^1 = 0, \quad (i,j) \in \omega, \tag{8.44}$$

$$u_{ij}^1 = 0, \quad (i,j) \in \partial\Omega. \tag{8.45}$$

用 u^1 与 (8.44) 的两边作内积, 可得

$$\frac{1}{\tau}\|u^1\|^2 - (\nu + \mathrm{i}\alpha)(\Delta_h u^1, u^1) + (\kappa + \mathrm{i}\beta)(|u^0|^2 u^1, u^1) - \gamma\|u^1\|^2 = 0.$$

注意到 $-(\Delta_h u^1, u^1) = \|\nabla_h u^1\|^2$, 上式两边取实部, 可得

$$\frac{1}{\tau}\|u^1\|^2 + \nu\|\nabla_h u^1\|^2 + \kappa(|u^0|^2 u^1, u^1) - \gamma\|u^1\|^2 = 0.$$

当 $\gamma \leqslant 0$, 或当 $\gamma > 0$ 且 $\tau < 1/\gamma$ 时 $\|u^1\| = 0$. 因而差分格式关于 u^1 是唯一可解的.

设 u^{k-2}, u^{k-1} 已唯一确定, 则有 (8.40) 和 (8.43) 可得关于 u^k 的线性方程组. 考虑其齐次线性方程组

$$\frac{3}{2\tau}u_{ij}^k - (\nu + \mathrm{i}\alpha)\Delta_h u^k + (\kappa + \mathrm{i}\beta)|2u_{ij}^{k-1} - u_{ij}^{k-2}|^2 u_{ij}^k - \gamma u_{ij}^k = 0, \quad (i,j) \in \omega, \tag{8.46}$$

$$u_{ij}^k = 0, \quad (i,j) \in \partial\omega, \tag{8.47}$$

用 u^k 与 (8.46) 作内积, 可得

$$\frac{3}{2\tau}\|u^k\|^2 + (\nu + \mathrm{i}\alpha)\|\nabla_h u^k\|^2 + (\kappa + \mathrm{i}\beta)(|2u^{k-1} - u^{k-2}|^2 u^k, u^k) - \gamma\|u^k\|^2 = 0.$$

两边取实部, 得

$$\frac{3}{2\tau}\|u^k\|^2 + \nu\|\nabla_h u^k\|^2 + \kappa(|2u^{k-1} - u^{k-2}|^2 u^k, u^k) - \gamma\|u^k\|^2 = 0,$$

当 $\gamma \leqslant 0$ 或当 $\gamma > 0$ 且 $\tau < \dfrac{3}{2\gamma}$ 时 $\|u^k\| = 0$. 因而差分格式关于 u^k 是唯一可解的. 由数学归纳原理, 定理证毕. □

8.3.3 差分格式解的有界性

定理 8.6 设 $\{u_{ij}^k \,|\, (i,j) \in \overline{\omega}, 0 \leqslant k \leqslant n\}$ 为差分格式 (8.40)–(8.43) 的解, 则当 $\max\{0,\gamma\}\tau \leqslant \dfrac{1}{8}$ 时, 有

$$\|u^k\| \leqslant \frac{4\sqrt{6}}{3} e^{4T\max\{0,\gamma\}} \|u^0\|, \quad 0 \leqslant k \leqslant n.$$

证明 (I) 用 u^1 与 (8.41) 的两边作内积, 得

$$(\nabla_\tau u^1, u^1) - (\nu + \mathrm{i}\alpha)(\Delta_h u^1, u^1) + (\kappa + \mathrm{i}\beta)(|u^0|^2 u^1, u^1) - \gamma\|u^1\|^2 = 0. \tag{8.48}$$

注意到

$$\begin{aligned}
\mathrm{Re}(\nabla_\tau u^1, u^1) &= \frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2 + \|u^1 - u^0\|^2) \\
&= \frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2) + \frac{\tau}{2}\|\nabla_\tau u^1\|^2, \\
-(\Delta_h u^1, u^1) &= \|\nabla_h u\|^2,
\end{aligned}$$

在 (8.48) 两边取实部, 可得

$$\frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2) + \frac{\tau}{2}\|\nabla_\tau u^1\|^2 + \nu\|\nabla_h u\|^2 + \kappa(|u^0|^2 u^1, u^1) - \gamma\|u^1\|^2 = 0. \tag{8.49}$$

当 $\gamma \leqslant 0$ 时, 易得

$$\|u^1\| \leqslant \|u^0\|.$$

当 $\gamma > 0$, 由 (8.49) 得到

$$(1 - 2\gamma\tau)\|u^1\| \leqslant \|u^0\|.$$

因而 $2\gamma\tau \leqslant \dfrac{1}{4}$ 时,

$$\|u^1\|^2 \leqslant \frac{1}{1-2\gamma\tau}\|u^0\|^2 \leqslant \frac{4}{3}\|u^0\|^2. \tag{8.50}$$

(II) 用 u^k 与 (8.40) 的两边作内积, 得

$$(2\nabla_\tau u^k - \nabla_{2\tau} u^k, u^k) - (\nu + \mathrm{i}\alpha)(\Delta_h u^k, u^k) + (\kappa + \mathrm{i}\beta)(|2u^{k-1} - u^{k-2}|^2 u^k, u^k) - \gamma\|u^k\|^2 = 0. \tag{8.51}$$

注意到

$$\begin{aligned}
&\mathrm{Re}\big(2\nabla_\tau u^k - \nabla_{2\tau} u^k, u^k\big) \\
&= \frac{1}{4\tau}\big[(\|u^k\|^2 + \|2u^k - u^{k-1}\|^2) - (\|u^{k-1}\|^2 + \|2u^{k-1} - u^{k-2}\|^2) \\
&\quad + \|u^k - 2u^{k-1} + u^{k-2}\|^2\big],
\end{aligned} \tag{8.52}$$

对 (8.51) 两边取实部, 得到

$$\frac{1}{4\tau}[(\|u^k\|^2+\|2u^k-u^{k-1}\|^2)-(\|u^{k-1}\|^2+\|2u^{k-1}-u^{k-2}\|^2)$$
$$+\|u^k-2u^{k-1}+u^{k-2}\|^2]+\nu\|\nabla_h u^k\|^2+\kappa(|2u^{k-1}-u^{k-2}|^2u^k,u^k)=\gamma\|u^k\|^2.$$

由上式, 得

$$\begin{aligned}&\frac{1}{4\tau}[(\|u^k\|^2+\|2u^k-u^{k-1}\|^2)-(\|u^{k-1}\|^2+\|2u^{k-1}-u^{k-2}\|^2)]\leqslant\gamma\|u^k\|^2\\ \leqslant\ &|\gamma|\big(\|u^k\|^2+\|2u^k-u^{k-1}\|^2\big),\quad 2\leqslant k\leqslant n.\end{aligned}\tag{8.53}$$

当 $\gamma\leqslant 0$ 时, 易得

$$\|u^k\|^2+\|2u^k-u^{k-1}\|^2\leqslant\|u^{k-1}\|^2+\|2u^{k-1}-u^{k-2}\|^2,\quad 2\leqslant k\leqslant n.\tag{8.54}$$

当 $\gamma>0$ 时, 由 (8.53) 得到

$$(1-4\gamma\tau)(\|u^k\|^2+\|2u^k-u^{k-1}\|^2)\leqslant\|u^{k-1}\|^2+\|2u^{k-1}-u^{k-2}\|^2,\quad 2\leqslant k\leqslant n.$$

因而当 $4\gamma\tau\leqslant\dfrac{1}{2}$ 时, 有

$$\begin{aligned}&\|u^k\|^2+\|2u^k-u^{k-2}\|\\ \leqslant\ &\frac{1}{1-4\gamma\tau}(\|u^{k-1}\|^2+2\|u^{k-1}-u^{k-2}\|^2)\\ \leqslant\ &(1+8\gamma\tau)(\|u^{k-1}\|^2+2\|u^{k-1}-u^{k-2}\|^2),\quad 2\leqslant k\leqslant n.\end{aligned}\tag{8.55}$$

综合 (8.54) 和 (8.55) 得到

$$\|u^k\|^2+\|2u^k-u^{k-2}\|\leqslant(1+8\max\{0,\gamma\}\tau)(\|u^{k-1}\|^2+2\|u^{k-1}-u^{k-2}\|^2),\quad 2\leqslant k\leqslant n.$$

由 Gronwall 不等式得

$$\|u^k\|^2+\|2u^k-u^{k-2}\|^2\leqslant e^{8\max\{0,\gamma\}k\tau}(\|u^1\|^2+2\|u^1-u^0\|^2),\quad 2\leqslant k\leqslant n.$$

再由 (8.50) 得

$$\begin{aligned}\|u^k\|^2&\leqslant e^{8\max\{0,\gamma\}T}[\|u^1\|^2+4(\|u^1\|^2+\|u^0\|^2)]\\ &\leqslant\frac{32}{3}e^{8\max\{0,\gamma\}T}\|u^0\|^2,\quad 1\leqslant k\leqslant n.\end{aligned}$$

□

8.3.4 差分格式解的收敛性

定理 8.7 设 $\{U_{ij}^k\,|\,(i,j)\in\overline{\omega},0\leqslant k\leqslant n\}$ 为问题 (8.1)–(8.3) 的解, $\{u_{ij}^k\,|\,(i,j)\in\overline{\omega},0\leqslant k\leqslant n\}$ 为差分格式 (8.40)–(8.43) 的解. 定义

$$e_{ij}^k=U_{ij}^k-u_{ij}^k,\quad (i,j)\in\omega,\ 0\leqslant k\leqslant n,$$

则当 τ 适当小时, 存在常数 c_8 使得

$$\|e^k\|\leqslant c_8(\tau^2+h_1^2+h_2^2),\quad 0\leqslant k\leqslant n. \tag{8.56}$$

证明 将 (8.34), (8.36), (8.38)–(8.39) 与 (8.40)–(8.43) 依次相减, 得

$$\begin{aligned}&2\nabla_\tau e_{ij}^k-\nabla_{2\tau}e_{ij}^k-(\nu+\mathrm{i}\alpha)\Delta_h e_{ij}^k\\&+(\kappa+\mathrm{i}\beta)\big(|2U_{ij}^{k-1}-U_{ij}^{k-2}|^2U_{ij}^k-|2u_{ij}^{k-1}-u_{ij}^{k-2}|^2u_{ij}^k\big)-\gamma e_{ij}^k=R_{ij}^k,\\&\quad(i,j)\in\omega,\ 2\leqslant k\leqslant n,\end{aligned} \tag{8.57}$$

$$\nabla_\tau e_{ij}^1-(\nu+\mathrm{i}\alpha)\Delta_h e_{ij}^1+(\kappa+\mathrm{i}\beta)|U_{ij}^0|^2e_{ij}^1-\gamma e_{ij}^1=R_{ij}^1,\quad (i,j)\in\omega, \tag{8.58}$$

$$e_{ij}^0=0,\quad (i,j)\in\omega, \tag{8.59}$$

$$e_{ij}^k=0,\quad (i,j)\in\partial\omega,\ 0\leqslant k\leqslant n. \tag{8.60}$$

(I) 用 e^1 与 (8.58) 的两边作内积, 并注意到 (8.59)–(8.60), 得

$$\frac{1}{\tau}\|e^1\|^2+(\nu+\mathrm{i}\alpha)\|\nabla_h e^1\|^2+(\kappa+\mathrm{i}\beta)(|U^0|^2e^1,e^1)-\gamma\|e^1\|^2=(R^1,e^1).$$

两边取实部, 得

$$\frac{1}{\tau}\|e^1\|^2+\nu\|\nabla_h e^1\|^2+\kappa(|U^0|^2e^1,e^1)=\gamma\|e^1\|^2+(R^1,e^1). \tag{8.61}$$

因而

$$\frac{1}{\tau}\|e^1\|^2\leqslant|\gamma|\cdot\|e^1\|^2+\|R^1\|\cdot\|e^1\|.$$

两边约去 $\|e^1\|$, 得

$$\frac{1}{\tau}\|e^1\|\leqslant|\gamma|\cdot\|e^1\|+\|R^1\|.$$

当 $|\gamma|\tau\leqslant\dfrac{1}{2}$ 时, 有

$$\|e^1\|\leqslant\frac{\tau}{1-|\gamma|\tau}\|R^1\|\leqslant2\tau\|R^1\|.$$

注意到 (8.37), 可得

$$\|e^1\| \leqslant 2\tau c_7(\tau + h_1^2 + h_2^2) \leqslant 2c_7(\tau^2 + h_1^2 + h_2^2). \tag{8.62}$$

(II) 用 e^k 与 (8.57) 的两边作内积, 得

$$\begin{aligned}&(2\nabla_\tau e^k - \nabla_{2\tau}e^k, e^k) + (\nu + \mathrm{i}\alpha)\|\nabla_h e^k\|^2 + (\kappa + \mathrm{i}\beta)(|2U^{k-1} - U^{k-2}|^2 U^k \\ &-|2u^{k-1} - u^{k-2}|^2 u^k, e^k) - \gamma\|e^k\|^2 = (R^k, e^k).\end{aligned} \tag{8.63}$$

注意到

$$\begin{aligned}&|2U^{k-1} - U^{k-2}|^2 U^k - |2u^{k-1} - u^{k-2}|^2 u^k \\ =&|2u^{k-1} - u^{k-2}|^2(U^k - u^k) + (|2U^{k-1} - U^{k-2}|^2 - |2u^{k-1} - u^{k-2}|^2)U^k \\ =&|2u^{k-1} - u^{k-2}|^2 e^k + \big\{(2u^{k-1} - u^{k-2})[(2\bar{U}^{k-1} - \bar{U}^{k-2}) - (2\bar{u}^{k-1} - \bar{u}^{k-2})] \\ &+ [(2U^{k-1} - U^{k-2}) - (2u^{k-1} - u^{k-2})](2\bar{U}^{k-1} - \bar{U}^{k-2})\big\}U^k \\ =&|2u^{k-1} - u^{k-2}|^2 e^k + \big[(2u^{k-1} - u^{k-2})(2\bar{e}^{k-1} - \bar{e}^{k-2}) \\ &+ (2e^{k-1} - e^{k-2})(2\bar{U}^{k-1} - \bar{U}^{k-2})\big]U^k,\end{aligned}$$

以及 (8.52), 对 (8.63) 两边取实部, 得到

$$\begin{aligned}A^k \equiv& \frac{1}{4\tau}[(\|e^k\|^2 + \|2e^k - e^{k-1}\|^2) - (\|e^{k-1}\|^2 + \|2e^{k-1} - e^{k-2}\|^2) \\ &+\|e^k - 2e^{k-1} + e^{k-2}\|^2] + \nu\|\nabla_h e^k\|^2 - \gamma\|e^k\|^2 \\ \leqslant& -\mathrm{Re}\Big\{(\kappa + \mathrm{i}\beta)\big[((2u^{k-1} - u^{k-2})(2\bar{e}^{k-1} - \bar{e}^{k-2})U^k, e^k) \\ &+((2e^{k-1} - e^{k-2})(2\bar{U}^{k-1} - \bar{U}^{k-2})U^k, e^k)\big]\Big\} + \|R^k\| \cdot \|e^k\|.\end{aligned} \tag{8.64}$$

再注意到

$$\begin{aligned}&\|U^k\|_\infty \leqslant c_0, \\ &\|u^k\| \leqslant \frac{4\sqrt{6}}{3}e^{4|\gamma|T}\|u^0\| \equiv c_9,\end{aligned}$$

有

$$\begin{aligned}A^k \leqslant& \sqrt{\kappa^2 + \beta^2}\Big(c_0\|2u^{k-1} - u^{k-2}\| \cdot \|2e^{k-1} - e^{k-2}\|_4\|e^k\|_4 \\ &+ 3c_0^2\|2e^{k-1} - e^{k-2}\| \cdot \|e^k\|\Big) + \|R^k\| \cdot \|e^k\| \\ \leqslant& \sqrt{\kappa^2 + \beta^2}c_0 \cdot 3c_9\big(2\|e^{k-1}\|_4 + \|e^{k-2}\|_4\big)\|e^k\|_4 \\ &+ \sqrt{\kappa^2 + \beta^2} \cdot 3c_0^2(2\|e^{k-1}\| + \|e^{k-2}\|)\|e^k\| + \|R^k\| \cdot \|e^k\|\end{aligned}$$

$$
\begin{aligned}
&\leqslant\sqrt{\kappa^2+\beta^2}3c_0c_9\frac{\sqrt{5}}{2}(\|e^k\|_4^2+\|e^{k-1}\|_4^2+\|e^{k-2}\|_4^2)\\
&\quad+\sqrt{\kappa^2+\beta^2}\cdot 3c_0^2\cdot\frac{\sqrt{5}}{2}(\|e^k\|^2+\|e^{k-1}\|^2+\|e^{k-2}\|^2)+\|R^k\|\cdot\|e^k\|.
\end{aligned}
$$

应用引理 8.5 得

$$
\begin{aligned}
A^k\leqslant&\sqrt{\kappa^2+\beta^2}\frac{3\sqrt{5}c_0c_9}{2}\left\{\left[4\|\nabla_h e^k\|^2+\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\|e^k\|^2\right]^{1/2}\|e^k\|\right.\\
&+\left[4\|\nabla_h e^{k-1}\|^2+\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\|e^{k-1}\|^2\right]^{1/2}\|e^{k-1}\|\\
&\left.+\left[4\|\nabla_h e^{k-2}\|^2+\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\|e^{k-2}\|^2\right]^{1/2}\|e^{k-2}\|\right\}\\
&+\sqrt{\kappa^2+\beta^2}\cdot\frac{3\sqrt{5}c_0^2}{2}(\|e^k\|^2+\|e^{k-1}\|^2+\|e^{k-2}\|^2)+\|R^k\|\cdot\|e^k\|\\
\leqslant&\sqrt{\kappa^2+\beta^2}\cdot\frac{3\sqrt{5}c_0c_9}{2}\left[\varepsilon\left(4\|\nabla_h e^k\|^2+\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\|e^k\|^2\right)+\frac{1}{4\varepsilon}\|e^k\|^2\right.\\
&+\varepsilon\left(4\|\nabla_h e^{k-1}\|^2+\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\|e^{k-1}\|^2\right)+\frac{1}{4\varepsilon}\|e^{k-1}\|^2\\
&\left.+\varepsilon\left(4\|\nabla_h e^{k-2}\|^2+\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\|e^{k-2}\|^2\right)+\frac{1}{4\varepsilon}\|e^{k-2}\|^2\right]\\
&+\sqrt{\kappa^2+\beta^2}\cdot\frac{3\sqrt{5}c_0^2}{2}(\|e^k\|^2+\|e^{k-1}\|^2+\|e^{k-2}\|^2)+\|R^k\|\cdot\|e^k\|.
\end{aligned}
$$

取 $6\sqrt{5}\sqrt{\kappa^2+\beta^2}c_0c_9\varepsilon=\dfrac{\nu}{3}$, 则有

$$
\begin{aligned}
A^k\leqslant&\frac{\nu}{3}(\|\nabla_h e^k\|^2+\|\nabla_h e^{k-1}\|^2+\|\nabla_h e^{k-2}\|^2)\\
&+\frac{3}{\nu}\left(\sqrt{\kappa^2+\beta^2}\cdot 3\sqrt{5}c_0c_9\right)^2(\|e^k\|^2+\|e^{k-1}\|^2+\|e^{k-2}\|^2)\\
&+\frac{\nu}{12}(\frac{1}{L_1^2}+\frac{1}{L_2^2})(\|e^k\|^2+\|e^{k-1}\|^2+\|e^{k-2}\|^2)\\
&+\sqrt{\kappa^2+\beta^2}\cdot\frac{3\sqrt{5}c_0^2}{2}(\|e^k\|^2+\|e^{k-1}\|^2+\|e^{k-2}\|^2)+\|R^k\|\cdot\|e^k\|.
\end{aligned}
$$

记

$$
c_{10}=\frac{3}{\nu}(\kappa^2+\beta^2)\cdot 45c_0^2c_9^2+\frac{\nu}{12}\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)+\sqrt{\kappa^2+\beta^2}\cdot\frac{3\sqrt{5}c_0^2}{2},
$$

则

$$
\begin{aligned}
A^k \leqslant & \frac{\nu}{3}(\|\nabla_h e^k\|^2 + \|\nabla_h e^{k-1}\|^2 + \|\nabla_h e^{k-2}\|^2) \\
& + c_{10}(\|e^k\|^2 + \|e^{k-1}\|^2 + \|e^{k-2}\|^2) + \|R^k\| \cdot \|e^k\|.
\end{aligned}
$$

将上式代入 (8.64), 得到

$$
\begin{aligned}
& \frac{1}{4\tau}[(\|e^k\|^2 + \|2e^k - e^{k-1}\|^2) - (\|e^{k-1}\|^2 + \|2e^{k-1} - e^{k-2}\|^2)] + \frac{2\nu}{3}\|\nabla_h e^k\|^2 \\
\leqslant & \frac{\nu}{3}(\|\nabla_h e^{k-1}\|^2 + \|\nabla_h e^{k-2}\|^2) + c_{10}(\|e^k\|^2 + \|e^{k-1}\|^2 + \|e^{k-2}\|^2) \\
& + |\gamma| \cdot \|e^k\|^2 + \|R^k\| \cdot \|e^k\|, \quad 2 \leqslant k \leqslant n.
\end{aligned} \tag{8.65}
$$

记

$$
F^k = \|e^k\|^2 + \|2e^k - e^{k-1}\|^2, \quad 1 \leqslant k \leqslant n.
$$

将 (8.65) 中 k 换为 l, 并对 l 从 2 到 k 求和, 得

$$
\begin{aligned}
& \frac{1}{4\tau}(F^k - F^1) + \frac{2\nu}{3}\sum_{l=2}^{k}\|\nabla_h e^l\|^2 \\
\leqslant & \frac{\nu}{3}\sum_{l=2}^{k}(\|\nabla_h e^{l-1}\|^2 + \|\nabla_h e^{l-2}\|^2) + c_{10}\sum_{l=2}^{k}(\|e^l\|^2 + \|e^{l-1}\|^2 + \|e^{l-2}\|^2) \\
& + |\gamma|\sum_{l=2}^{k}\|e^l\|^2 + \frac{1}{2}\sum_{l=2}^{k}(\|R^l\|^2 + \|e^l\|^2), \quad 2 \leqslant k \leqslant n.
\end{aligned} \tag{8.66}
$$

由 (8.61) 得

$$
\frac{1}{\tau}\|e^1\|^2 + \nu\|\nabla_h e^1\|^2 \leqslant |\gamma| \cdot \|e^1\|^2 + \frac{1}{2}(\|R^1\|^2 + \|e^1\|^2). \tag{8.67}
$$

注意到 $\|e^0\| = 0$, $F^1 = 5\|e^1\|^2$, 由 (8.66) 和 (8.67) 可得

$$
\begin{aligned}
\frac{1}{4\tau}F^k \leqslant & c_{10}\sum_{l=2}^{k}(\|e^l\|^2 + \|e^{l-1}\|^2 + \|e^{l-2}\|^2) \\
& + |\gamma|\sum_{l=2}^{k}\|e^l\|^2 + \frac{1}{2}\sum_{l=2}^{k}(\|R^l\|^2 + \|e^l\|^2) \\
& + \frac{5}{4}\left[|\gamma|\|e^1\|^2 + \frac{1}{2}(\|R^1\|^2 + \|e^1\|^2)\right],
\end{aligned}
$$

或

$$\begin{aligned}&\frac{1}{4\tau}\|e^k\|^2\\ \leqslant&\left(c_{10}+|\gamma|+\frac{1}{2}\right)\|e^k\|^2+(2c_{10}+|\gamma|+\frac{1}{2})\sum_{l=2}^{k-1}\|e^l\|^2\\&+\left(2c_{10}+\frac{5}{4}|\gamma|+\frac{5}{8}\right)\|e^1\|^2+\frac{1}{2}\sum_{l=2}^{k}\|R^l\|^2+\frac{5}{8}\|R^1\|^2,\quad 2\leqslant k\leqslant n.\end{aligned}\tag{8.68}$$

观察 (8.67) 知 (8.68) 对于 $k=1$ 也是成立的.

由 (8.68) 得

$$\begin{aligned}&\left[1-4\left(c_{10}+|\gamma|+\frac{1}{2}\right)\tau\right]\|e^k\|^2\\ \leqslant&4\left(2c_{10}+\frac{5}{4}|\gamma|+\frac{5}{8}\right)\tau\sum_{l=1}^{k-1}\|e^l\|^2+4\tau\left(\frac{1}{2}\sum_{l=2}^{k}\|R^k\|^2+\frac{5}{8}\|R^1\|^2\right),\quad 1\leqslant k\leqslant n.\end{aligned}$$

当 $4\left(c_{10}+|\gamma|+\frac{1}{2}\right)\tau\leqslant\frac{1}{2}$ 时,

$$\begin{aligned}\|e^k\|^2\leqslant&(16c_{10}+10|\gamma|+5)\tau\sum_{l=1}^{k-1}\|e^l\|^2\\&+4\tau\left(\sum_{l=2}^{k}\|R^k\|^2+\frac{5}{4}\|R^1\|^2\right),\quad 1\leqslant k\leqslant n.\end{aligned}$$

应用 Gronwall 不等式, 并注意到 (8.35), (8.37), 得到

$$\|e^k\|^2\leqslant e^{(16c_{10}+10|\gamma|+5)T}\cdot(4Tc_6^2+5c_7^2)L_1L_2(\tau^2+h_1^2+h_2^2)^2,\quad 1\leqslant k\leqslant n.$$

因而

$$\|e^k\|\leqslant e^{(8c_{10}+5|\gamma|+\frac{5}{2})T}\sqrt{(4T_1c_6^2+5c_7^2)L_1L_2}\,(\tau^2+h_1^2+h_2^2),\quad 1\leqslant k\leqslant n.\qquad\Box$$

8.4 小结与延拓

本章研究了 Ginzburg-Landau 方程的有限差分方法. 介绍了二层非线性差分格式和三层线性化差分格式. 证明了差分格式解的存在性和有界性. 利用差分格式解的 L_2 范数的有界性, 证明了差分格式解在 L_2 范数下的无条件收敛性. 本章内

容是在 [35] 和 [42] 工作的基础上发展而成的. 文 [16] 对 Ginzburg-Landau 方程构造了紧差分格式和紧交替方向格式, 证明了差分格式的无条件收敛性.

本章建立三层线性化差分格式时, 采用了二阶向后差商公式 (8.32) 逼近时间导数, 外推公式 (8.33) 逼近非线性项. 可对问题 (8.1)—(8.3) 建立如下三层 Crank-Nicolson 型线性化差分格式

$$\begin{aligned}
&\Delta_t u_{ij}^k - (\nu + \mathrm{i}\alpha)\Delta_h u_{ij}^{\bar{k}} + (\kappa + \mathrm{i}\beta)|u_{ij}^k|^2 u_{ij}^{\bar{k}} - \gamma u_{ij}^{\bar{k}} = 0,\\
&\qquad (i,j) \in \omega, \quad 1 \leqslant k \leqslant n-1,\\
&\nabla_\tau u_{ij}^1 - (\nu + \mathrm{i}\alpha)\Delta_h u_{ij}^1 + (\kappa + \mathrm{i}\beta)|u_{ij}^0|^2 u_{ij}^1 - \gamma u_{ij}^1 = 0, \quad (i,j) \in \omega,\\
&u_{ij}^0 = \varphi(x_i, y_j), \quad (i,j) \in \omega,\\
&u_{ij}^k = 0, \quad (i,j) \in \partial\omega,\ 0 \leqslant k \leqslant n.
\end{aligned}$$

第 9 章　Cahn-Hilliard 方程的差分方法

Cahn-Hilliard 方程是一类典型的四阶非线性扩散方程. 它最初是由 Cahn 和 Hilliard 于 1958 年在研究热力学中两相物质 (如合金、聚合物等) 之间的相互扩散现象时提出来的. 后来, 在描述生物种群的竞争与排斥现象、河床的迁移过程、固体表面上微滴的扩散等许多扩散现象的研究中也提出了同样的数学模型.

9.1 引　　言

考虑如下 Cahn-Hilliard 方程的初边值问题

$$u_t = \Delta(\phi(u) - \alpha\Delta u), \quad (x,y) \in \Omega, \quad 0 < t \leqslant T, \tag{9.1}$$

$$\frac{\partial u}{\partial \nu} = 0, \quad \frac{\partial(\phi(u) - \alpha\Delta u)}{\partial \nu} = 0, \quad 0 < t \leqslant T, \tag{9.2}$$

$$u(x,y,0) = \varphi(x,y), \quad (x,y) \in \bar{\Omega}, \tag{9.3}$$

其中 $\Omega = (0, L_1) \times (0, L_2), \nu$ 为边界 Ω 的单位外法向矢量, $\phi(u) = \psi'(u)$, $\psi(u) = \gamma(u^2 - \beta^2)^2/4$, α, β, γ 为正常数.

问题 (9.1)–(9.3) 的解具有如下守恒律.

定理 9.1　设 $u(x,y,t)$ 为 (9.1)–(9.3) 的解. 记

$$F(t) = \frac{1}{2}\iint\limits_{\Omega} u^2(x,y,t)\mathrm{d}x\mathrm{d}y + \int_0^t \left[\iint\limits_{\Omega} \phi'(u(x,y,s))\left|\nabla u(x,y,s)\right|^2 \mathrm{d}x\mathrm{d}y + \alpha\iint\limits_{\Omega} \left[\Delta u(x,y,s)\right]^2 \mathrm{d}x\mathrm{d}y\right]\mathrm{d}s,$$

则有

$$F(t) \equiv F(0), \quad 0 < t \leqslant T. \tag{9.4}$$

证明　用 u 与 (9.1) 的两边相乘, 并关于 (x,y) 在 Ω 上积分, 得

$$\iint\limits_{\Omega} uu_t \mathrm{d}x\mathrm{d}y - \iint\limits_{\Omega} [\Delta(\phi(u) - \alpha\Delta u)]u\mathrm{d}x\mathrm{d}y = 0. \tag{9.5}$$

现在分析上式左端的每一项.

第一项

$$\iint\limits_{\Omega} uu_t \mathrm{d}x\mathrm{d}y = \frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega} u^2\mathrm{d}x\mathrm{d}y; \tag{9.6}$$

第二项

$$\begin{aligned}
&-\iint\limits_{\Omega}\Big[\Delta\big(\phi(u)-\alpha\Delta u\big)\Big]u\mathrm{d}x\mathrm{d}y\\
=&-\int_0^{L_2}\Big[\int_0^{L_1}\big(\phi(u)-\alpha\Delta u\big)_{xx}u\mathrm{d}x\Big]\mathrm{d}y-\int_0^{L_1}\Big[\int_0^{L_2}\big(\phi(u)-\alpha\Delta u\big)_{yy}u\mathrm{d}y\Big]\mathrm{d}x\\
=&\int_0^{L_2}\Big[-\big(\phi(u)-\alpha\Delta u\big)_x u\Big|_{x=0}^{L_1}+\int_0^{L_1}\big(\phi(u)-\alpha\Delta u\big)_x u_x\mathrm{d}x\Big]\mathrm{d}y\\
&+\int_0^{L_1}\Big[-\big(\phi(u)-\alpha\Delta u\big)_y u\Big|_{y=0}^{L_2}+\int_0^{L_2}\big(\phi(u)-\alpha\Delta u\big)_y u_y\mathrm{d}y\Big]\mathrm{d}x\\
=&\int_0^{L_2}\Big[\int_0^{L_1}\phi'(u)u_x^2\mathrm{d}x-\alpha(\Delta u)u_x\Big|_{x=0}^{L_1}+\alpha\int_0^{L_1}(\Delta u)u_{xx}\mathrm{d}x\Big]\mathrm{d}y\\
&+\int_0^{L_1}\Big[\int_0^{L_2}\phi'(u)u_y^2\mathrm{d}y-\alpha(\Delta u)u_y\Big|_{y=0}^{L_2}+\alpha\int_0^{L_2}(\Delta u)u_{yy}\mathrm{d}y\Big]\mathrm{d}x\\
=&\iint\limits_{\Omega}\phi'(u)|\nabla u|^2\mathrm{d}x\mathrm{d}y+\alpha\iint\limits_{\Omega}(\Delta u)^2\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{9.7}$$

将 (9.6)–(9.7) 代入 (9.5) 得到

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega} u^2(x,y,t)\mathrm{d}x\mathrm{d}y+\iint\limits_{\Omega}\phi'(u)|\nabla u|^2\mathrm{d}x\mathrm{d}y+\alpha\iint\limits_{\Omega}(\Delta u)^2\mathrm{d}x\mathrm{d}y=0,\quad 0<t\leqslant T, \tag{9.8}$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t)=0,\quad 0<t\leqslant T.$$

□

注 9.1 注意到 $\phi'(u)=\gamma(3u^2-\beta^2)\geqslant-\gamma\beta^2$ 及 $|u|_1^2\leqslant|u|_2\cdot\|u\|$, 由 (9.8) 得

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega} u^2(x,y,t)dx+\alpha|u|_2^2=&-\iint\limits_{\Omega}\phi'(u)|\nabla u|^2\mathrm{d}x\mathrm{d}y\\
&\leqslant\gamma\beta^2|u|_1^2\leqslant\gamma\beta^2|u|_2\cdot\|u\|\\
&\leqslant\alpha|u|_2^2+\frac{\gamma^2\beta^4}{4\alpha}\|u\|^2.
\end{aligned}$$

因而

$$\frac{1}{2}\frac{d}{dt}\|u\|^2\leqslant\frac{\gamma^2\beta^4}{4\alpha}\|u\|^2.$$

由上式易得

$$\|u(\cdot,\cdot,t)\|^2 \leqslant \|u(\cdot,\cdot,0)\|^2 e^{\frac{\gamma^2\beta^4}{2\alpha}t}, \quad 0 < t \leqslant T,$$

或

$$\|u(\cdot,\cdot,t)\| \leqslant e^{\frac{\gamma^2\beta^4}{4\alpha}T}\|u(\cdot,\cdot,0)\|, \quad 0 \leqslant t \leqslant T.$$

定理 9.2 设 $u(x,y,t)$ 为 (9.1)–(9.3) 的解. 记

$$\begin{aligned}
&v = \phi(u) - \alpha\Delta u,\\
&E(t) = \iint\limits_{\Omega} \psi(u(x,y,t))\mathrm{d}x\mathrm{d}y + \frac{\alpha}{2}\iint\limits_{\Omega} |\nabla u(x,y,t)|^2\mathrm{d}x\mathrm{d}y\\
&\qquad\quad + \int_0^t \left[\iint\limits_{\Omega} |\nabla v(x,y,s)|^2\mathrm{d}x\mathrm{d}y\right]\mathrm{d}s,
\end{aligned}$$

则有

$$E(t) = E(0), \quad 0 < t \leqslant T.$$

证明 用 $\phi(u) - \alpha\Delta u$ 乘以 (9.1) 的两边, 得

$$u_t\big(\phi(u) - \alpha\Delta u\big) = \big[\Delta(\phi(u) - \alpha\Delta u)\big]\cdot\big[\phi(u) - \alpha\Delta u\big] = (\Delta v)v.$$

两边关于 x, y 在 Ω 上积分, 得

$$\iint\limits_{\Omega} \phi(u)u_t\mathrm{d}x\mathrm{d}y - \alpha\iint\limits_{\Omega}(\Delta u)u_t\mathrm{d}x\mathrm{d}y - \iint\limits_{\Omega}(\Delta v)v\mathrm{d}x\mathrm{d}y = 0. \tag{9.9}$$

由 $\phi(u)$ 的定义知

$$\iint\limits_{\Omega} \phi(u)u_t\mathrm{d}x\mathrm{d}y = \iint\limits_{\Omega} \psi'(u)u_t\mathrm{d}x\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega}\psi(u)\mathrm{d}x\mathrm{d}y. \tag{9.10}$$

注意到 (9.2) 的前一式, 由分部积分公式, 得

$$-\alpha\iint\limits_{\Omega}(\Delta u)u_t\mathrm{d}x\mathrm{d}y = \alpha\iint\limits_{\Omega}\nabla u\cdot\nabla u_t\mathrm{d}x\mathrm{d}y = \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega}|\nabla u|^2\mathrm{d}x\mathrm{d}y. \tag{9.11}$$

注意到 (9.2) 的后一式, 由分部积分公式, 得

$$-\iint\limits_{\Omega}(\Delta v)v\mathrm{d}x\mathrm{d}y = \iint\limits_{\Omega}|\nabla v|^2\mathrm{d}x\mathrm{d}y, \tag{9.12}$$

将 (9.10)–(9.12) 代入 (9.9), 得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega}\psi(u(x,y,t))dt+\frac{\alpha}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega}|\nabla u(x,y,t)|^2\mathrm{d}x\mathrm{d}y$$
$$+\iint\limits_{\Omega}|\nabla v(x,y,t)|^2\mathrm{d}x\mathrm{d}y=0,\quad 0<t\leqslant T.$$

易知

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)=0,\quad 0<t\leqslant T.$$ □

9.2　二层非线性差分格式

对于 $v\in\mathcal{V}_h$, 引进如下记号

$$\delta_x^2 v_{ij}=\begin{cases}\dfrac{2}{h_1}\delta_x v_{\frac{1}{2},j}, & i=0,\\ \dfrac{1}{h_1}\left(\delta_x v_{i+\frac{1}{2},j}-\delta_x v_{i-\frac{1}{2},j}\right), & 1\leqslant i\leqslant m_1-1,\\ \dfrac{2}{h_1}\left(-\delta_x v_{m_1-\frac{1}{2},j}\right), & i=m_1,\end{cases}$$

$$\delta_y^2 v_{ij}=\begin{cases}\dfrac{2}{h_2}\delta_y v_{i,\frac{1}{2}}, & j=0,\\ \dfrac{1}{h_2}\left(\delta_y v_{i,j+\frac{1}{2}}-\delta_y v_{i,j-\frac{1}{2}}\right), & 1\leqslant j\leqslant m_2-1,\\ \dfrac{2}{h_2}\left(-\delta_y v_{i,m_2-\frac{1}{2}}\right), & j=m_2.\end{cases}$$

引理 9.1　设 $u\in\mathcal{V}_h$, 则有

$$-(\Delta_h u,u)=|u|_1^2.$$

证明　由分部求和公式可得

$$\begin{aligned}&-(\Delta_h u,u)\\ =&-h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\omega_i\bar{\omega}_j(\delta_x^2 u_{i,j}+\delta_y^2 u_{ij})u_{ij}\\ =&\,h_2\sum_{j=0}^{m_2}\bar{\omega}_j\left[-h_1\sum_{i=0}^{m_1}\omega_i(\delta_x^2 u_{i,j})u_{ij}\right]+h_1\sum_{i=1}^{m_1}\omega_i\left[-h_2\sum_{j=0}^{m_2}\bar{\omega}_j(\delta_y^2 u_{ij})u_{ij}\right]\\ =&\,h_2\sum_{j=0}^{m_2}\bar{\omega}_j\left[h_1\sum_{i=1}^{m_1}(\delta_x u_{i-\frac{1}{2},j})^2\right]+h_1\sum_{i=0}^{m_1}\omega_i\left[h_2\sum_{j=1}^{m_2}(\delta_y u_{i,j-\frac{1}{2}})^2\right]\\ =&\,|u|_1^2.\end{aligned}$$ □

引理 9.2 设 $u \in \mathcal{V}_h$, 对任意的 $\varepsilon > 0$ 有

$$\|u\|_\infty \leqslant \varepsilon\|\Delta_h u\| + \sqrt{3}\left[\frac{1}{\varepsilon}+\frac{1}{2}\left(\frac{1}{L_1}+\frac{1}{L_2}\right)\right]\|u\|.$$

证明 设

$$\|u\|_\infty = |u_{i_0,j_0}|.$$

由引理 1.1(e), 有

$$\begin{aligned}
u_{i_0,j_0}^2 \leqslant& \varepsilon h_1 \sum_{i=1}^{m_1}(\delta_x u_{i-\frac{1}{2},j_0})^2 + \left(\frac{1}{\varepsilon}+\frac{1}{L_1}\right) h_1 \sum_{i=0}^{m_1}\omega_i u_{i,j_0}^2 \\
\leqslant& \varepsilon h_1 \sum_{i=1}^{m_1}\left[\varepsilon h_2 \sum_{j=1}^{m_2}(\delta_y\delta_x u_{i-\frac{1}{2},j-\frac{1}{2}})^2 + \left(\frac{1}{\varepsilon}+\frac{1}{L_2}\right) h_2 \sum_{j=0}^{m_2}\bar{\omega}_j(\delta_x u_{i-\frac{1}{2},j})^2\right] \\
&+ \left(\frac{1}{\varepsilon}+\frac{1}{L_1}\right) h_1 \sum_{i=0}^{m_1}\omega_i\left[\varepsilon h_2 \sum_{j=1}^{m_2}(\delta_y u_{i,j-\frac{1}{2}})^2 + \left(\frac{1}{\varepsilon}+\frac{1}{L_2}\right) h_2 \sum_{j=1}^{m_2}\bar{\omega}_j u_{i,j}^2\right] \\
\leqslant& \varepsilon^2\|\delta_y\delta_x u\|^2 + \left[\varepsilon\left(\frac{1}{L_1}+\frac{1}{L_2}\right)+2\right]|u|_1^2 \\
&+ \left(\frac{1}{\varepsilon}+\frac{1}{L_1}\right)\left(\frac{1}{\varepsilon}+\frac{1}{L_2}\right)\|u\|^2.
\end{aligned} \tag{9.13}$$

此外, 由

$$\begin{aligned}
&\|\Delta_h u\|^2 \\
=& h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\omega_i\bar{\omega}_j\left(\delta_x^2 u_{ij}+\delta_y^2 u_{ij}\right)^2 \\
=& h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\omega_i\bar{\omega}_j\left(\delta_x^2 u_{ij}\right)^2 + 2h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\omega_i\bar{\omega}_j\left(\delta_x^2 u_{ij}\right)\left(\delta_y^2 u_{ij}\right) \\
&+h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\omega_i\bar{\omega}_j\left(\delta_y^2 u_{ij}\right)^2
\end{aligned}$$

及

$$\begin{aligned}
&h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}\omega_i\bar{\omega}_j\left(\delta_x^2 u_{ij}\right)\left(\delta_y^2 u_{ij}\right) \\
=& h_2\sum_{j=0}^{m_2}\bar{\omega}_j\left[h_1\sum_{i=0}^{m_1}\omega_i\left(\delta_x^2 u_{ij}\right)\left(\delta_y^2 u_{ij}\right)\right] \\
=& h_2\sum_{j=0}^{m_2}\bar{\omega}_j\left[-h_1\sum_{i=0}^{m_1-1}\left(\delta_x u_{i+\frac{1}{2},j}\right)\left(\delta_x\delta_y^2 u_{i+\frac{1}{2},j}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= -h_1 \sum_{i=0}^{m_1-1} \left[h_2 \sum_{j=0}^{m_2} \bar{\omega}_j \left(\delta_y^2 \delta_x u_{i+\frac{1}{2},j}\right)\left(\delta_x u_{i+\frac{1}{2},j}\right) \right] \\
&= -h_1 \sum_{i=0}^{m_1-1} \left[-h_2 \sum_{j=0}^{m_2} \left(\delta_y \delta_x u_{i+\frac{1}{2},j+\frac{1}{2}}\right)^2 \right] \\
&= h_1 h_2 \sum_{i=0}^{m_1-1} \sum_{j=0}^{m_2} \left(\delta_y \delta_x u_{i+\frac{1}{2},j+\frac{1}{2}}\right)^2 \\
&= \|\delta_y \delta_x u\|^2,
\end{aligned}
$$

可得

$$\|\Delta_h u\|^2 = \|\delta_x^2 u\|^2 + 2\|\delta_y \delta_x u\|^2 + \|\delta_y^2 u\|^2. \tag{9.14}$$

易知

$$\|\delta_y \delta_x u\|^2 \leqslant \frac{1}{2}\|\Delta_h u\|^2. \tag{9.15}$$

应用引理 9.1, 对任意的 $\varepsilon_1 > 0$, 可得

$$|u|_1^2 = -(\Delta_h u, u) \leqslant \varepsilon_1 \|\Delta_h u\|^2 + \frac{1}{4\varepsilon_1}\|u\|^2 \tag{9.16}$$

将 (9.15) 和 (9.16) 代入 (9.13) 得到

$$
\begin{aligned}
u_{i_0,j_0}^2 \leqslant & \frac{1}{2}\varepsilon^2 \|\Delta_h u\|^2 + \left[\varepsilon\left(\frac{1}{L_1} + \frac{1}{L_2}\right) + 2\right]\left(\varepsilon_1 \|\Delta_h u\|^2 + \frac{1}{4\varepsilon_1}\|u\|^2\right) \\
& + \left(\frac{1}{\varepsilon} + \frac{1}{L_1}\right)\left(\frac{1}{\varepsilon} + \frac{1}{L_2}\right)\|u\|^2.
\end{aligned}
$$

取 ε_1 使得 $\left[\varepsilon\left(\frac{1}{L_1} + \frac{1}{L_2}\right) + 2\right]\varepsilon_1 = \frac{1}{2}\varepsilon^2$, 并注意到

$$\frac{4}{L_1 L_2} \leqslant \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^2,$$

可得

$$
\begin{aligned}
u_{i_0,j_0}^2 \leqslant & \varepsilon^2 \|\Delta_h u\|^2 + \left\{ \frac{\left[\varepsilon\left(\frac{1}{L_1} + \frac{1}{L_2}\right) + 2\right]^2}{2\varepsilon^2} + \left(\frac{1}{\varepsilon} + \frac{1}{L_1}\right)\left(\frac{1}{\varepsilon} + \frac{1}{L_2}\right) \right\}\|u\|^2 \\
= & \varepsilon^2 \|\Delta_h u\|^2 + \left[\frac{3}{\varepsilon^2} + \frac{3}{\varepsilon}\left(\frac{1}{L_1} + \frac{1}{L_2}\right) + \frac{1}{2}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)^2 + \frac{1}{L_1 L_2}\right]\|u\|^2 \\
\leqslant & \varepsilon^2 \|\Delta_h u\|^2 + 3\left[\frac{1}{\varepsilon} + \frac{1}{2}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)\right]^2 \|u\|^2.
\end{aligned}
$$

两边开平方得到

$$\|u\|_\infty \leqslant \varepsilon\|\Delta_h u\| + \sqrt{3}\left[\frac{1}{\varepsilon} + \frac{1}{2}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)\right]\|u\|. \qquad \square$$

9.2.1 差分格式的建立

令

$$v = \phi(u) - \alpha\Delta u,$$

则 (9.1)–(9.3) 等价于

$$u_t = \Delta v, \quad (x,y)\in\Omega,\ 0<t\leqslant T, \tag{9.17}$$
$$v = \phi(u) - \alpha\Delta u, \quad (x,y)\in\Omega,\ 0<t\leqslant T, \tag{9.18}$$
$$\frac{\partial u}{\partial \nu} = 0, \quad \frac{\partial v}{\partial \nu} = 0, \quad (x,y)\in\partial\Omega,\ 0<t\leqslant T, \tag{9.19}$$
$$u(x,y,0) = \varphi(x,y), \quad (x,y)\in\bar{\Omega}. \tag{9.20}$$

将 (9.17) 关于 x 求导, 可得

$$(u_x)_t = v_{xxx} + (v_x)_{yy}.$$

令 $x = 0, L_1$, 并利用 (9.19), 可得

$$v_{xxx}|_{x=0} = 0, \quad v_{xxx}|_{x=L_1} = 0. \tag{9.21}$$

将 (9.17) 对 y 求导, 令 $y = 0, L_2$, 并利用 (9.19) 可得

$$v_{yyy}|_{y=0} = 0, \quad v_{yyy}|_{y=L_2} = 0. \tag{9.22}$$

同理对 (9.18) 分别关于 x 及 y 求导, 并利用 (9.19) 可得

$$u_{xxx}|_{x=0} = 0, \quad u_{xxx}|_{x=L_1} = 0, \tag{9.23}$$
$$u_{yyy}|_{y=0} = 0, \quad u_{yyy}|_{y=L_2} = 0. \tag{9.24}$$

定义网格函数

$$U_{ij}^k = u(x_i,y_j,t_k), \quad V_{ij}^k = v(x_i,y_j,t_k), \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 0\leqslant k\leqslant n.$$

在点 $(x_i, y_i, t_{k+\frac{1}{2}})$ 处考虑方程 (9.17)–(9.18), 应用引理 1.2, (9.19), (9.21)–(9.24), 可得

$$\delta_t U_{ij}^{k+\frac{1}{2}} = \Delta_h V_{ij}^{k+\frac{1}{2}} + P_{ij}^{k+\frac{1}{2}},$$

$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.25}$$

$$V_{ij}^{k+\frac{1}{2}} = \phi(U_{ij}^{k+\frac{1}{2}}) - \alpha\Delta_h U_{ij}^{k+\frac{1}{2}} + Q_{ij}^{k+\frac{1}{2}},$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.26}$$

存在常数 c_1 使得

$$\left|P_{ij}^{k+\frac{1}{2}}\right| \leqslant c_1(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.27}$$

$$\left|Q_{ij}^{k+\frac{1}{2}}\right| \leqslant c_1(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.28}$$

$$\frac{1}{\tau}\left|Q_{ij}^{k+\frac{1}{2}} - Q_{ij}^{k-\frac{1}{2}}\right| \leqslant c_1(\tau^2 + h_1^2 + h_2^2),$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.29}$$

在 (9.25)–(9.26) 中略去小量项, 并注意到初值条件

$$U_{ij}^0 = \varphi(x_i, y_j), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.30}$$

对 (9.17)–(9.20) 建立如下差分格式

$$\delta_t u_{ij}^{k+\frac{1}{2}} = \Delta_h v_{ij}^{k+\frac{1}{2}}, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.31}$$

$$v_{ij}^{k+\frac{1}{2}} = \phi\left(u_{ij}^{k+\frac{1}{2}}\right) - \alpha\Delta_h u_{ij}^{k+\frac{1}{2}},$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.32}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2. \tag{9.33}$$

将 (9.32) 代入 (9.31) 可得

$$\delta_t u_{ij}^{k+\frac{1}{2}} = \Delta_h\big(\phi(u_{ij}^{k+\frac{1}{2}}) - \alpha\Delta_h u_{ij}^{k+\frac{1}{2}}\big),$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.34}$$

$$u_{ij}^0 = \varphi(x_i, y_i), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.35}$$

9.2.2 差分格式解的存在性

定理 9.3 设 $\dfrac{\gamma^2\beta^4}{8\alpha}\tau < 1$, 则差分格式 (9.34)–(9.35) 的解是存在的.

证明 由 (9.35) 知第 0 层解 u^0 已给定.

设已求得第 k 层的值 u^k. 则由 (9.34) 可得关于第 $k+1$ 层解 u^{k+1} 的非线性方程组. 令

$$w_{ij} = u_{ij}^{k+\frac{1}{2}}, \quad 0 \leqslant i \leqslant m_1, \quad 0 \leqslant j \leqslant m_2.$$

由 (9.34) 可得

$$\frac{2}{\tau}\left(w_{ij}-u_{ij}^{k}\right)=\Delta_h\big(\phi(w_{ij})-\alpha\Delta_h w_{ij}\big),\quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\tag{9.36}$$

当 w 求得时,

$$u_{ij}^{k+1}=2w_{ij}-u_{ij}^{k},\quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2.$$

故只要证明 (9.36) 存在解.

令

$$\Pi(w)_{ij}=\frac{2}{\tau}(w_{ij}-u_{ij}^{k})-\Delta_h\big(\phi(w_{ij})-\alpha\Delta_h w_{ij}\big),\quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2.$$

则

$$\begin{aligned}(\Pi(w),w)=&\frac{2}{\tau}\left[(w,w)-(u^k,w)\right]-(\Delta_h(\phi(w)-\alpha\Delta_h w),w)\\=&\frac{2}{\tau}\left[\|w\|^2-(u^k,w)\right]-(\Delta_h\phi(w),w)+\alpha\|\Delta_h w\|^2.\end{aligned}\tag{9.37}$$

注意到存在 $\xi_{i+\frac12,j}$ 介于 w_{ij} 与 $w_{i+1,j}$ 之间, $\eta_{i,j+\frac12}$ 介于 w_{ij} 与 $w_{i,j+1}$ 之间使得

$$\begin{aligned}-\big(\Delta_h\phi(w),w\big)=&\big(\delta_x\phi(w),\delta_x w\big)+\big(\delta_y\phi(w),\delta_y w\big)\\=&h_1h_2\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2}\overline{\omega}_j\left(\delta_x\phi(w)_{i+\frac12,j}\right)\left(\delta_x w_{i+\frac12,j}\right)\\&+h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2-1}\omega_i\left(\delta_y\phi(w)_{i,j+\frac12}\right)\left(\delta_y w_{i,j+\frac12}\right)\\=&h_1h_2\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2}\overline{\omega}_j\phi'(\xi_{i+\frac12,j})\left(\delta_x w_{i+\frac12,j}\right)^2\\&+h_1h_2\sum_{i=0}^{m_1}\sum_{j=0}^{m_2-1}\omega_i\phi'(\eta_{i,j+\frac12})\left(\delta_y w_{i,j+\frac12}\right)^2\end{aligned}$$

及

$$\phi'(u)=\gamma(3u^2-\beta^2)\geqslant-\gamma\beta^2$$

知

$$-\big(\Delta_h\phi(w),w\big)\geqslant-\gamma\beta^2\|\nabla_h w\|^2\geqslant-\gamma\beta^2\|\Delta_h w\|\cdot\|w\|\geqslant-\left(\alpha\|\Delta_h w\|^2+\frac{\gamma^2\beta^4}{4\alpha}\|w\|^2\right).\tag{9.38}$$

将 (9.38) 代入 (9.37) 得

$$(\Pi(w),w)\geqslant\frac{2}{\tau}[\|w\|^2-\|u^k\|\cdot\|w\|]-\frac{1}{4\alpha}\gamma^2\beta^4\|w\|^2$$

$$=\frac{2}{\tau}\|w\|\left(\|w\|-\|u^k\|-\frac{\tau}{8\alpha}\gamma^2\beta^4\|w\|\right)$$
$$=\frac{2}{\tau}\|w\|\left[\left(1-\frac{\tau}{8\alpha}\gamma^2\beta^4\right)\|w\|-\|u\|^k\right],$$

当 $\frac{\gamma^2\beta^4}{8\alpha}\tau<1$ 且 $\|w\|=\dfrac{\|u^k\|}{1-\frac{\gamma^2\beta^4}{8\alpha}\tau}$ 时, $(\Pi(w),w)\geqslant 0$. 由 Browder 定理 (定理 1.3) 知 (9.36) 存在解. □

9.2.3 差分格式解的有界性

定理 9.4 设 $\{u_{ij}^k\,|\,0\leqslant i\leqslant m_1, 0\leqslant j\leqslant m_2, 0\leqslant k\leqslant n\}$ 为差分格式 (9.34)-(9.35) 的解, 则当 $\frac{3\gamma^2\beta^4}{8\alpha}\tau\leqslant 1$ 时, 有

$$\|u^k\|\leqslant e^{\frac{3\gamma^2\beta^4}{8\alpha}T}\|u^0\|,\quad 1\leqslant k\leqslant n.$$

证明 用 $u^{k+\frac{1}{2}}$ 与 (9.34) 的两边作内积, 得

$$\left(\delta_t u^{k+\frac{1}{2}},u^{k+\frac{1}{2}}\right)=\left(\Delta_h\left(\phi(u^{k+\frac{1}{2}})-\alpha\Delta_h u^{k+\frac{1}{2}}\right),u^{k+\frac{1}{2}}\right)$$
$$=\left(\Delta_h\phi(u^{k+\frac{1}{2}}),u^{k+\frac{1}{2}}\right)-\alpha\left(\Delta_h u^{k+\frac{1}{2}},\Delta_h u^{k+\frac{1}{2}}\right).$$

利用 (9.38) 可得

$$\frac{1}{2\tau}(\|u^{k+1}\|^2-\|u^k\|^2)\leqslant\alpha\left\|\Delta_h u^{k+\frac{1}{2}}\right\|^2+\frac{\gamma^2\beta^4}{4\alpha}\left\|u^{k+\frac{1}{2}}\right\|^2-\alpha\left\|\Delta_h u^{k+\frac{1}{2}}\right\|^2$$
$$=\frac{\gamma^2\beta^4}{4\alpha}\left\|u^{k+\frac{1}{2}}\right\|^2$$
$$\leqslant\frac{\gamma^2\beta^4}{4\alpha}\left(\frac{\|u^{k+1}\|+\|u^k\|}{2}\right)^2,\quad 0\leqslant k\leqslant n-1.$$

两边约去 $\frac{1}{2}(\|u\|^{k+1}+\|u^k\|)$, 得到

$$\frac{1}{\tau}(\|u^{k+1}\|-\|u^k\|)\leqslant\frac{\gamma^2\beta^4}{8\alpha}(\|u^{k+1}\|+\|u^k\|),\quad 0\leqslant k\leqslant n-1.$$

整理得

$$\left(1-\frac{\gamma^2\beta^4}{8\alpha}\tau\right)\|u^{k+1}\|\leqslant\left(1+\frac{\gamma^2\beta^4}{8\alpha}\tau\right)\|u^k\|,\quad 0\leqslant k\leqslant n-1.$$

当 $\frac{\gamma^2\beta^4}{8\alpha}\tau\leqslant\frac{1}{3}$ 时,

$$\|u^{k+1}\|\leqslant\left(1+\frac{3\gamma^2\beta^4}{8\alpha}\tau\right)\|u^k\|,\quad 0\leqslant k\leqslant n-1.$$

递推, 得到

$$\|u^k\| \leqslant e^{\frac{3\gamma^2\beta^4}{8\alpha}T}\|u^0\|, \quad 1 \leqslant k \leqslant n.$$

9.2.4 差分格式解的收敛性

记

$$c_0 = \max_{(x,y)\in\bar{\Omega},0\leqslant t\leqslant T} |u(x,y,t)|.$$

作二阶光滑函数

$$\Phi(u) = \begin{cases} \phi(u), & |u| \leqslant c_0+1, \\ 0, & |u| \geqslant c_0+2, \end{cases}$$

则存在常数 c_2 使得

$$|\Phi'(u)| \leqslant c_2, \quad |\Phi''(u)| \leqslant c_2.$$

注 9.2 当 $u \in [c_0+1, c_0+2]$ 时, 可取 $\Phi(u)$ 为满足下列插值条件的 5 次插值多项式

$$\Phi^{(l)}(c_0+1) = \phi^{(l)}(c_0+1), \quad \Phi^{(l)}(c_0+2) = 0, \qquad l = 0,1,2.$$

当 $u \in [-c_0-2, -c_0-1]$ 时, 可取 $\Phi(u)$ 为满足下列插值条件的 5 次插值多项式

$$\Phi^{(l)}(-c_0-2) = 0, \quad \Phi^{(l)}(-c_0-1)) = \phi^{(l)}(-c_0-1), \quad l = 0,1,2.$$

注 9.3 通常称所构造的光滑函数 $\Phi(u)$ 为截断函数 (Cut Off Function), 并称使用截断函数论证差分格式收敛性的方法为截断函数法.

记

$$c_3 = \max_{(x,y)\in\bar{\Omega},0\leqslant t\leqslant T} |u_t(x,y,t)|.$$

考虑差分格式

$$\delta_t u_{ij}^{k+\frac{1}{2}} = \Delta_h v_{ij}^{k+\frac{1}{2}}, \quad 0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.39}$$

$$v_{ij}^{k+\frac{1}{2}} = \Phi(u_{ij}^{k+\frac{1}{2}}) - \alpha\Delta_h u_{ij}^{k+\frac{1}{2}}, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.40}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2. \tag{9.41}$$

定理 9.5 设 $\{U_{ij}^k \,|\, 0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 为 (9.17)–(9.20) 的解, $\{u_{ij}^k \,|\, 0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 为 (9.39)–(9.41) 的解. 记

$$e_{ij}^k = U_{ij}^k - u_{ij}^k, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n.$$

则存在常数 c_5 使得

$$\|e^k\|_\infty \leqslant c_5(\tau^2 + h_1^2 + h_2^2), \quad 1 \leqslant k \leqslant n. \tag{9.42}$$

证明　记

$$f_{ij}^k = V_{ij}^k - v_{ij}^k,\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n.$$

注意到 $\Phi(U_{ij}^{k+\frac{1}{2}}) = \phi(U_{ij}^{k+\frac{1}{2}})$, 将 (9.25)–(9.26), (9.30) 与 (9.39)–(9.41) 依次相减, 得到误差方程组

$$\delta_t e_{ij}^{k+\frac{1}{2}} = \Delta_h f_{ij}^{k+\frac{1}{2}} + P_{ij}^{k+\frac{1}{2}},$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.43}$$
$$f_{ij}^{k+\frac{1}{2}} = \Phi(U_{ij}^{k+\frac{1}{2}}) - \Phi(u_{ij}^{k+\frac{1}{2}}) - \alpha\Delta_h e_{ij}^{k+\frac{1}{2}} + Q_{ij}^{k+\frac{1}{2}},$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{9.44}$$
$$e_{ij}^0 = 0\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2. \tag{9.45}$$

(I) 用 $e^{k+\frac{1}{2}}$ 与 (9.43) 的两边作内积, 得到

$$\begin{aligned}\frac{1}{2\tau}\left(\|e^{k+1}\|^2 - \|e^k\|^2\right) &= \left(\Delta_h f^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}\right) + \left(P^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}\right)\\ &= \left(f^{k+\frac{1}{2}}, \Delta_h e^{k+\frac{1}{2}}\right) + \left(P^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}\right).\end{aligned} \tag{9.46}$$

用 $\dfrac{1}{\alpha}f^{k+\frac{1}{2}}$ 与 (9.44) 作内积, 得到

$$\begin{aligned}\frac{1}{\alpha}\left\|f^{k+\frac{1}{2}}\right\|^2 =& \frac{1}{\alpha}\left(\Phi(U^{k+\frac{1}{2}}) - \Phi(u^{k+\frac{1}{2}}), f^{k+\frac{1}{2}}\right)\\ &- \left(\Delta_h e^{k+\frac{1}{2}}, f^{k+\frac{1}{2}}\right) + \frac{1}{\alpha}\left(Q^{k+\frac{1}{2}}, f^{k+\frac{1}{2}}\right).\end{aligned} \tag{9.47}$$

将 (9.46) 和 (9.47) 相加, 得到

$$\begin{aligned}&\frac{1}{2\tau}\left(\|e^{k+1}\|^2 - \|e^k\|^2\right) + \frac{1}{\alpha}\left\|f^{k+\frac{1}{2}}\right\|^2\\ =&\frac{1}{\alpha}\left(\Phi(U^{k+\frac{1}{2}}) - \Phi(u^{k+\frac{1}{2}}), f^{k+\frac{1}{2}}\right) + \left(P^{k+\frac{1}{2}}, e^{k+\frac{1}{2}}\right) + \frac{1}{\alpha}\left(Q^{k+\frac{1}{2}}, f^{k+\frac{1}{2}}\right)\\ \leqslant&\frac{1}{\alpha}c_2\left\|e^{k+\frac{1}{2}}\right\|\cdot\left\|f^{k+\frac{1}{2}}\right\| + \left\|P^{k+\frac{1}{2}}\right\|\cdot\left\|e^{k+\frac{1}{2}}\right\| + \frac{1}{\alpha}\left\|Q^{k+\frac{1}{2}}\right\|\cdot\left\|f^{k+\frac{1}{2}}\right\|\\ \leqslant&\left(\frac{1}{2\alpha}\left\|f^{k+\frac{1}{2}}\right\|^2 + \frac{c_2^2}{2\alpha}\left\|e^{k+\frac{1}{2}}\right\|^2\right) + \left(\frac{1}{2}\left\|P^{k+\frac{1}{2}}\right\|^2 + \frac{1}{2}\left\|e^{k+\frac{1}{2}}\right\|^2\right)\\ &+\left(\frac{1}{2\alpha}\left\|f^{k+\frac{1}{2}}\right\|^2 + \frac{1}{2\alpha}\left\|Q^{k+\frac{1}{2}}\right\|^2\right).\end{aligned} \tag{9.48}$$

注意到 (9.27), (9.28), 可得

$$\frac{1}{2\tau}\left(\left\|e^{k+1}\right\|^2 - \left\|e^k\right\|^2\right)$$

$$\leqslant\frac{1}{2}\left(1+\frac{c_2^2}{\alpha}\right)\left\|e^{k+\frac{1}{2}}\right\|^2+\frac{1}{2}\left\|P^{k+\frac{1}{2}}\right\|^2+\frac{1}{2\alpha}\left\|Q^{k+\frac{1}{2}}\right\|^2$$
$$\leqslant\frac{1}{2}\left(1+\frac{c_2^2}{\alpha}\right)\left\|e^{k+\frac{1}{2}}\right\|^2+\frac{1}{2}\left(1+\frac{1}{\alpha}\right)L_1L_2c_1^2\tau(\tau^2+h_1^2+h_2^2)^2,\quad 0\leqslant k\leqslant n-1.$$

当 $\frac{1}{2}\left(1+\frac{c_2^2}{\alpha}\right)\tau\leqslant\frac{1}{3}$ 时, 有

$$\left\|e^{k+1}\right\|^2\leqslant\left[1+\frac{3}{2}\left(1+\frac{c_2^2}{\alpha}\right)\tau\right]\left\|e^k\right\|^2+\frac{3}{2}\left(1+\frac{1}{\alpha}\right)L_1L_2c_1^2\tau(\tau^2+h_1^2+h_2^2)^2,\quad 0\leqslant k\leqslant n-1.$$

由 Gronwall 不等式, 可得

$$\|e^k\|^2\leqslant e^{\frac{3}{2}\left(1+\frac{c_2^2}{\alpha}\right)T}\frac{\left(1+\frac{1}{\alpha}\right)L_1L_2c_1^2}{1+\frac{c_2^2}{\alpha}}(\tau^2+h_1^2+h_2^2)^2,\quad 1\leqslant k\leqslant n. \tag{9.49}$$

(II) 用 $\delta_te^{k+\frac{1}{2}}$ 与 (9.43) 的两边作内积, 得

$$\left\|\delta_te^{k+\frac{1}{2}}\right\|^2=\left(\Delta_hf^{k+\frac{1}{2}},\delta_te^{k+\frac{1}{2}}\right)+\left(P^{k+\frac{1}{2}},\delta_te^{k+\frac{1}{2}}\right),\quad 0\leqslant k\leqslant n-1.$$

用 $\Delta_h\delta_te^{k+\frac{1}{2}}$ 与 (9.44) 作内积, 得

$$\begin{aligned}\left(f^{k+\frac{1}{2}},\Delta_h\delta_te^{k+\frac{1}{2}}\right)=&\left(\Phi(U^{k+\frac{1}{2}})-\Phi(u^{k+\frac{1}{2}}),\Delta_h\delta_te^{k+\frac{1}{2}}\right)\\&-\alpha\left(\Delta_he^{k+\frac{1}{2}},\Delta_h\delta_te^{k+\frac{1}{2}}\right)\\&+\left(Q^{k+\frac{1}{2}},\Delta_h\delta_te^{k+\frac{1}{2}}\right),\quad 0\leqslant k\leqslant n-1.\end{aligned}$$

将以上两式相加, 得

$$\begin{aligned}&\frac{\alpha}{2\tau}\left(\|\Delta_he^{k+1}\|^2-\|\Delta_he^k\|^2\right)+\left\|\delta_te^{k+\frac{1}{2}}\right\|^2\\=&\left(\Phi(U^{k+\frac{1}{2}})-\Phi(u^{k+\frac{1}{2}}),\Delta_h\delta_te^{k+\frac{1}{2}}\right)+\left(Q^{k+\frac{1}{2}},\Delta_h\delta_te^{k+\frac{1}{2}}\right)\\&+\left(P^{k+\frac{1}{2}},\delta_te^{k+\frac{1}{2}}\right),\quad 0\leqslant k\leqslant n-1.\end{aligned}$$

将上式中的 k 换为 l, 并对 l 从 0 到 k 求和, 得

$$\begin{aligned}&\frac{\alpha}{2\tau}\left(\|\Delta_he^{k+1}\|^2-\|\Delta_he^0\|^2\right)+\sum_{l=0}^k\left\|\delta_te^{l+\frac{1}{2}}\right\|^2\\=&\sum_{l=0}^k\left(\Phi(U^{l+\frac{1}{2}})-\Phi(u^{l-\frac{1}{2}}),\Delta_h\delta_te^{l+\frac{1}{2}}\right)\end{aligned}$$

$$+\sum_{l=0}^{k}\left(Q^{l+\frac{1}{2}},\Delta_h\delta_t e^{l+\frac{1}{2}}\right)+\sum_{l=0}^{k}\left(P^{l+\frac{1}{2}},\delta_t e^{l+\frac{1}{2}}\right),\quad 0\leqslant k\leqslant n-1. \tag{9.50}$$

对于 (9.50) 右端的第二项, 有

$$\begin{aligned}
&\sum_{l=0}^{k}\left(Q^{l+\frac{1}{2}},\Delta_h\delta_t e^{l+\frac{1}{2}}\right)\\
=&\frac{1}{\tau}\left[\sum_{l=0}^{k}\left(Q^{l+\frac{1}{2}},\Delta_h e^{l+1}\right)-\sum_{l=0}^{k}\left(Q^{l+\frac{1}{2}},\Delta_h e^{l}\right)\right]\\
=&\frac{1}{\tau}\left[\left(Q^{k+\frac{1}{2}},\Delta_h e^{k+1}\right)-\left(Q^{\frac{1}{2}},\Delta_h e^{0}\right)\right]-\sum_{l=1}^{k}\left(\frac{Q^{l+\frac{1}{2}}-Q^{l-\frac{1}{2}}}{\tau},\Delta_h e^{l}\right)\\
\leqslant&\frac{1}{\tau}\left(\left\|Q^{k+\frac{1}{2}}\right\|\cdot\|\Delta_h e^{k+1}\|+\left\|Q^{\frac{1}{2}}\right\|\cdot\|\Delta_h e^{0}\|\right)\\
&+\sum_{l=1}^{k}\left\|\frac{Q^{l+\frac{1}{2}}-Q^{l-\frac{1}{2}}}{\tau}\right\|\cdot\|\Delta_h e^{l}\|.
\end{aligned}\tag{9.51}$$

对于 (9.50) 右端的第一项, 有

$$\begin{aligned}
&\sum_{l=0}^{k}\left(\Phi(U^{l+\frac{1}{2}})-\Phi(u^{l+\frac{1}{2}}),\Delta_h\delta_t e^{l+\frac{1}{2}}\right)\\
=&\frac{1}{\tau}\left[\left(\Phi(U^{k+\frac{1}{2}})-\Phi(u^{k+\frac{1}{2}}),\Delta_h e^{k+1}\right)-\left(\Phi(U^{\frac{1}{2}})-\Phi(u^{\frac{1}{2}}),\Delta_h e^{0}\right)\right]\\
&-\sum_{l=1}^{k}\frac{\left[\Phi(U^{l+\frac{1}{2}})-\Phi(u^{l+\frac{1}{2}})\right]-\left[\Phi(U^{l-\frac{1}{2}})-\Phi(u^{l-\frac{1}{2}})\right]}{\tau}\Delta_h e^{l}\\
\leqslant&\frac{1}{\tau}\left(\left\|\Phi(U^{k+\frac{1}{2}})-\Phi(u^{k+\frac{1}{2}})\right\|\cdot\left\|\Delta_h e^{k+1}\right\|+\left\|\Phi(U^{\frac{1}{2}})-\Phi(u^{\frac{1}{2}})\right\|\cdot\|\Delta_h e^{0}\|\right)\\
&+\sum_{l=1}^{k}\left\|\frac{\left[\Phi(U^{l+\frac{1}{2}})-\Phi(u^{l+\frac{1}{2}})\right]-\left[\Phi(U^{l-\frac{1}{2}})-\Phi(u^{l-\frac{1}{2}})\right]}{\tau}\right\|\cdot\|\Delta_h e^{l}\|.
\end{aligned}$$

注意到存在 $\rho\in(0,1),\theta\in(0,1)$, 使得

$$\begin{aligned}
&\left[\Phi(U^{l+\frac{1}{2}})-\Phi(u^{l+\frac{1}{2}})\right]-\left[\Phi(U^{l-\frac{1}{2}})-\Phi(u^{l-\frac{1}{2}})\right]\\
=&\left[\Phi(U^{l-\frac{1}{2}}+\tau\Delta_t U^{l})-\Phi(u^{l-\frac{1}{2}}+\tau\Delta_t u^{l})\right]-\left[\Phi(U^{l-\frac{1}{2}})-\Phi(u^{l-\frac{1}{2}})\right]\\
=&\Phi'(U^{l-\frac{1}{2}}+\tau\Delta_t U^{l})\tau\Delta_t U^{l}-\Phi'(u^{l-\frac{1}{2}}+\tau\rho\Delta_t u^{l})\tau\Delta_t u^{l}\\
=&\Phi'(u^{l-\frac{1}{2}}+\tau\rho\Delta_t u^{l})\tau(\Delta_t U^{l}-\Delta_t u^{l})\\
&+\left[\Phi'(U^{l-\frac{1}{2}}+\tau\rho\Delta_t U^{l})-\Phi'(u^{l-\frac{1}{2}}+\tau\rho\Delta_t u^{l})\right]\tau\Delta_t U^{l}
\end{aligned}$$

$$=\Phi'(u^{l-\frac{1}{2}}+\tau\rho\Delta_t u^l)\tau\Delta_t e^l$$
$$+\Phi''\big(\theta(U^{l-\frac{1}{2}}+\tau\rho\Delta_t U^l)+(1-\theta)(u^{l-\frac{1}{2}}+\tau\rho\Delta_t u^l)\big)\big(e^{l-\frac{1}{2}}+\tau\rho\Delta_t e^l\big)\tau\Delta_t U^l,$$

有

$$\left\|\frac{\big[\Phi(U^{l+\frac{1}{2}})-\Phi(u^{l+\frac{1}{2}})\big]-\big[\Phi(U^{l-\frac{1}{2}})-\Phi(u^{l-\frac{1}{2}})\big]}{\tau}\right\|$$
$$\leqslant c_2\|\Delta_t e^l\|+c_2c_3\left(\left\|e^{l-\frac{1}{2}}\right\|+\tau\|\Delta_t e^l\|\right),$$

于是

$$\begin{aligned}&\sum_{l=0}^{k}\left(\Phi(U^{l+\frac{1}{2}})-\Phi(u^{l+\frac{1}{2}}),\Delta_h\delta_t e^{l+\frac{1}{2}}\right)\\ \leqslant&\frac{1}{\tau}\left(\left\|\Phi(U^{k+\frac{1}{2}})-\Phi(u^{k+\frac{1}{2}})\right\|\cdot\|\Delta_h e^{k+1}\|+\left\|\Phi(U^{\frac{1}{2}})-\Phi(u^{\frac{1}{2}})\right\|\cdot\|\Delta_h e^0\|\right)\\ &+\sum_{l=1}^{k}\left((c_2+c_2c_3\tau)\|\Delta_t e^l\|+c_2c_3\|e^{l-\frac{1}{2}}\|\right)\|\Delta_h e^l\|\\ \leqslant&\frac{1}{\tau}c_2\|e^{k+\frac{1}{2}}\|\cdot\|\Delta_h e^{k+1}\|+\sum_{l=1}^{k}(c_2+c_2c_3\tau)\|\Delta_t e^l\|\cdot\|\Delta_h e^l\|\\ &+\sum_{l=1}^{k}c_2c_3\|e^{l-\frac{1}{2}}\|\cdot\|\Delta_h e^l\|.\end{aligned}\tag{9.52}$$

将 (9.51) 和 (9.52) 代入 (9.50), 得

$$\begin{aligned}&\frac{\alpha}{2\tau}\left\|\Delta_h e^{k+1}\right\|^2+\sum_{l=0}^{k}\left\|\delta_t e^{l+\frac{1}{2}}\right\|^2\\ \leqslant&\frac{1}{\tau}\left\|Q^{k+\frac{1}{2}}\right\|\cdot\|\Delta_h e^{k+1}\|+\sum_{l=1}^{k}\left\|\frac{Q^{l+\frac{1}{2}}-Q^{l-\frac{1}{2}}}{\tau}\right\|\cdot\|\Delta_h e^l\|\\ &+\frac{1}{\tau}c_2\left\|e^{k+\frac{1}{2}}\right\|\cdot\|\Delta_h e^{k+1}\|+(c_2+c_2c_3\tau)\sum_{l=1}^{k}\|\Delta_t e^l\|\cdot\|\Delta_h e^l\|\\ &+c_2c_3\sum_{l=1}^{k}\left\|e^{l-\frac{1}{2}}\right\|\cdot\|\Delta_h e^l\|+\sum_{l=0}^{k}\left\|P^{l+\frac{1}{2}}\right\|\cdot\left\|\delta_t e^{l+\frac{1}{2}}\right\|\\ \leqslant&\frac{1}{\tau}\left[\frac{\alpha}{8}\|\Delta_h e^{k+1}\|^2+\frac{2}{\alpha}\|Q^{k+1}\|^2\right]\\ &+\frac{1}{2}\sum_{l=1}^{k}\|\Delta_h e^l\|^2+\frac{1}{2}\sum_{l=1}^{k}\left\|\frac{Q^{l+\frac{1}{2}}-Q^{l-\frac{1}{2}}}{\tau}\right\|^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\tau}\left[\frac{\alpha}{8}\|\Delta_h e^{k+1}\|^2+\frac{2c_2^2}{\alpha}\left\|e^{k+\frac{1}{2}}\right\|^2\right]\\
&+\frac{1}{2}\sum_{l=1}^{k}\|\Delta_t e^l\|^2+\frac{1}{2}(c_2+c_2c_3)^2\sum_{l=1}^{k}\|\Delta_h e^l\|^2\\
&+\frac{c_2c_3}{2}\sum_{l=1}^{k}\left\|e^{l-\frac{1}{2}}\right\|^2+\frac{c_2c_3}{2}\sum_{l=1}^{k}\|\Delta_h e^l\|^2\\
&+\frac{1}{2}\sum_{l=0}^{k}\left\|\delta_t e^{l+\frac{1}{2}}\right\|^2+\frac{1}{2}\sum_{l=0}^{k}\left\|P^{l+\frac{1}{2}}\right\|^2.
\end{aligned}
$$

上式两边乘以 $\dfrac{4\tau}{\alpha}$, 得

$$
\begin{aligned}
\|\Delta_h e^{k+1}\|^2\leqslant&\frac{8c_2^2}{\alpha^2}\left\|e^{k+\frac{1}{2}}\right\|^2+\frac{8}{\alpha^2}\|Q^{k+1}\|^2\\
&+\frac{2\tau}{\alpha}\left(1+c_2c_3+(c_2+c_2c_3)^2\right)\sum_{l=1}^{k}\|\Delta_h e^l\|^2+\frac{2c_2c_3}{\alpha}\tau\sum_{l=1}^{k}\|e^{l-\frac{1}{2}}\|^2\\
&+\frac{2\tau}{\alpha}\cdot\left(\sum_{l=0}^{k}\left\|P^{l+\frac{1}{2}}\right\|^2+\sum_{l=1}^{k}\left\|\frac{Q^{l+\frac{1}{2}}-Q^{l-\frac{1}{2}}}{\tau}\right\|^2\right),\quad 0\leqslant k\leqslant n-1.
\end{aligned}
$$

将 (9.49) 代入上式右端并注意到 (9.27) 和 (9.29), 利用 Gronwall 不等式知存在常数 c_4 使得

$$
\|\Delta_h e^k\|\leqslant c_4(\tau^2+h_1^2+h_2^2),\quad 1\leqslant k\leqslant n. \tag{9.53}
$$

由 (9.49), (9.53) 和引理 9.2 可得存在常数 c_5 使得

$$
\|e^k\|_\infty\leqslant c_5(\tau^2+h_1^2+h_2^2),\quad 1\leqslant k\leqslant n. \tag{9.54}
$$

至此, 我们证得了差分格式 (9.39)–(9.41) 的解收敛到 (9.17)–(9.20) 的解, 且有估计 (9.42) 式. □

注意到当 $c_5(\tau^2+h_1^2+h_2^2)\leqslant 1$ 时,

$$
|u_{ij}^k|\leqslant|U_{ij}^k|+|e_{ij}^k|\leqslant c_0+1,\quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 0\leqslant k\leqslant n.
$$

因而

$$
\Phi(u_{ij}^k)=\phi(u_{ij}^k),\quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 0\leqslant k\leqslant n.
$$

也就是说差分格式 (9.39)–(9.41) 和差分格式 (9.31)–(9.33) 完全一致. 因而差分格式 (9.31)–(9.33) 的解收敛到 (9.17)–(9.20) 的解, 且有估计式 (9.42). 于是得到如下定理.

定理 9.6 设 $\{U_{ij}^k \,|\, 0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 为 (9.17)–(9.20) 的解, $\{u_{ij}^k \,|\, 0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 为 (9.31)–(9.33) 的解. 记

$$e_{ij}^k = U_{ij}^k - u_{ij}^k, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n.$$

则存在常数 c_5, 当 $\tau^2 + h_1^2 + h_2^2 \leqslant 1/c_5$ 时, 有

$$\|e^k\|_\infty \leqslant c_5(\tau^2 + h_1^2 + h_2^2), \quad 1 \leqslant k \leqslant n.$$

9.3 三层线性化差分格式

9.3.1 差分格式的建立

在点 $\left(x_i, y_i, t_{\frac{1}{2}}\right)$ 处考虑方程 (9.17), (9.18), 利用引理 1.2 及 (9.19), (9.21)–(9.24), 可得

$$\delta_t U_{ij}^{\frac{1}{2}} = \Delta_h V_{ij}^{\frac{1}{2}} + p_{ij}^0, \quad 0 \leqslant i \leqslant m_1, \quad 0 \leqslant j \leqslant m_2, \tag{9.55}$$

$$V_{ij}^{\frac{1}{2}} = \phi(\hat{u}_{ij}) - \alpha \Delta_h U_{ij}^{\frac{1}{2}} + q_{ij}^0, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.56}$$

其中

$$\hat{u}_{ij} = u(x_i, y_j, 0) + \frac{\tau}{2} u_t(x_i, y_j, 0), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,$$

存在常数 c_6 使得

$$|p_{ij}^0| \leqslant c_6(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2 \tag{9.57}$$

$$|q_{ij}^0| \leqslant c_6(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2. \tag{9.58}$$

在点 (x_i, y_i, t_k) 处考虑方程 (9.17), (9.18), 利用引理 1.2 及 (9.19), (9.21)–(9.24), 可得

$$\Delta_t U_{ij}^k = \Delta_h V_{ij}^k + p_{ij}^k, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.59}$$

$$V_{ij}^k = \phi(U_{ij}^k) - \alpha \Delta_h U_{ij}^{\bar{k}} + q_{ij}^k, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.60}$$

存在常数 c_7 使得

$$|p_{ij}^k| \leqslant c_7(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.61}$$

$$|q_{ij}^k| \leqslant c_7(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.62}$$

$$|\Delta_t q_{ij}^k| \leqslant c_7(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 2 \leqslant k \leqslant n-2. \tag{9.63}$$

由 (9.20) 可知

$$U_{ij}^0 = \varphi(x_i, y_j), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.64}$$

在 (9.55)–(9.56), (9.59)–(9.60) 中略去小量项, 并注意到 (9.64), 对问题 (9.17)–(9.20) 建立如下差分格式

$$\delta_t u_{ij}^{\frac{1}{2}} = \Delta_h v_{ij}^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.65}$$

$$v_{ij}^{\frac{1}{2}} = \phi(\hat{u}_{ij}) - \alpha \Delta_h u_{ij}^{\frac{1}{2}}, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.66}$$

$$\Delta_t u_{ij}^k = \Delta_h v_{ij}^k, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.67}$$

$$v_{ij}^k = \phi(u_{ij}^k) - \alpha \Delta_h u_{ij}^{\bar{k}}, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.68}$$

$$u_{ij}^0 = \varphi(x_i, y_i), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2. \tag{9.69}$$

将 (9.66) 代入 (9.65), 将 (9.68) 代入 (9.67), 可得

$$\delta_t u_{ij}^{\frac{1}{2}} = \Delta_h \left(\phi(\hat{u}_{ij}) - \alpha \Delta_h u_{ij}^{\frac{1}{2}} \right), \quad 0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2, \tag{9.70}$$

$$\Delta_t u_{ij}^k = \Delta_h \left(\phi(u_{ij}^k) - \alpha \Delta_h u_{ij}^{\bar{k}} \right), \\ 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.71}$$

$$u_{ij}^0 = \varphi_0(x_i, y_i), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.72}$$

容易看出 (9.70)–(9.72) 是一个三层线性化差分格式. 对 (9.1)–(9.3) 建立差分格式 (9.70)–(9.72).

9.3.2 差分格式解的存在性和唯一性

定理 9.7 差分格式 (9.70)–(9.72) 是唯一可解的.

证明 由 (9.72) 知 u^0 唯一确定.

由 (9.70) 可得关于 u^1 的线性方程组. 考虑其齐次方程组

$$\frac{1}{\tau} u_{ij}^1 = -\frac{\alpha}{2} \Delta_h^2 u_{ij}^1, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2. \tag{9.73}$$

用 u^1 与 (9.73) 的两边作内积, 得

$$\frac{1}{\tau} \|u^1\|^2 + \frac{\alpha}{2} \|\Delta_h u^1\|^2 = 0.$$

易知 $\|u^1\| = 0$, 即 (9.73) 只有零解. 因而 (9.70) 唯一确定 u^1.

设 u^{k-1}, u^k 已唯一确定. 则由 (9.71) 可得关于 u^{k+1} 的线性方程组. 其齐次方程组为

$$\frac{1}{2\tau} u_{ij}^{k+1} = -\frac{\alpha}{2} \Delta_h^2 u_{ij}^{k+1}, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2.$$

类似地可证 $\|u^{k+1}\| = 0$. 因而 (9.71) 唯一确定 u^{k+1}. □

9.3.3 差分格式解的收敛性

引理 9.3 设 $u=(u^0,u^1,\cdots,u^n)$ 和 $U=(U^0,U^2,\cdots,U^n)$ 是 Ω_τ 上的两个网格函数, 则存在 $\rho\in(0,1)$ 和 $\xi\in(a,b)$ 使得

$$\begin{aligned}\Delta_t[\phi(U^k)-\phi(u^k)]=&\phi'(\rho u^{k+1}+(1-\rho)u^{k-1})\Delta_t(U^k-u^k)\\&+\phi''(\xi)\big[\rho(U^{k+1}-u^{k+1})+(1-\rho)(U^{k-1}-u^{k-1})\big]\Delta_t U^k,\end{aligned}$$

其中

$$\begin{aligned}a=&\min\{\rho u^{k+1}+(1-\rho)u^{k-1},\rho U^{k+1}+(1-\rho)U^{k-1}\},\\b=&\max\{\rho u^{k+1}+(1-\rho)u^{k-1},\rho U^{k+1}+(1-\rho)U^{k-1}\}.\end{aligned}$$

证明

$$\begin{aligned}&\Delta_t(\phi(U^k)-\phi(u^k))\\=&\frac{1}{2\tau}\big\{[\phi(U^{k+1})-\phi(u^{k+1})]-[\phi(U^{k-1})-\phi(u^{k-1})]\big\}\\=&\frac{1}{2\tau}\big\{[\phi(U^{k-1}+2\tau\Delta_t U^k)-\phi(u^{k-1}+2\tau\Delta_t u^k)]-[\phi(U^{k-1})-\phi(u^{k-1})]\big\}\\=&\phi'(U^{k-1}+2\rho\tau\Delta_t U^k)\Delta_t U^k-\phi'(u^{k-1}+2\rho\tau\Delta_t u^k)\Delta_t u^k \qquad (9.74)\\=&\phi'(u^{k-1}+2\rho\tau\Delta_t u^k)\Delta_t(U^k-u^k)\\&+[\phi'(\rho U^{k-1}+2\rho\tau\Delta_t U^k)-\phi'(u^{k-1}+2\rho\tau\Delta_t u^k)]\Delta_t U^k\\=&\phi'(\rho u^{k+1}+(1-\rho)u^{k-1})\Delta_t(U^k-u^k)\\&+\big[\phi'(\rho U^{k+1}+(1-\rho)U^{k-1})-\phi'(\rho u^{k+1}+(1-\rho)u^{k-1})\big]\Delta_t U^k\\=&\phi'(\rho u^{k+1}+(1-\rho)u^{k-1})\Delta_t(U^k-u^k)\\&+\phi''(\xi)\big[\rho(U^{k+1}-u^{k+1})+(1-\rho)(U^{k-1}-u^{k-1})\big]\Delta_t U^k. \qquad (9.75)\end{aligned}$$

在得到 (9.74) 时, 将 $\phi(U^{k-1}+2\tau\rho\Delta_t U^k)-\phi(u^{k-1}+2\tau\rho\Delta_t u^k)$ 看成 $\rho\in[0,1]$ 的函数, 然后用微分中值定理. 得到 (9.75) 是再次应用微分中值定理. □

引理 9.4

$$\sum_{l=1}^k u^l\Delta_t v^l=\frac{1}{2\tau}(u^kv^{k+1}+u^{k-1}v^k-u^0v^1-u^1v^0)-\sum_{l=1}^{k-1}(\Delta_t u^l)v^l,\quad 2\leqslant k\leqslant n-1;$$

$$\sum_{l=1}^k u^l\Delta_t v^l=\frac{1}{2\tau}(u^kv^{k+1}+u^{k-1}v^k-u^2v^1-u^1v^0)-\sum_{l=2}^{k-1}(\Delta_t u^l)v^l,\quad 2\leqslant k\leqslant n-1.$$

证明 注意到

$$\begin{aligned}\sum_{l=1}^{k} u^l \Delta_t v^l =&\frac{1}{2\tau}\sum_{l=1}^{k} u^l(v^{l+1}-v^{l-1})\\ =&\frac{1}{2\tau}\left(\sum_{l=1}^{k} u^l v^{l+1}-\sum_{l=1}^{k} u^l v^{l-1}\right)=\frac{1}{2\tau}\left(\sum_{l=2}^{k+1} u^{l-1} v^{l}-\sum_{l=0}^{k-1} u^{l+1} v^{l}\right),\end{aligned}$$

有

$$\begin{aligned}\sum_{l=1}^{k} u^l \Delta_t v^l =&\frac{1}{2\tau}(u^k v^{k+1}+u^{k-1}v^k-u^0v^1-u^1v^0)-\sum_{l=1}^{k-1}(\Delta_t u^l)v^l\\ =&\frac{1}{2\tau}(u^k v^{k+1}+u^{k-1}v^k-u^2v^1-u^1v^0)-\sum_{l=2}^{k-1}(\Delta_t u^l)v^l.\end{aligned}$$

□

定理 9.8 设 $\{U_{ij}^k\,|\,0\leqslant i\leqslant m_1, 0\leqslant j\leqslant m_2, 0\leqslant k\leqslant n\}$ 为 (9.1)–(9.3) 的解, $\{u_{ij}^k\,|\,0\leqslant i\leqslant m_1, 0\leqslant j\leqslant m_2, 0\leqslant k\leqslant n\}$ 为 (9.70)–(9.72) 的解. 记

$$e_{ij}^k=U_{ij}^k-u_{ij}^k,\quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 0\leqslant k\leqslant n.$$

则存在常数 c_8, 当 $\tau^2+h_1^2+h_2^2\leqslant \dfrac{1}{c_8}$ 时,

$$\|e^k\|_\infty\leqslant c_8(\tau^2+h_1^2+h_2^2),\quad 0\leqslant k\leqslant n. \tag{9.76}$$

证明 记

$$f_{ij}^k=V_{ij}^k-v_{ij}^k,\quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 0\leqslant k\leqslant n.$$

$$c_0=\max_{(x,y)\in\bar\Omega,t\in[0,T]}|u(x,y,t)|,\quad c_9=\max_{|u|\leqslant c_0+1}|\varphi'(u)|,\quad c_{10}=\max_{|u|\leqslant c_0+1}|\varphi''(u)|.$$

将 (9.55)–(9.56), (9.59)–(9.60), (9.64) 与 (9.65)–(9.69) 相减, 得到误差方程组

$$\delta_t e_{ij}^{\frac12}=\Delta_h f_{ij}^{\frac12}+p_{ij}^0,\quad 0\leqslant i\leqslant m_1,\quad 0\leqslant j\leqslant m_2, \tag{9.77}$$

$$f_{ij}^{\frac12}=-\alpha\Delta_h e_{ij}^{\frac12}+q_{ij}^0,\quad 0\leqslant i\leqslant m_1,\quad 0\leqslant j\leqslant m_2, \tag{9.78}$$

$$\Delta_t e_{ij}^k=\Delta_h f_{ij}^{\bar k}+p_{ij}^k,\quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n-1, \tag{9.79}$$

$$\begin{aligned}f_{ij}^{\bar k}=&\phi(U_{ij}^k)-\phi(u_{ij}^k)-\alpha\Delta_h e_{ij}^{\bar k}+q_{ij}^k,\\ &0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n-1,\end{aligned} \tag{9.80}$$

$$e_{ij}^0=0,\quad 0\leqslant i\leqslant m_1, 0\leqslant j\leqslant m_2, \tag{9.81}$$

由 (9.81) 知

$$\|e^0\|_\infty=0.$$

(I) $\|e^1\|$ 和 $\|\Delta_h e^1\|$ 的估计.

(a) 用 $e^{\frac{1}{2}}$ 与 (9.77) 的两边作内积, 得

$$\left(\delta_t e^{\frac{1}{2}}, e^{\frac{1}{2}}\right) = \left(\Delta_h f^{\frac{1}{2}}, e^{\frac{1}{2}}\right) + \left(p^0, e^{\frac{1}{2}}\right);$$

用 $\dfrac{1}{\alpha} f^{\frac{1}{2}}$ 与 (9.78) 的两边作内积, 得

$$\frac{1}{\alpha}\left\|f^{\frac{1}{2}}\right\|^2 = -\left(\Delta_h e^{\frac{1}{2}}, f^{\frac{1}{2}}\right) + \frac{1}{\alpha}\left(q^0, f^{\frac{1}{2}}\right).$$

将以上二式相加, 得到

$$\begin{aligned}\frac{1}{2\tau}(\|e^1\|^2 - \|e^0\|^2) + \frac{1}{\alpha}\left\|f^{\frac{1}{2}}\right\|^2 &= \left(p^0, e^{\frac{1}{2}}\right) + \frac{1}{\alpha}\left(q^0, f^{\frac{1}{2}}\right) \\ &\leqslant \|p^0\| \cdot \left\|e^{\frac{1}{2}}\right\| + \frac{1}{\alpha}\left\|f^{\frac{1}{2}}\right\|^2 + \frac{1}{4\alpha}\|q^0\|^2.\end{aligned}$$

注意到 $e^0 = 0$, 得

$$\begin{aligned}\frac{1}{2\tau}\|e^1\|^2 &\leqslant \|p^0\| \cdot \frac{1}{2}\|e^1\| + \frac{1}{4\alpha}\|q^0\|^2 \\ &\leqslant \frac{1}{4\tau}\|e^1\|^2 + \frac{1}{4}\tau\|p^0\|^2 + \frac{1}{4\alpha}\|q^0\|^2.\end{aligned}$$

因而

$$\begin{aligned}\|e^1\|^2 &\leqslant \tau(\tau\|p^0\|^2 + \frac{1}{\alpha}\|q^0\|^2) \\ &\leqslant \tau\left(\tau + \frac{1}{\alpha}\right) c_6^2 L_1 L_2 (\tau^2 + h_1^2 + h_2^2)^2. \end{aligned} \tag{9.82}$$

(b) 用 $\delta_t e^{\frac{1}{2}}$ 与 (9.77) 的两边作内积, 得到

$$\left\|\delta_t e^{\frac{1}{2}}\right\|^2 = \left(\Delta_h f^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}\right) + \left(p^0, \delta_t e^{\frac{1}{2}}\right),$$

用 $\Delta_h \delta_t e^{\frac{1}{2}}$ 与 (9.78) 的两边作内积, 得到

$$\left(f^{\frac{1}{2}}, \Delta_h \delta_t e^{\frac{1}{2}}\right) = -\alpha\left(\Delta_h e^{\frac{1}{2}}, \Delta_h \delta_t e^{\frac{1}{2}}\right) + \left(q^0, \Delta_h \delta_t e^{\frac{1}{2}}\right).$$

将以上二式相加, 得到

$$\left\|\delta_t e^{\frac{1}{2}}\right\|^2 + \alpha\left(\Delta_h e^{\frac{1}{2}}, \Delta_t \delta_t e^{\frac{1}{2}}\right) = \left(p^0, \delta_t e^{\frac{1}{2}}\right) + \left(q^0, \Delta_h \delta_t e^{\frac{1}{2}}\right).$$

注意到 $e^0 = 0$, 得

$$\frac{1}{\tau^2}\|e^1\|^2 + \frac{\alpha}{2\tau}\|\Delta_h e^1\|^2 = \frac{1}{\tau}(p^0, e^1) + \frac{1}{\tau}(q^0, \Delta_h e^1)$$

$$\leqslant \frac{1}{\tau^2}\|e^1\|^2 + \frac{1}{4}\|p^0\|^2 + \frac{\alpha}{4\tau}\|\Delta_h e^1\|^2 + \frac{1}{\alpha\tau}\|q^0\|^2.$$

将上式乘以 $\frac{4\tau}{\alpha}$, 得

$$\begin{aligned}\|\Delta_h e^1\|^2 &\leqslant \frac{4\tau}{\alpha}\left(\frac{1}{4}\|p^0\|^2 + \frac{1}{\alpha\tau}\|q^0\|^2\right)\\ &= \frac{\tau}{\alpha}\|p^0\|^2 + \frac{4}{\alpha^2}\|q^0\|^2\\ &\leqslant \left(\frac{\tau}{\alpha} + \frac{4}{\alpha^2}\right) c_6^2 L_1 L_2(\tau^2 + h_1^2 + h_2^2)^2. \end{aligned} \tag{9.83}$$

(II) 设 (9.76) 对 $1 \leqslant k \leqslant m$ 成立. 则当 $\tau^2 + h_1^2 + h_2^2 \leqslant \frac{1}{c_8}$ 时,

$$\|e^k\|_\infty \leqslant 1, \quad 1 \leqslant k \leqslant m.$$

于是

$$\|u^k\|_\infty \leqslant \|U^k\|_\infty + \|e^k\|_\infty \leqslant c_0 + 1, \quad 1 \leqslant k \leqslant m,$$

$$|\phi(U_{ij}^k) - \phi(u_{ij}^k)| \leqslant c_9|e_{ij}^k|, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant m. \tag{9.84}$$

应用引理 9.3, 得

$$\begin{aligned}&|\Delta_t[\phi(U_{ij}^k) - \phi(u_{ij}^k)]| \leqslant c_9|\Delta_t e_{ij}^k| + c_{10}(|e_{ij}^{k+1}| + |e_{ij}^{k-1}|),\\ &\qquad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant m-1. \end{aligned} \tag{9.85}$$

现在来证明 (9.76) 对 $k = m+1$ 也成立.

(a) 用 $e^{\bar{k}}$ 与 (9.79) 的两边作内积, 得到

$$(\Delta_t e^k, e^{\bar{k}}) = (\Delta_h f^{\bar{k}}, e^{\bar{k}}) + (p^k, e^{\bar{k}}), \quad 1 \leqslant k \leqslant m.$$

用 $\frac{1}{\alpha} f^{\bar{k}}$ 与 (9.80) 的两边作内积, 得

$$\frac{1}{\alpha}\|f^{\bar{k}}\|^2 = \frac{1}{\alpha}(\phi(U^k) - \phi(u^k), f^{\bar{k}}) - (\Delta_h e^{\bar{k}}, f^{\bar{k}}) + \frac{1}{\alpha}(q^k, f^{\bar{k}}), \quad 1 \leqslant k \leqslant m.$$

将以上二式相加, 得

$$\begin{aligned}&\frac{1}{4\tau}(\|e^{k+1}\|^2 - \|e^{k-1}\|^2) + \frac{1}{\alpha}\|f^{\bar{k}}\|^2\\ =&\frac{1}{\alpha}(\phi(U^k) - \phi(u^k), f^{\bar{k}}) + (p^k, e^{\bar{k}}) + \frac{1}{\alpha}(q^k, f^{\bar{k}}), \quad 1 \leqslant k \leqslant m.\end{aligned}$$

应用 (9.84), 得

$$\begin{aligned}
&\frac{1}{4\tau}(\|e^{k+1}\|^2-\|e^{k-1}\|^2)+\frac{1}{\alpha}\|f^{\bar{k}}\|^2\\
\leqslant&\frac{c_9}{\alpha}\|e^k\|\cdot\|f^{\bar{k}}\|+\|p^k\|\cdot\|e^{\bar{k}}\|+\frac{1}{\alpha}\|q^k\|\cdot\|f^{\bar{k}}\|\\
\leqslant&\frac{1}{2\alpha}\|f^{\bar{k}}\|^2+\frac{1}{2\alpha}c_9^2\|e^k\|^2+\frac{1}{2}\|p^k\|^2\\
&+\frac{1}{2}\|e^{\bar{k}}\|^2+\frac{1}{2\alpha}\|f^{\bar{k}}\|^2+\frac{1}{2\alpha}\|q^k\|^2,\quad 1\leqslant k\leqslant m,
\end{aligned}$$

即

$$\begin{aligned}
\frac{1}{4\tau}(\|e^{k+1}\|^2-\|e^{k-1}\|^2)\leqslant&\frac{1}{2\alpha}c_9^2\|e^k\|^2+\frac{1}{2}\|e^{\bar{k}}\|^2+\frac{1}{2}\|p^k\|^2+\frac{1}{2\alpha}\|q^k\|^2\\
\leqslant&\frac{1}{2\alpha}c_9^2\|e^k\|^2+\frac{1}{4}(\|e^{k+1}\|^2+\|e^{k-1}\|^2)\\
&+\frac{1}{2}\|p^k\|^2+\frac{1}{2\alpha}\|q^k\|^2,\quad 1\leqslant k\leqslant m.
\end{aligned}\tag{9.86}$$

两边乘以 4τ, 得到

$$\begin{aligned}
(1-\tau)\|e^{k+1}\|^2\leqslant&(1+\tau)\|e^{k-1}\|^2+\frac{2c_9^2}{\alpha}\tau\|e^k\|^2\\
&+2\tau\left(\|p^k\|^2+\frac{1}{\alpha}\|q^k\|^2\right),\quad 1\leqslant k\leqslant m.
\end{aligned}$$

当 $\tau\leqslant\frac{1}{3}$ 时,

$$\begin{aligned}
\|e^{k+1}\|^2\leqslant&(1+3\tau)\|e^{k-1}\|^2+\frac{3c_9^2}{\alpha}\tau\|e^k\|^2\\
&+3\tau\left(\|p^k\|^2+\frac{1}{\alpha}\|q^k\|^2\right),\quad 1\leqslant k\leqslant m.
\end{aligned}$$

易知

$$\begin{aligned}
\max\{\|e^{k+1}\|^2,\|e^k\|^2\}\leqslant&\left[1+3\left(1+\frac{c_9^2}{\alpha}\right)\tau\right]\max\{\|e^k\|^2,\|e^{k-1}\|^2\}\\
&+3\tau\left(\|p^k\|^2+\frac{1}{\alpha}\|q^k\|^2\right),\quad 1\leqslant k\leqslant m.
\end{aligned}$$

由 (9.61) 和 (9.62) 得

$$\begin{aligned}
\max\{\|e^{k+1}\|^2,\|e^k\|^2\}\leqslant&\left[1+3\left(1+\frac{c_9^2}{\alpha}\right)\tau\right]\max\{\|e^k\|^2,\|e^{k-1}\|^2\}\\
&+3\left(1+\frac{1}{\alpha}\right)L_1L_2c_7^2(\tau^2+h_1^2+h_2^2)^2,\quad 1\leqslant k\leqslant m.
\end{aligned}$$

由 Gronwall 不等式以及 (9.81) 和 (9.82), 知存在常数 c_{11} 使得

$$\max\{\|e^{k+1}\|^2, \|e^k\|^2\} \leqslant c_{11}^2(\tau^2 + h_1^2 + h_2^2)^2, \quad 1 \leqslant k \leqslant m,$$

即

$$\|e^k\| \leqslant c_{11}(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant k \leqslant m+1. \tag{9.87}$$

(b) 用 $\Delta_t e^k$ 与 (9.79) 的两边作内积, 得

$$\|\Delta_t e^k\|^2 = (\Delta_h f^{\bar{k}}, \Delta_t e^k) + (p^k, \Delta_t e^k), \quad 1 \leqslant k \leqslant m.$$

用 $\Delta_h \Delta_t e^k$ 与 (9.80) 的两边作内积, 得

$$\begin{aligned}(f^{\bar{k}}, \Delta_h \Delta_t e^k) =& (\phi(U^k) - \phi(u^k), \Delta_h \Delta_t e^k) - \alpha(\Delta_h e^{\bar{k}}, \Delta_h \Delta_t e^k) \\ &+ (q^k, \Delta_h \Delta_t e^k), \quad 1 \leqslant k \leqslant m.\end{aligned}$$

将以上二式相加, 得

$$\begin{aligned}&\|\Delta_t e^k\|^2 + \alpha(\Delta_h e^{\bar{k}}, \Delta_h \Delta_t e^k) \\ =& (\phi(U^k) - \phi(u^k), \Delta_h \Delta_t e^k) + (q^k, \Delta_h \Delta_t e^k) + (p^k, \Delta_t e^k), \quad 1 \leqslant k \leqslant m.\end{aligned}$$

易知

$$\begin{aligned}&\|\Delta_t e^k\|^2 + \frac{\alpha}{4\tau}(\|\Delta_h e^{k+1}\|^2 - \|\Delta_h e^{k-1}\|^2) \\ \leqslant& (\phi(U^k) - \phi(u^k), \Delta_t \Delta_h e^k) + (q^k, \Delta_t \Delta_h e^k) + (p^k, \Delta_t e^k) \\ \leqslant& (\phi(U^k) - \phi(u^k), \Delta_t \Delta_h e^k) + (q^k, \Delta_t \Delta_h e^k) \\ &+ \frac{1}{2}\|\Delta_t e^k\|^2 + \frac{1}{2}\|p^k\|^2, \quad 1 \leqslant k \leqslant m.\end{aligned}$$

将上式中的 k 换为 l, 并对 l 从 1 到 k 求和, 得

$$\begin{aligned}&\frac{1}{2}\sum_{l=1}^{k}\|\Delta_t e^l\|^2 + \frac{\alpha}{4\tau}(\|\Delta_h e^{k+1}\|^2 + \|\Delta_h e^k\|^2 - \|\Delta_h e^1\|^2 - \|\Delta_h e^0\|^2) \\ \leqslant& \sum_{l=1}^{k}(\phi(U^l) - \phi(u^l), \Delta_t \Delta_h e^l) + \sum_{l=1}^{k}(q^l, \Delta_t \Delta_h e^l) + \frac{1}{2}\sum_{l=1}^{k}\|p^l\|^2, \\ &\qquad 1 \leqslant k \leqslant m.\end{aligned} \tag{9.88}$$

应用引理 9.4 的第一式, 得

$$\sum_{l=1}^{k}(\phi(U^l) - \phi(u^l), \Delta_t \Delta_h e^k)$$

$$
\begin{aligned}
&=\frac{1}{2\tau}\left[(\phi(U^{k-1})-\phi(u^{k-1}),\Delta_h e^k)+(\phi(U^k)-\phi(u^k),\Delta_h e^{k+1})\right]\\
&\quad-\sum_{l=1}^{k-1}(\Delta_t(\phi(U^l)-\phi(u^l)),\Delta_h e^l),\quad 1\leqslant k\leqslant m.
\end{aligned}
$$

注意到 (9.84)–(9.85), (9.81) 及引理 9.3 , 得

$$
\begin{aligned}
&\sum_{l=1}^{k}(\phi(U^l)-\phi(u^l),\Delta_t\Delta_h e^l)\\
\leqslant&\frac{1}{2\tau}(\|\phi(U^{k-1})-\phi(u^{k-1})\|\cdot\|\Delta_h e^k\|+\|\phi(U^k)-\phi(u^k)\|\cdot\|\Delta_h e^{k+1}\|)\\
&+\sum_{l=1}^{k-1}\left|(\Delta_t(\phi(U^l)-\phi(u^l)),\Delta_h e^l)\right|\\
\leqslant&\frac{1}{2\tau}\big(c_9\|e^{k-1}\|\cdot\|\Delta_h e^k\|+c_9\|e^k\|\cdot\|\Delta_h e^{k+1}\|\big)\\
&+\sum_{l=1}^{k-1}c_3\big(c_9\|\Delta_t e^l\|+c_{10}(\|e^{l+1}\|+\|e^{l-1}\|)\big)\|\Delta_h e^l\|,\quad 1\leqslant k\leqslant m.
\end{aligned}\tag{9.89}
$$

应用引理 9.4 的第二式, 有

$$
\begin{aligned}
&\sum_{l=1}^{k}(q^l,\Delta_t\Delta_h e^l)\\
=&\frac{1}{2\tau}\left[(q^k,\Delta_h e^{k+1})+(q^{k-1},\Delta_h e^k)-(q^2,\Delta_h e^1)\right]-\sum_{l=2}^{k-1}(\Delta_t q^l,\Delta_h e^l)\\
\leqslant&\frac{1}{2\tau}(\|q^k\|\cdot\|\Delta_h e^{k+1}\|+\|q^{k-1}\|\cdot\|\Delta_h e^k\|+\|q^2\|\cdot\|\Delta_h e^1\|)\\
&+\sum_{l=2}^{k-1}\|\Delta_t q^l\|\cdot\|\Delta_h e^l\|,\quad 1\leqslant k\leqslant m.
\end{aligned}\tag{9.90}
$$

将 (9.89) 和 (9.90) 代入 (9.88), 得到

$$
\begin{aligned}
&\frac{1}{2}\sum_{l=1}^{k}\|\Delta_t e^l\|^2+\frac{\alpha}{4\tau}\big(\|\Delta_h e^{k+1}\|^2+\|\Delta_h e^k\|^2-\|\Delta_h e^1\|^2\big)\\
\leqslant&\frac{1}{2\tau}\big(c_9\|e^{k-1}\|\cdot\|\Delta_h e^k\|+c_9\|e^k\|\cdot\|\Delta_h e^{k+1}\|\big)\\
&+\sum_{l=1}^{k-1}c_3\big(c_9\|\Delta_t e^l\|+c_{10}(\|e^{l+1}\|+\|e^{l-1}\|)\big)\|\Delta_h e^l\|\\
&+\frac{1}{2\tau}\big(\|q^k\|\cdot\|\Delta_h e^{k+1}\|+\|q^{k-1}\|\cdot\|\Delta_h e^k\|+\|q^2\|\cdot\|\Delta_h e^l\|\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=2}^{k-1}\|\Delta_t q^l\|\cdot\|\Delta_h e^l\|+\frac{1}{2}\sum_{l=1}^{k}\|p^l\|^2\\
\leqslant&\frac{1}{2\tau}\left(\frac{\alpha}{8}\|\Delta_h e^k\|^2+\frac{2c_9^2}{\alpha}\|e^{k-1}\|^2+\frac{\alpha}{8}\|\Delta_h e^{k+1}\|^2+\frac{2c_9^2}{\alpha}\|e^k\|^2\right)\\
&+\sum_{l=1}^{k-1}\left(\frac{1}{2}\|\Delta_t e^l\|^2+\frac{c_3^2c_9^2}{2}\|\Delta_h e^l\|^2+\frac{1}{2}\|e^{l+1}\|^2\right.\\
&\left.+\frac{c_3^2c_{10}^2}{2}\|\Delta_h e^l\|^2+\frac{1}{2}\|e^{l-1}\|^2+\frac{c_3^2c_{10}^2}{2}\|\Delta_h e^l\|^2\right)\\
&+\frac{1}{2\tau}\left(\frac{\alpha}{8}\|\Delta_h e^{k+1}\|^2+\frac{2}{\alpha}\|q^k\|^2+\frac{\alpha}{8}\|\Delta_h e^k\|^2\right.\\
&\left.+\frac{2}{\alpha}\|q^{k-1}\|^2+\frac{\alpha}{2}\|\Delta_h e^1\|^2+\frac{1}{2\alpha}\|q^2\|^2\right)+\frac{1}{2}\sum_{l=2}^{k-1}(\|\Delta_t q^l\|^2+\|\Delta_h e^l\|^2)\\
&+\frac{1}{2}\sum_{l=1}^{k}\|p^l\|^2,\quad 1\leqslant k\leqslant m.
\end{aligned}
$$

由上式可得

$$
\begin{aligned}
&\frac{\alpha}{8\tau}(\|\Delta_h e^{k+1}\|^2+\|\Delta_h e^k\|^2)\\
\leqslant&\frac{\alpha}{4\tau}\|\Delta_h e^1\|^2+\frac{1}{2\tau}\left(\frac{2c_9^2}{\alpha}\|e^{k-1}\|^2+\frac{2c_9^2}{\alpha}\|e^k\|^2+\frac{2}{\alpha}\|q^k\|^2+\frac{2}{\alpha}\|q^{k-1}\|^2\right.\\
&\left.+\frac{\alpha}{2}\|\Delta_h e^1\|^2+\frac{1}{2\alpha}\|q^2\|^2\right)+\sum_{l=1}^{k-1}\left[c_3^2\left(\frac{1}{2}c_9^2+c_{10}^2\right)\|\Delta_h e^l\|^2+\frac{1}{2}\|e^{l+1}\|^2+\frac{1}{2}\|e^{l-1}\|^2\right]\\
&+\frac{1}{2}\sum_{l=2}^{k-1}(\|\Delta_t q^l\|^2+\|\Delta_h e^l\|^2)+\frac{1}{2}\sum_{l=1}^{k}\|p^l\|^2,\quad 1\leqslant k\leqslant m,
\end{aligned}
$$

将上式乘以 $\dfrac{8\tau}{\alpha}$, 得到

$$
\begin{aligned}
&\|\Delta_h e^{k+1}\|^2+\|\Delta_h e^k\|^2\\
\leqslant&\frac{8c_9^2}{\alpha^2}(\|e^{k-1}\|^2+\|e^k\|^2)\\
&+\frac{8\tau}{\alpha}\left[c_3^2\left(\frac{1}{2}c_9^2+c_{10}^2\right)\|\Delta_h e^1\|^2+\sum_{l=2}^{k-1}\left(\frac{1}{2}c_3^2c_9^2+c_3^2c_{10}^2+\frac{1}{2}\right)\|\Delta_h e^l\|^2\right]\\
&+\frac{4\tau}{\alpha}\left[\|e^0\|^2+\|e^1\|^2+\|e^{k-1}\|^2+\|e^k\|^2+2\sum_{l=2}^{k-2}\|e^l\|^2\right]\\
&+2\|\Delta_h e^1\|^2+\frac{4}{\alpha}\left(\frac{2}{\alpha}\|q^k\|^2+\frac{2}{\alpha}\|q^{k-1}\|^2+\frac{\alpha}{2}\|\Delta_h e^1\|^2+\frac{1}{2\alpha}\|q^2\|^2\right)
\end{aligned}
$$

$$+\frac{4\tau}{\alpha}\sum_{l=2}^{k-1}\|\Delta_t q^l\|^2+\frac{4\tau}{\alpha}\sum_{l=1}^{k}\|p^l\|^2,\quad 1\leqslant k\leqslant m.$$

由 (9.87), (9.61)–(9.63), (9.81), (9.83) 知存在常数 c_{12}, c_{13} 使得

$$\begin{aligned}&\|\Delta_h e^{k+1}\|^2+\|\Delta_h e^k\|^2\\ \leqslant& c_{12}\tau\sum_{l=2}^{k}(\|\Delta_h e^l\|^2+\|\Delta_h e^{l-1}\|^2)+c_{13}(\tau^2+h_1^2+h_2^2)^2,\quad 1\leqslant k\leqslant m.\end{aligned}$$

由 Gronwall 不等式, 得

$$\|\Delta_h e^{k+1}\|^2+\|\Delta_h e^k\|^2\leqslant e^{c_{12}k\tau}\cdot c_{13}(\tau^2+h_1^2+h_2^2)^2,\quad 1\leqslant k\leqslant m.$$

因而

$$\|\Delta_h e^k\|\leqslant e^{\frac{1}{2}c_{12}T}\sqrt{c_{13}}(\tau^2+h_1^2+h_2^2),\quad 1\leqslant k\leqslant m+1. \tag{9.91}$$

由引理 9.2, (9.87), (9.91), 得

$$\begin{aligned}\|e^{m+1}\|_\infty\leqslant&\|\Delta_h e^{m+1}\|+\sqrt{3}\left[1+\frac{1}{2}\left(\frac{1}{L_1}+\frac{1}{L_2}\right]\|e^{m+1}\|\right.\\ \leqslant&\left\{e^{\frac{1}{2}c_{12}T}\sqrt{c_{13}}+\sqrt{3}\left[1+\frac{1}{2}\left(\frac{1}{L_1}+\frac{1}{L_2}\right]c_{11}\right\}\right.(\tau^2+h_1^2+h_2^2)\\ \equiv& c_8(\tau^2+h_1^2+h_2^2),\end{aligned}$$

即 (9.76) 对 $k=m+1$ 成立. □

9.4 三层线性化紧致差分格式

对于 $u\in\mathcal{V}_h$ 引进如下记号

$$\mathcal{A}_1u_{ij}=\begin{cases}\dfrac{5}{6}u_{0j}+\dfrac{1}{6}u_{1j}, & i=0,\\ \dfrac{1}{12}(u_{i-1,j}+10u_{ij}+u_{i+1,j}), & 1\leqslant i\leqslant m_1-1,\\ \dfrac{1}{6}u_{m_1-1,j}+\dfrac{5}{6}u_{m_1,j}, & i=m_1,\end{cases}$$

$$\mathcal{A}_2u_{ij}=\begin{cases}\dfrac{5}{6}u_{i0}+\dfrac{1}{6}u_{i1}, & j=0,\\ \dfrac{1}{12}(u_{i,j-1}+10u_{ij}+u_{i,j+1}), & 1\leqslant j\leqslant m_2-1,\\ \dfrac{1}{6}u_{i,m_2-1}+\dfrac{5}{6}u_{i,m_2}, & j=m_2.\end{cases}$$

记

$$\mathcal{A}_h=\mathcal{A}_1\mathcal{A}_2,\quad \Lambda_h=\mathcal{A}_2\delta_x^2+\mathcal{A}_1\delta_y^2,$$

引理 9.5　对任意的 $u \in \mathcal{V}_h$, 有

$$\frac{5}{12}\|u\|^2 \leqslant \|\mathcal{A}_1 u\|^2 \leqslant \|u\|^2,$$

$$\frac{5}{12}\|u\|^2 \leqslant \|\mathcal{A}_2 u\|^2 \leqslant \|u\|^2,$$

$$\frac{25}{144}\|u\|^2 \leqslant \|\mathcal{A}_h u\|^2 \leqslant \|u\|^2.$$

引理 9.6　对任意的 $u \in \mathcal{V}_h$, 有

$$\|\Delta_h u\|^2 \leqslant \frac{3}{7}(3+\sqrt{2})\|\Lambda_h u\|^2.$$

证明　由 (9.14) 知

$$\|\delta_x^2 u\|^2 + \|\delta_y^2 u\|^2 \leqslant \|\Delta_h u\|^2. \tag{9.92}$$

由

$$\Delta_h u_{ij} = \Lambda_h u_{ij} - \frac{1}{12}h_2^2\delta_y^2\delta_x^2 u_{ij} - \frac{1}{12}h_1^2\delta_x^2\delta_y^2 u_{ij},$$

得

$$\begin{aligned}
\|\Delta_h u\| &\leqslant \|\Lambda_h u\| + \frac{1}{12}h_2^2\|\delta_y^2\delta_x^2 u\| + \frac{1}{12}h_1^2\|\delta_x^2\delta_y^2 u\| \\
&\leqslant \|\Lambda_h u\| + \frac{1}{3}\|\delta_x^2 u\| + \frac{1}{3}\|\delta_y^2 u\| \\
&\leqslant \|\Lambda_h u\| + \frac{1}{3}(\|\delta_x^2 u\| + \|\delta_y^2 u\|) \\
&\leqslant \|\Lambda_h u\| + \frac{1}{3}\sqrt{2(\|\delta_x^2 u\|^2 + \|\delta_y^2 u\|^2)} \\
&\leqslant \|\Lambda_h u\| + \frac{\sqrt{2}}{3}\|\Delta_h u\|.
\end{aligned}$$

易得

$$\|\Delta_h u\| \leqslant \frac{1}{1-\dfrac{\sqrt{2}}{3}}\|\Lambda_h u\| \leqslant \frac{3(3+\sqrt{2})}{7}\|\Lambda_h u\|.$$

□

9.4.1　差分格式的建立

对方程 (9.17) 关于 x 求三阶导数, 并利用 (9.21), (9.23), 得

$$v_{xxxxx}|_{x=0} = 0, \quad v_{xxxxx}|_{x=L_1} = 0. \tag{9.93}$$

对方程 (9.18) 关于 y 求三阶导数, 并利用 (9.22), (9.24), 得

$$v_{yyyyy}|_{y=0} = 0, \quad v_{yyyyy}|_{y=L_2} = 0. \tag{9.94}$$

对方程 (9.18) 关于 x 求三阶导数, 并利用 (9.19),(9.21),(9.23), 得

$$u_{xxxxx}|_{x=0}=0, \quad u_{xxxxx}|_{x=L_1}=0. \tag{9.95}$$

对方程 (9.18) 关于 y 求三阶导数, 并利用 (9.19), (9.22),(9.24), 得

$$u_{yyyyy}|_{y=0}=0, \quad u_{yyyyy}|_{y=L_2}=0. \tag{9.96}$$

在点 $(x_i,y_j,t_{\frac{1}{2}})$ 处考虑方程 (9.17)–(9.18) 可得

$$u_t(x_i,y_i,t_{\frac{1}{2}})=v_{xx}(x_i,y_i,t_{\frac{1}{2}})+v_{yy}(x_i,y_i,t_{\frac{1}{2}}), \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,$$

$$v(x_i,y_i,t_{\frac{1}{2}})=\phi\big(u(x_i,y_i,t_{\frac{1}{2}})\big)-\alpha\Big[u_{xx}(x_i,y_i,t_{\frac{1}{2}})+u_{yy}(x_i,y_i,t_{\frac{1}{2}})\Big],$$
$$0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2.$$

用 $\mathcal{A}_1\mathcal{A}_2$ 作用以上两式, 并应用引理 1.4 以及 (9.19), (9.21)–(9.24), (9.93)–(9.96), 可得

$$\mathcal{A}_1\mathcal{A}_2\delta_t U_{ij}^{\frac{1}{2}}=\mathcal{A}_2\delta_x^2V_{ij}^{\frac{1}{2}}+\mathcal{A}_1\delta_y^2V_{ij}^{\frac{1}{2}}+\hat{p}_{ij}^0, \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2, \tag{9.97}$$

$$\mathcal{A}_1\mathcal{A}_2V_{ij}^{\frac{1}{2}}=\mathcal{A}_1\mathcal{A}_2\phi(\hat{u}_{ij})-\alpha(\mathcal{A}_2\delta_x^2U_{ij}^{\frac{1}{2}}+\mathcal{A}_1\delta_y^2U_{ij}^{\frac{1}{2}})+\hat{q}_{ij}^0,$$
$$0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2, \tag{9.98}$$

其中

$$\hat{u}_{ij}=u(x_i,y_j,0)+\frac{\tau}{2}u_t(x_i,y_j,0), \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2, \tag{9.99}$$

存在常数 c_{14} 使得

$$|\hat{p}_{ij}^0|\leqslant c_{14}(\tau^2+h_1^4+h_2^4), \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2, \tag{9.100}$$

$$|\hat{q}_{ij}^0|\leqslant c_{14}(\tau^2+h_1^4+h_2^4), \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2. \tag{9.101}$$

在点 (x_i,y_i,t_k) 处考虑方程 (9.17), (9.18) 可得

$$u_t(x_i,y_j,t_k)=v_{xx}(x_i,y_j,t_k)+v_{yy}(x_i,y_j,t_k),$$
$$0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n-1,$$

$$v(x_i,y_j,t_k)=\phi(u(x_i,y_j,t_k))-\alpha[u_{xx}(x_i,y_j,t_k)+u_{yy}(x_i,y_j,t_k)],$$
$$0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n-1.$$

用 $\mathcal{A}_1\mathcal{A}_2$ 作用以上两式, 并再次应用引理 1.4 以及 (9.19), (9.21)–(9.24), (9.93)–(9.96), 可得

$$\mathcal{A}_1\mathcal{A}_2\Delta_tU_{ij}^k=\mathcal{A}_2\delta_x^2V_{ij}^k+\mathcal{A}_1\delta_y^2V_{ij}^k+\hat{p}_{ij}^k,$$

$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.102}$$

$$\mathcal{A}_1\mathcal{A}_2 V_{ij}^k = \mathcal{A}_1\mathcal{A}_2\phi(U_{ij}^k) - \alpha(\mathcal{A}_2\delta_x^2 U_{ij}^{\bar{k}} + \mathcal{A}_1\delta_y^2 U_{ij}^{\bar{k}}) + \hat{q}_{ij}^k,$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.103}$$

存在常数 c_{15} 使得

$$|\hat{p}_{ij}^k| \leqslant c_{15}(\tau^2 + h_1^4 + h_2^4),\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.104}$$

$$|\hat{q}_{ij}^k| \leqslant c_{15}(\tau^2 + h_1^4 + h_2^4),\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.105}$$

$$|\Delta_t\hat{q}_{ij}^k| \leqslant c_{15}(\tau^2 + h_1^4 + h_2^4),\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 2 \leqslant k \leqslant n-2. \tag{9.106}$$

在 (9.97)–(9.98),(9.102)–(9.103) 中略去小量项, 并注意到初值条件

$$U_{ij}^0 = \varphi(x_i, y_i),\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.107}$$

对 (9.17)–(9.20) 建立如下差分格式

$$\mathcal{A}_1\mathcal{A}_2\delta_t u_{ij}^{\frac{1}{2}} = \mathcal{A}_2\delta_x^2 v_{ij}^{\frac{1}{2}} + \mathcal{A}_1\delta_y^2 v_{ij}^{\frac{1}{2}},\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.108}$$

$$\mathcal{A}_1\mathcal{A}_2 v_{ij}^{\frac{1}{2}} = \mathcal{A}_1\mathcal{A}_2\phi(\hat{u}_{ij}) - \alpha\left(\mathcal{A}_2\delta_x^2 u_{ij}^{\frac{1}{2}} + \mathcal{A}_1\delta_y^2 u_{ij}^{\frac{1}{2}}\right),$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.109}$$

$$\mathcal{A}_1\mathcal{A}_2\Delta_t u_{ij}^k = \mathcal{A}_2\delta_x^2 v_{ij}^k + \mathcal{A}_1\delta_y^2 v_{ij}^k,$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.110}$$

$$\mathcal{A}_1\mathcal{A}_2 v_{ij}^k = \mathcal{A}_1\mathcal{A}_2\phi(u_{ij}^k) - \alpha\left(\mathcal{A}_2\delta_x^2 u_{ij}^{\bar{k}} + \mathcal{A}_1\delta_y^2 u_{ij}^{\bar{k}}\right),$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.111}$$

$$u_{ij}^0 = \varphi(x_i, y_j),\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2. \tag{9.112}$$

由 (9.108)–(9.111) 可得

$$\mathcal{A}_h\delta_t u_{ij}^{\frac{1}{2}} = \Lambda_h v_{ij}^{\frac{1}{2}},\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.113}$$

$$\mathcal{A}_h v_{ij}^{\frac{1}{2}} = \mathcal{A}_h\phi(\hat{u}_{ij}) - \alpha\Lambda_h u_{ij}^{\frac{1}{2}},\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.114}$$

$$\mathcal{A}_h\Delta_t u_{ij}^k = \Lambda_h v_{ij}^k,\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.115}$$

$$\mathcal{A}_h v_{ij}^k = \mathcal{A}_h\phi(u_{ij}^k) - \alpha\Lambda_h u_{ij}^{\bar{k}},$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1. \tag{9.116}$$

用 $\mathcal{A}_h$ 作用 (9.113), 并将 (9.114) 代入所得等式, 得到

$$\mathcal{A}_h^2\delta_t u_{ij}^{\frac{1}{2}} = \Lambda_h\big(\mathcal{A}_h\phi(\hat{u}_{ij}) - \alpha\Lambda_h u_{ij}^{\frac{1}{2}}\big),\quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,$$

用 $\mathcal{A}_h$ 作用 (9.115), 并将 (9.116) 代入所得等式, 得到

$$\mathcal{A}_h^2\Delta_t u_{ij}^k = \Lambda_h\big(\mathcal{A}_h\phi(u_{ij}^k) - \alpha\Lambda_h u_{ij}^{\bar{k}}\big), \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n-1.$$

于是得到求解 (9.1)–(9.3) 的差分格式

$$\mathcal{A}_h^2\delta_t u_{ij}^{\frac{1}{2}} = \Lambda_h\big(\mathcal{A}_h\phi(\hat{u}_{ij}) - \alpha\Lambda_h u_{ij}^{\frac{1}{2}}\big), \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2, \tag{9.117}$$

$$\mathcal{A}_h^2\Delta_t u_{ij}^k = \Lambda_h\big(\mathcal{A}_h\phi(u_{ij}^k) - \alpha\Lambda_h u_{ij}^{\bar{k}}\big),$$
$$0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n-1, \tag{9.118}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2. \tag{9.119}$$

9.4.2 差分格式解的存在性和唯一性

定理 9.9 差分格式 (9.117)–(9.119) 是唯一可解的.

证明 记 $u^k = \{u_{ij}^k \,|\, 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2\}$.

由 (9.119) 知 u^0 唯一给定.

由 (9.117) 可得关于 u^1 的线性方程组. 考虑其齐次方程组

$$\mathcal{A}_h^2\left(\frac{1}{\tau}u_{ij}^1\right) = \Lambda_h\left(-\alpha\Lambda_h\left(\frac{1}{2}u_{ij}^1\right)\right), \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2.$$

用 u^1 与上式两边作内积, 得

$$\frac{1}{\tau}(\mathcal{A}_h^2u^1, u^1) + \frac{1}{2}\alpha(\Lambda_h^2u^1, u^1) = 0,$$

即

$$\frac{1}{\tau}(\mathcal{A}_hu^1, \mathcal{A}_hu^1) + \frac{\alpha}{2}(\Lambda_hu^1, \Lambda_hu^1) = 0,$$

或

$$\frac{1}{\tau}\|\mathcal{A}_hu^1\|^2 + \frac{\alpha}{2}\|\Lambda_hu^1\|^2 = 0.$$

易知

$$\|u^1\| = 0,$$

因而差分格式 (9.117) 关于 u^1 唯一可解.

设 u^{k-1}, u^k 已唯一确定. 则由 (9.118) 可得关于 u^{k+1} 的线性方程组. 考虑其齐次方程组

$$\mathcal{A}_h^2\left(\frac{1}{2\tau}u_{ij}^{k+1}\right) = \Lambda_h\left(-\alpha\Lambda_h\left(\frac{1}{2}u_{ij}^{k+1}\right)\right), \quad 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2.$$

用 u^{k+1} 与其两边作内积, 得

$$\frac{1}{2\tau}\|\mathcal{A}_hu^{k+1}\|^2 + \frac{\alpha}{2}\|\Lambda_hu^{k+1}\|^2 = 0.$$

易得

$$\|u^{k+1}\| = 0.$$

因而差分格式 (9.115) 唯一确定 u^{k+1}. □

9.4.3 差分格式解的收敛性

定理 9.10 设 $\{U_{ij}^k \mid 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n\}$ 是 (9.1)–(9.3) 的解, $\{u_{ij}^k \mid 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n\}$ 为 (9.117)–(9.119) 的解. 记

$$e_{ij}^k = U_{ij}^k - u_{ij}^k, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n.$$

则存在常数 c_{16} 使得当 $\tau^2 + h_1^4 + h_2^4 \leqslant \dfrac{1}{c_{16}}$ 时, 有

$$\|e^k\|_\infty \leqslant c_{16}(\tau^2 + h_1^4 + h_2^4), \quad 0 \leqslant k \leqslant n, \tag{9.120}$$

证明 记

$$f_{ij}^k = V_{ij}^k - v_{ij}^k, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n,$$

将 (9.97), (9.98), (9.102), (9.103), (9.107) 与 (9.108)–(9.112) 依次相减, 得误差方程组

$$\mathcal{A}_h \delta_t e_{ij}^{\frac{1}{2}} = \Lambda_h f_{ij}^{\frac{1}{2}} + \hat{p}_{ij}^0, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.121}$$

$$\mathcal{A}_h f_{ij}^{\frac{1}{2}} = -\alpha \Lambda_h e_{ij}^{\frac{1}{2}} + \hat{q}_{ij}^0, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.122}$$

$$\mathcal{A}_h \Delta_t e_{ij}^k = \Lambda_h f_{ij}^k + \hat{p}_{ij}^k, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.123}$$

$$\mathcal{A}_h f_{ij}^k = \mathcal{A}_h(\phi(U_{ij}^k) - \phi(u_{ij}^k)) - \alpha \Lambda_h e_{ij}^{\bar{k}} + \hat{q}_{ij}^k,$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.124}$$

$$e_{ij}^0 = 0, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2. \tag{9.125}$$

由 (9.125) 知 $\|e^0\|_\infty = 0$. 因而 (9.120) 对 $k = 0$ 成立.

(I) 估计 $|\mathcal{A}_h e^1|$ 和 $\|\Delta_h e^1\|$.

(a) 用 $\mathcal{A}_h e^{\frac{1}{2}}$ 与 (9.121) 的两边作内积, 得

$$\left(\mathcal{A}_h \delta_t e^{\frac{1}{2}}, \mathcal{A}_h e^{\frac{1}{2}}\right) = \left(\mathcal{A}_h e^{\frac{1}{2}}, \Lambda_h f^{\frac{1}{2}}\right) + \left(\mathcal{A}_h e^{\frac{1}{2}}, \hat{p}^0\right).$$

用 $\dfrac{1}{\alpha}\mathcal{A}_h f^{\frac{1}{2}}$ 与 (9.122) 的两边作内积, 得

$$\frac{1}{\alpha}\|\mathcal{A}_h f^{\frac{1}{2}}\|^2 = -\left(\mathcal{A}_h f^{\frac{1}{2}}, \Lambda_h e^{\frac{1}{2}}\right) + \frac{1}{\alpha}\left(\mathcal{A}_h f^{\frac{1}{2}}, \hat{q}^0\right).$$

将以上两式相加, 得

$$\begin{aligned}
&\frac{1}{2\tau}(\|\mathcal{A}_h e^1\|^2-\|\mathcal{A}_h e^0\|^2)+\frac{1}{\alpha}\|\mathcal{A}_h f^{\frac12}\|^2\\
=&\left(\mathcal{A}_h e^{\frac12},\hat{p}^0\right)+\frac{1}{\alpha}(\mathcal{A}_h f^{\frac12},\hat{q}^0)\\
=&\frac{1}{2}(\mathcal{A}_h e^1,\hat{p}^0)+\frac{1}{\alpha}\left(\mathcal{A}_h f^{\frac12},\hat{q}^0\right)\\
\leqslant&\frac{1}{2}\left(\frac{1}{2\tau}\|\mathcal{A}_h e^1\|^2+\frac{\tau}{2}\|\hat{p}^0\|^2\right)+\frac{1}{\alpha}\left(\left\|\mathcal{A}_h f^{\frac12}\right\|^2+\frac{1}{4}\|\hat{q}^0\|^2\right),
\end{aligned}$$

即

$$\|\mathcal{A}_h e^1\|^2\leqslant\tau^2\|\hat{p}^0\|^2+\frac{\tau}{\alpha}\|\hat{q}^0\|^2.$$

由 (9.100)–(9.101), 得

$$\|\mathcal{A}_h e^1\|^2\leqslant\left(\tau^2+\frac{\tau}{\alpha}\right)c_{14}^2L_1L_2(\tau^2+h_1^4+h_2^4)^2. \tag{9.126}$$

(b) 用 $\mathcal{A}_h\delta_t e^{\frac12}$ 与 (9.121) 的两边作内积, 得

$$\left\|\mathcal{A}_h\delta_t e^{\frac12}\right\|^2=\left(\Lambda_h f^{\frac12},\mathcal{A}_h\delta_t e^{\frac12}\right)+\left(\hat{p}^0,\mathcal{A}_h\delta_t e^{\frac12}\right).$$

用 $\Lambda_h\delta_t e^{\frac12}$ 与 (9.122) 作内积, 得

$$\left(\mathcal{A}_h f^{\frac12},\Lambda_h\delta_t e^{\frac12}\right)=-\alpha\left(\Lambda_h e^{\frac12},\Lambda_h\delta_t e^{\frac12}\right)+\left(\Lambda_h\delta_t e^{\frac12},\hat{q}^0\right).$$

将以上两式相加, 得

$$\left\|\mathcal{A}_h\delta_t e^{\frac12}\right\|^2+\alpha\left(\Lambda_h e^{\frac12},\wedge_t\delta_t e^{\frac12}\right)=\left(\hat{p}^0,\mathcal{A}_h\delta_t e^{\frac12}\right)+\left(\Lambda_h\delta_t e^{\frac12},\hat{q}^0\right),$$

即

$$\begin{aligned}
\frac{1}{\tau^2}\|\mathcal{A}_h e^1\|^2+\frac{\alpha}{2\tau}\|\Lambda_h e^1\|^2=&\frac{1}{\tau}(\hat{p}^0,\mathcal{A}_h e^1)+\frac{1}{\tau}(\Lambda_h e^1,\hat{q}^0)\\
\leqslant&\frac{1}{\tau^2}\|\mathcal{A}_h e^1\|^2+\frac{1}{4}\|\hat{p}^0\|^2+\frac{\alpha}{4\tau}\|\Lambda_h e^1\|^2+\frac{1}{\alpha\tau}\|\hat{q}^0\|^2,
\end{aligned}$$

因而

$$\|\Lambda_h e^1\|^2\leqslant\frac{4\tau}{\alpha}\left(\frac{1}{4}\|\hat{p}^0\|^2+\frac{1}{\alpha\tau}\|\hat{q}^0\|^2\right)=\frac{\tau}{\alpha}\|\hat{p}^0\|^2+\frac{4}{\alpha^2}\|\hat{q}^0\|^2.$$

由 (9.100)–(9.101), 得

$$\|\Lambda_h e^1\|^2\leqslant\left(\frac{\tau}{\alpha}+\frac{4}{\alpha^2}\right)c_{14}^2L_1L_2(\tau^2+h_1^4+h_2^4)^2. \tag{9.127}$$

(II) 假设 (9.120) 对 k 从 0 到 $m(1 \leqslant m \leqslant n-1)$ 成立. 则当 $\tau^2+h_1^4+h_2^4 \leqslant \dfrac{1}{c_{16}}$ 时

$$\|e^k\|_\infty \leqslant c_{16}(\tau^2+h_1^4+h_2^4) \leqslant 1, \quad 1 \leqslant k \leqslant m.$$

由此推得

$$\|u^k\|_\infty \leqslant \|U^k-(U^k-u^k)\|_\infty \leqslant \|U^k\|_\infty+\|e^k\|_\infty \leqslant c_0+1,\ 1 \leqslant k \leqslant m, \tag{9.128}$$

$$|\phi(U_{ij}^k)-\phi(u_{ij}^k)| \leqslant c_9|e_{ij}^k|, \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant m. \tag{9.129}$$

应用引理 9.3, 得

$$\begin{aligned}&|\Delta_t[\phi(U_{ij}^k)-\phi(u_{ij}^k)]| \leqslant c_3\Big[c_9|\Delta_t e_{ij}^k|+c_{10}(|e_{ij}^{k+1}|+|e_{ij}^{k-1}|)\Big],\\ &\qquad\qquad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant m-1.\end{aligned} \tag{9.130}$$

现在来证明 (9.120) 对 $k=m+1$ 也成立.

(a) 用 $\mathcal{A}_h e^{\bar{k}}$ 与 (9.123) 的两边作内积, 得

$$(\mathcal{A}_h e^{\bar{k}}, \mathcal{A}_h \Delta_t e^k) = (\mathcal{A}_h e^{\bar{k}}, \Lambda_h f^k)+(\mathcal{A}_h e^{\bar{k}}, \hat{p}^k), \quad 1 \leqslant k \leqslant m.$$

用 $\dfrac{1}{\alpha}\mathcal{A}_h f^k$ 与 (9.124) 的两边作内积, 得

$$\begin{aligned}\frac{1}{\alpha}\|\mathcal{A}_h f^k\|^2 =& \frac{1}{\alpha}(\mathcal{A}_h(\phi(U^k)-\phi(u^k)), \mathcal{A}_h f^k)\\ &-(\mathcal{A}_h f^k, \Lambda_h e^{\bar{k}})+\frac{1}{\alpha}(\mathcal{A}_h f^k, \hat{q}^k), \quad 1 \leqslant k \leqslant m.\end{aligned}$$

将以上两式相加, 得

$$\begin{aligned}&\frac{1}{4\tau}(\|\mathcal{A}_h e^{k+1}\|^2-\|\mathcal{A}_h e^{k-1}\|^2)+\frac{1}{\alpha}\|\mathcal{A}_h f^k\|^2\\ =&\frac{1}{\alpha}(\mathcal{A}_h(\phi(U^k)-\phi(u^k)), \mathcal{A}_h f^k)+(\mathcal{A}_h e^{\bar{k}}, \hat{p}^k)+\frac{1}{\alpha}(\mathcal{A}_h f^k, \hat{q}^k)\\ \leqslant&\frac{1}{2\alpha}\|\mathcal{A}_h(\phi(U^k)-\phi(u^k))\|^2+\frac{1}{2\alpha}\|\mathcal{A}_h f^k\|^2\\ &+\frac{1}{2}\|\mathcal{A}_h e^{\bar{k}}\|^2+\frac{1}{2}\|\hat{p}^k\|^2+\frac{1}{2\alpha}\|\mathcal{A}_h f^k\|^2+\frac{1}{2\alpha}\|\hat{q}^k\|^2, \quad 1 \leqslant k \leqslant m.\end{aligned} \tag{9.131}$$

由 (9.129) 以及引理 9.5 得

$$\|\mathcal{A}_h(\phi(U^k)-\phi(u^k))\|^2 \leqslant c_9^2\|e^k\|^2 \leqslant \frac{144}{25}c_9^2\|\mathcal{A}_h e^k\|^2, \quad 1 \leqslant k \leqslant m. \tag{9.132}$$

将 (9.132) 代入 (9.131), 得

$$\frac{1}{4\tau}(\|\mathcal{A}_h e^{k+1}\|^2-\|\mathcal{A}_h e^{k-1}\|^2)$$

$$\leqslant \frac{1}{2\alpha}\cdot\frac{144}{25}\|\mathcal{A}_h e^k\|^2 + \frac{1}{4}(\|\mathcal{A}_h e^{k+1}\|^2 + \|\mathcal{A}_h e^{k-1}\|^2) + \frac{1}{2}\|\hat{p}^k\|^2 + \frac{1}{2\alpha}\|\hat{q}^k\|^2, \quad 1 \leqslant k \leqslant m. \tag{9.133}$$

由 Gronwall 不等式, 并注意到 (9.126), (9.104), (9.105), 知存在常数 c_{17} 使得

$$\|\mathcal{A}_h e^l\| \leqslant c_{17}(\tau^2 + h_1^2 + h_2^2), \quad 1 \leqslant l \leqslant m+1. \tag{9.134}$$

(b) 用 $\mathcal{A}_h\Delta_t e^k$ 与 (9.123) 的两边作内积, 得

$$\|\mathcal{A}_h\Delta_t e^k\|^2 = (\mathcal{A}_h\Delta_t e^k, \Lambda_h f^k) + (\mathcal{A}_h\Delta_t e^k, \hat{p}^k), \quad 1 \leqslant k \leqslant m.$$

用 $\Lambda_h\Delta_t e^k$ 与 (9.124) 的两边作内积, 得

$$(\Lambda_h\Delta_t e^k, \mathcal{A}_h f^k) = \big(\mathcal{A}_h\big(\phi(U^k) - \phi(u^k)\big), \Lambda_h\Delta_t e^k\big) - \alpha(\Lambda_h e^{\bar{k}}, \Lambda_h\Delta_t e^k) + (\Lambda_h\Delta_t e^k, \hat{q}^k), \quad 1 \leqslant k \leqslant m.$$

将以上两式相加, 得

$$\|\mathcal{A}_h\Delta_t e^k\|^2 + \frac{\alpha}{4\tau}(\|\Lambda_h e^{k+1}\|^2 - \|\Lambda_h e^{k-1}\|^2) = \big(\mathcal{A}_h(\phi(U^k) - \phi(u^k)), \Lambda_h\Delta_t e^k\big) + (\mathcal{A}_h\Delta_t e^k, \hat{p}^k) + (\Lambda_h\Delta_t e^k, \hat{q}^k), \quad 1 \leqslant k \leqslant m.$$

由

$$|(\mathcal{A}_h\Delta_t e^k, \hat{p}^k)| \leqslant \frac{1}{2}\|\mathcal{A}_h\Delta_t e^k\|^2 + \frac{1}{2}\|\hat{p}^k\|^2,$$

得

$$\frac{1}{2}\|\mathcal{A}_h\Delta_t e^k\|^2 + \frac{\alpha}{4\tau}(\|\Lambda_h e^{k+1}\|^2 - \|\Lambda_h e^{k-1}\|^2) \leqslant \big(\mathcal{A}_h(\phi(U^k) - \phi(u^k)), \Lambda_h\Delta_t e^k\big) + \frac{1}{2}\|\hat{p}^k\|^2 + (\Lambda_h\Delta_t e^k, \hat{q}^k), \quad 1 \leqslant k \leqslant m.$$

将上式中的 k 换为 l, 并对 l 从 1 到 k 求和, 得

$$\frac{1}{2}\sum_{l=1}^{k}\|\mathcal{A}_h\Delta_t e^l\|^2 + \frac{\alpha}{4\tau}(\|\Lambda_h e^{k+1}\|^2 + \|\Lambda_h e^k\|^2 - \|\Lambda_h e^1\|^2 - \|\Lambda_h e^0\|^2) \leqslant \sum_{l=1}^{k}\big(\mathcal{A}_h(\phi(U^l) - \phi(u^l)), \Lambda_h\Delta_t e^l\big) + \frac{1}{2}\sum_{l=1}^{k}\|\hat{p}^l\|^2 + \sum_{l=1}^{k}(\Lambda_h\Delta_t e^l, \hat{q}^l), \quad 1 \leqslant k \leqslant m. \tag{9.135}$$

由引理 9.4 的第一式得

$$\sum_{l=1}^{k}(\mathcal{A}_h(\phi(U^l) - \phi(u^l)), \Lambda_h\Delta_t e^l)$$

$$
\begin{aligned}
&= \frac{1}{2\tau}\left[(\mathcal{A}_h(\phi(U^k)-\phi(u^k)),\Lambda_h e^{k+1}) + (\mathcal{A}_h(\phi(U^{k-1})-\phi(u^{k-1})),\Lambda_h e^k)\right.\\
&\quad \left. -(\mathcal{A}_h(\phi(U^0)-\phi(u^0)),\Lambda_h e^1) - (\mathcal{A}_h(\phi(U^1)-\phi(u^1)),\Lambda_h e^0)\right]\\
&\quad -\sum_{l=1}^{k-1}(\Delta_t\mathcal{A}_h(\phi(U^l)-\phi(u^l)),\Lambda_h e^l), \quad 1\leqslant k\leqslant m.
\end{aligned}
$$

应用 Cauchy-Schwarz 不等式, (9.129), (9.130) 以及 (9.125), 得

$$
\begin{aligned}
&\sum_{l=1}^{k}\Big(\mathcal{A}_h(\phi(U^k)-\phi(u^k)),\Lambda_h\Delta_t e^k\Big)\\
\leqslant& \frac{1}{2\tau}(\|\mathcal{A}_h(\phi(U^k)-\phi(u^k))\|\cdot\|\Lambda_h e^{k+1}\| + \|\mathcal{A}_h(\phi(U^{k-1})-\phi(u^{k-1}))\|\cdot\|\Lambda_h e^k\|)\\
&+\sum_{l=1}^{k-1}\|\Delta_t\mathcal{A}_h(\phi(U^l)-\phi(u^l))\|\cdot\|\Lambda_h e^l\|\\
\leqslant& \frac{1}{2\tau}\big(c_9\|\mathcal{A}_h e^k\|\cdot\|\Lambda_h e^{k+1}\| + c_9\|\mathcal{A}_h e^{k-1}\|\cdot\|\Lambda_h e^k\|\big)\\
&+\sum_{l=1}^{k-1}c_3\big(c_9\|\mathcal{A}_h\Delta_t e^l\| + c_{10}\|\mathcal{A}_h e^{l+1}\| + c_{10}\|\mathcal{A}_h e^{l-1}\|\big)\|\Lambda_h e^l\|, \ 1\leqslant k\leqslant m. \quad (9.136)
\end{aligned}
$$

对于 (9.135) 的最后一式, 应用引理 9.4 的第二式, 得

$$
\begin{aligned}
&\sum_{l=1}^{k}(\Lambda_h\Delta_t e^l,\hat{q}^l) = \sum_{l=1}^{k}(\hat{q}^l,\Delta_t\Lambda_h e^l)\\
=&\frac{1}{2\tau}\left[(\hat{q}^k,\Lambda_h e^{k+1}) + (\hat{q}^{k-1},\Lambda_h e^k) - (\hat{q}^2,\Lambda_h e^1) - (\hat{q}^1,\Lambda_h e^0)\right]\\
&-\sum_{l=2}^{k-1}(\Delta_t\hat{q}^l,\Lambda_h e^l)\\
\leqslant&\frac{1}{2\tau}(\|\hat{q}^k\|\cdot\|\Lambda_h e^{k+1}\| + \|\hat{q}^{k-1}\|\cdot\|\Lambda_h e^k\| + \|\hat{q}^2\|\cdot\|\Lambda_h e^l\|)\\
&+\sum_{l=2}^{k-1}\|\Delta_t\hat{q}^l\|\cdot\|\Lambda_h e^l\|. \qquad (9.137)
\end{aligned}
$$

将 (9.136) 和 (9.137) 代入 (9.135), 并注意到 (9.125), 得

$$
\begin{aligned}
&\frac{1}{2}\sum_{l=1}^{k}\|\mathcal{A}_h\Delta_t e^l\|^2 + \frac{\alpha}{4\tau}(\|\Lambda_h e^{k+1}\|^2 + \|\Lambda_h e^k\|^2 - \|\Lambda_h e^1\|^2)\\
\leqslant&\frac{1}{2\tau}(c_9\|\mathcal{A}_h e^k\|\cdot\|\Lambda_h e^{k+1}\| + c_9\|\mathcal{A}_h e^{k-1}\|\cdot\|\Lambda_h e^k\|)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{k-1} c_3\big(c_9\|\mathcal{A}_h\Delta_t e^l\|+c_{10}\|\mathcal{A}_h e^{k+1}\|+c_{10}\|\mathcal{A}_h e^{l-1}\|\big)\|\Lambda_h e^l\| \\
&+\frac{1}{2}\sum_{l=1}^{k}\|\hat{p}^l\|^2+\frac{1}{2\tau}(\|\hat{q}^k\|\cdot\|\Lambda_h e^{k+1}\|+\|\hat{q}^{k-1}\|\cdot\|\Lambda_h e^k\|+\|\hat{q}^2\|\cdot\|\Lambda_h e^1\|) \\
&+\sum_{l=2}^{k-1}\|\Delta_t\hat{q}^l\|\cdot\|\Lambda_h e^l\| \\
\leqslant&\frac{1}{2\tau}\left[\left(\frac{\alpha}{8}\|\Lambda_h e^{k+1}\|^2+\frac{2}{\alpha}c_9^2\|\mathcal{A}_h e^k\|^2\right)+\left(\frac{\alpha}{8}\|\Lambda_h e^k\|^2+\frac{2}{\alpha}c_9^2\|\mathcal{A}_h e^{k-1}\|^2\right)\right] \\
&+\sum_{l=1}^{k-1}\left[\left(\frac{1}{2}\|\mathcal{A}_h\Delta_t e^l\|^2+\frac{1}{2}c_3^2c_9^2\|\Lambda_h e^l\|^2\right)\right. \\
&\left.+\left(\frac{1}{2}\|\mathcal{A}_h e^{l+1}\|^2+\frac{1}{2}c_3^2c_{10}^2\|\Lambda_h e^l\|^2\right)+\left(\frac{1}{2}\|\mathcal{A}_h e^{l-1}\|^2+\frac{1}{2}c_3^2c_{10}^2\|\Lambda_h e^l\|^2\right)\right] \\
&+\frac{1}{2}\sum_{l=1}^{k}\|\hat{p}^l\|^2+\frac{1}{2\tau}\left[\left(\frac{\alpha}{8}\|\Lambda_h e^{k+1}\|^2+\frac{2}{\alpha}\|\hat{q}^k\|^2\right)\right. \\
&\left.+\left(\frac{\alpha}{8}\|\Lambda_h e^k\|^2+\frac{2}{\alpha}\|\hat{q}^{k-1}\|^2\right)+\left(\frac{\alpha}{2}\|\Lambda_h e^1\|^2+\frac{1}{2\alpha}\|\hat{q}^2\|^2\right)\right] \\
&+\frac{1}{2}\sum_{l=2}^{k-1}(\|\Delta_t\hat{q}^l\|^2+\|\Lambda_h e^l\|^2).
\end{aligned}
$$

由上式可得

$$
\begin{aligned}
&\|\Lambda_h e^{k+1}\|^2+\|\Lambda_h e^k\|^2 \\
\leqslant&\|\Lambda_h e^1\|^2+\frac{4}{\alpha}\left[\frac{2c_9^2}{\alpha}\|\mathcal{A}_h e^k\|^2+\frac{2c_9^2}{\alpha}\|\mathcal{A}_h e^{k-1}\|^2\right. \\
&\left.+\frac{2}{\alpha}\|\hat{q}^k\|^2+\frac{2}{\alpha}\|\hat{q}^{k-1}\|^2+\frac{\alpha}{2}\|\Lambda_h e^1\|^2+\frac{1}{2\alpha}\|\hat{q}^2\|^2\right] \\
&+\frac{8\tau}{\alpha}\sum_{l=1}^{k-1}\left[\frac{1}{2}\|\mathcal{A}_h e^{l+1}\|^2+\frac{1}{2}\|\mathcal{A}_h e^{l-1}\|^2+c_3^2\Big(\frac{1}{2}c_9^2+c_{10}^2\Big)\|\Lambda_h e^l\|^2\right] \\
&+\frac{8\tau}{\alpha}\cdot\frac{1}{2}\sum_{l=2}^{k-1}\|\Lambda_h e^l\|^2+\frac{8\tau}{\alpha}\cdot\frac{1}{2}\sum_{l=1}^{k}(\|\hat{p}^l\|^2+\|\Delta_t\hat{q}^l\|^2),\quad 1\leqslant k\leqslant m.
\end{aligned}
$$

由 (9.134), (9.104), (9.106) 以及 (9.127) 知存在常数 c_{18} 使得

$$
\begin{aligned}
\|\Lambda_h e^{k+1}\|^2+\|\Lambda_h e^k\|^2\leqslant&\frac{4}{\alpha}(c_3^2c_9^2+2c_3^2c_{10}^2+1)\tau\sum_{l=1}^{k-1}\|\Lambda_h e^l\|^2 \\
&+c_{18}(\tau^2+h_1^4+h_2^4)^2,\quad 1\leqslant k\leqslant m.
\end{aligned}
$$

再由 Gronwall 不等式, 知存在常数 c_{19} 使得

$$\|\Lambda_h e^k\| \leqslant c_{19}(\tau^2 + h_1^4 + h_2^4), \quad 1 \leqslant k \leqslant m+1. \tag{9.138}$$

由引理 9.2, 引理 9.6 (9.134), (9.138) 知存在常数 c_{20} 使得

$$\|e^k\|_\infty \leqslant c_{20}(\tau^2 + h_1^2 + h_2^4), \quad 1 \leqslant k \leqslant m+1.$$

细致分析可知以上出现的 c_{17}, c_{18}, c_{19}, c_{20} 均是与 τ, h_1, h_2 无关的常数. 因而 (9.120) 对 $k = m+1$ 成立. □

9.5 小结与延拓

本章讨论了 Cahn-Hilliard 方程初边值问题的差分方法. 首先介绍了问题 (9.1)–(9.3) 的解满足能量守恒律. 接着分别构造了二层非线性差分格式、空间二阶三层线性化差分格式和空间四阶三层线性化紧致差分格式. 对每一差分格式, 分析了差分格式解的存在性和唯一性, 证明了差分格式解在无穷模下的收敛性.

本章内容由文 [20] 和 [26] 工作的基础上发展而成.

对问题 (9.1)–(9.3) 的另一个二层非线性差分格式如下.

$$\delta_t u_{ij}^{k+\frac{1}{2}} = \Delta_h \left(\frac{\psi(u_{ij}^{k+1}) - \psi(u_{ij}^k)}{u_{ij}^{k+1} - u_{ij}^k} - \alpha \Delta_h u_{ij}^{k+\frac{1}{2}} \right),$$
$$0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n-1, \tag{9.139}$$

$$u_{ij}^0 = \varphi(x_j, y_j), \quad 0 \leqslant i \leqslant m_1,\ 0 \leqslant j \leqslant m_2, \tag{9.140}$$

其中当 $u_{ij}^{k+1} = u_{ij}^k$ 时, 约定 $\dfrac{\psi(u_{ij}^{k+1}) - \psi(u_{ij}^k)}{u_{ij}^{k+1} - u_{ij}^k} = \phi(u_{ij}^k)$.

可以证明差分格式 (9.139)–(9.140) 的解满足能量守恒性、可解性和收敛性.

应用定理 9.6 的证明方法可以证明 1.2 节的二层非线性差分格式 (1.15)–(1.17) 在无穷范数下的收敛性.

第10章　外延增长模型方程的差分方法

10.1　引　　言

本章考虑二维外延增长模型问题

$$u_t + \delta\Delta^2 u - \nabla\cdot(|\nabla u|^2\nabla u) + \Delta u = 0, \quad (x,y)\in\mathcal{R}^2,\ 0<t\leqslant T, \tag{10.1}$$

$$u(x,y,0) = \varphi(x,y), \quad (x,y)\in\mathcal{R}^2 \tag{10.2}$$

的差分方法, 其中 δ 是一个正常数, ∇ 为梯度算子, Δ 为 Laplace 算子, $u(x,y,t)$ 关于 (x,y) 在 R^2 上关于盒子 $\Omega=(0,L_1)\times(0,L_2)$ 是周期的.

定理 10.1　设 $u(x,y,t)$ 为 (10.1)–(10.2) 的解. 则有

$$\|u(\cdot,\cdot,t)\|^2 \leqslant \|u(\cdot,\cdot,0)\|^2 + \frac{1}{2}L_1L_2t, \quad t>0. \tag{10.3}$$

证明　用 u 与 (10.1) 的两边在 Ω 上作内积, 得

$$(u_t,u) + \delta(\Delta^2 u,u) - (\nabla\cdot(|\nabla u|^2\nabla u),u) + (\Delta u,u) = 0.$$

应用分部求积公式和周期边界条件, 得

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(\cdot,\cdot,t)\|^2\right) + \delta\|\Delta u\|^2 + \left(|\nabla u|^2\nabla u,\nabla u\right) - (\nabla u,\nabla u) = 0,$$

即

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(\cdot,\cdot,t)\|^2\right) + \delta\|\Delta u\|^2 + \|\nabla u\|_4^4 - \|\nabla u\|^2 = 0.$$

上式也可以写为

$$\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(\cdot,\cdot,t)\|^2\right) + \delta\|\Delta u\|^2 + \iint\limits_{\Omega}\left(|\nabla u|^2-\frac{1}{2}\right)^2\mathrm{d}x\mathrm{d}y = \frac{1}{4}\iint\limits_{\Omega}1\,\mathrm{d}x\mathrm{d}y.$$

因而

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(\cdot,\cdot,t)\|^2\right) \leqslant \frac{1}{4}L_1L_2, \quad t>0.$$

易知

$$\|u(\cdot,\cdot,t)\|^2 \leqslant \|u(\cdot,\cdot,0)\|^2 + \frac{1}{2}L_1L_2t, \quad t>0.$$ □

定理 10.2　设 $u(x,y,t)$ 为 (10.1)–(10.2) 的解. 记

$$E(t)=\frac{\delta}{2}\|\Delta u(\cdot,\cdot,t)\|^2+\frac{1}{4}\iint\limits_{\Omega}\left(|\nabla u(x,y,t)|^2-1\right)^2\mathrm{d}x\mathrm{d}y+\int_0^t\|u_s(\cdot,\cdot,s)\|^2\mathrm{d}s,$$

则有

$$E(t)=E(0),\quad t>0. \tag{10.4}$$

证明　用 u_t 与 (10.1) 的两边在 Ω 上作内积, 得

$$(u_t,u_t)+\delta(\Delta^2 u,u_t)-\left(\nabla\cdot(|\nabla u|^2\nabla u),u_t\right)+(\Delta u,u_t)=0.$$

应用分部求积公式和周期边界条件, 得

$$\|u_t\|^2+\frac{\delta}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta u\|^2\right)+\frac{1}{4}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega}|\nabla u(x,y,t)|^4\mathrm{d}x\mathrm{d}y-\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega}|\nabla u(x,y,t)|^2\mathrm{d}x\mathrm{d}y=0.$$

因而

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\delta}{2}\|\Delta u\|^2+\frac{1}{4}\iint\limits_{\Omega}\left(|\nabla u(x,y,t)|^2-1\right)^2\mathrm{d}x\mathrm{d}y+\int_0^t\|u_s(\cdot,\cdot,s)\|^2\mathrm{d}s\right]=0,\quad t>0.$$

即

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t}=0,\quad t>0.$$

易知 (10.4) 成立. □

由定理 10.1 和定理 10.2 以及嵌入定理知存在常数 c_0 使得

$$\|u(\cdot,\cdot,t)\|_\infty\leqslant c_0,\quad 0\leqslant t\leqslant T. \tag{10.5}$$

10.2　记号与基本引理

取正整数 m_1,m_2,n. 记

$$h_1=\frac{L_1}{m_1},\quad h_2=\frac{L_2}{m_2},\quad \tau=\frac{T}{n},\quad x_i=ih_1,\quad y_j=jh_2,\quad t_k=k\tau,$$

$$\Omega_{h_1,h_2}=\{(x_i,y_j)\mid 0\leqslant i\leqslant m_1,\ 0\leqslant j\leqslant m_2\},\quad \Omega_\tau=\{t_k\mid 0\leqslant k\leqslant n\}.$$

记

$$\mathcal{W}_h=\left\{u\mid u=\{u_{ij}\},u_{i+m_1,j}=u_{ij},u_{i,j+m_2}=u_{ij}\right\}.$$

设 $u \in \mathcal{W}_h$, 引进如下记号

$$\delta_x u_{i+\frac{1}{2},j} = \frac{1}{h_1}(u_{i+1,j} - u_{ij}), \quad \delta_y u_{i,j+\frac{1}{2}} = \frac{1}{h_2}(u_{i,j+1} - u_{ij}),$$
$$\Delta_x u_{ij} = \frac{1}{2h_1}(u_{i+1,j} - u_{i-1,j}), \quad \Delta_y u_{ij} = \frac{1}{2h_2}(u_{i,j+1} - u_{i,j-1}),$$
$$\delta_x^2 u_{ij} = \frac{1}{h_1}(\delta_x u_{i+\frac{1}{2},j} - \delta_x u_{i-\frac{1}{2},j}), \quad \delta_y^2 u_{ij} = \frac{1}{h_2}(\delta_y u_{i,j+\frac{1}{2}} - \delta_y u_{i,j-\frac{1}{2}}),$$
$$\nabla_h u_{ij} = (\Delta_x u_{ij}, \Delta_y u_{ij})^{\mathrm{T}}, \quad \Delta_h u_{ij} = \delta_x^2 u_{ij} + \delta_y^2 u_{ij}, \quad \bar{\Delta}_h u_{ij} = (\Delta_x^2 + \Delta_y^2) u_{ij}.$$

显然

$$\Delta_x u_{ij} = \frac{1}{2}(\delta_x u_{i-\frac{1}{2},j} + \delta_x u_{i+\frac{1}{2},j}), \quad \Delta_y u_{ij} = \frac{1}{2}\left(\delta_y u_{i,j-\frac{1}{2}} + \delta_y u_{i,j+\frac{1}{2}}\right).$$

设 $u \in \mathcal{W}_h$, $v \in \mathcal{W}_h$, 定义内积

$$(u, v) = h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} u_{ij} v_{ij}$$

及 Sobolev 范数 (半范数)

$$\|\delta_x u\| = \sqrt{h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(\delta_x u_{i-\frac{1}{2},j}\right)^2}, \quad \|\delta_y u\| = \sqrt{h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(\delta_y u_{i,j-\frac{1}{2}}\right)^2},$$

$$\|u\| = \sqrt{(u, u)}, \quad |u|_1 = \sqrt{\|\delta_x u\|^2 + \|\delta_y u\|^2},$$

$$\|\nabla_h u\| = \sqrt{h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} |\nabla_h u_{ij}|^2}, \quad \|\nabla_h u\|_4 = \sqrt[4]{h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} |\nabla_h u_{ij}|^4},$$

$$\|\Delta_h u\| = \sqrt{h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} |\Delta_h u_{ij}|^2}.$$

记

$$\mathcal{S}_\tau = \{w \mid w = (w_0, w_1, \cdots, w_n)\text{为}\Omega_\tau\text{上的网格函数}\}.$$

设 $w \in \mathcal{S}_\tau$, 引进如下记号:

$$w^{k+\frac{1}{2}} = \frac{1}{2}(w^k + w^{k+1}), \qquad \delta_t w^{k+\frac{1}{2}} = \frac{1}{\tau}(w^{k+1} - w^k),$$
$$\nabla_\tau w^k = \frac{1}{\tau}(w^k - w^{k-1}), \qquad \nabla_{2\tau} w^k = \frac{1}{2\tau}(3w^k - 4w^{k-1} + w^{k-2}).$$

引理 10.1　设 $u \in \mathcal{W}_h$. 则有

$$\|\nabla_h u\|^2 \leqslant \|\Delta_h\| \cdot \|u\|. \tag{10.6}$$

证明　由

$$\begin{aligned}
&-(\Delta_h u, u)\\
&= -h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} (\Delta_h u_{ij}) u_{ij}\\
&= h_2 \sum_{j=1}^{m_2}\Big[-h_1 \sum_{i=1}^{m_1} (\delta_x^2 u_{ij}) u_{ij}\Big] + h_1 \sum_{i=1}^{m_1}\Big[-h_2 \sum_{j=1}^{m_2} (\delta_y^2 u_{ij}) u_{ij}\Big]\\
&= h_2 \sum_{j=1}^{m_2}\Big[h_1 \sum_{i=1}^{m_1} \big(\delta_x u_{i-\frac{1}{2},j}\big)^2\Big] + h_1 \sum_{i=1}^{m_1}\Big[h_2 \sum_{j=1}^{m_2} \big(\delta_y u_{i,j-\frac{1}{2}}\big)^2\Big]\\
&= \|\delta_x u\|^2 + \|\delta_y u\|^2\\
&= |u|_1^2,
\end{aligned}$$

得到

$$|u|_1^2 = -(\Delta_h u, u) \leqslant \|\Delta_h u\| \cdot \|u\|.$$

再注意到

$$\|\nabla_h u\| \leqslant |u|_1,$$

即得 (10.6). □

10.3　二层非线性向后 Euler 差分格式

10.3.1　差分格式的建立

令

$$\begin{pmatrix} v \\ w \end{pmatrix} = |\nabla u|^2 \nabla u,$$

则 (10.1) 等价于

$$u_t + \delta\Delta^2 u - v_x - w_y + \Delta u = 0, \quad (x,y) \in \mathcal{R}^2,\ 0 < t \leqslant T, \tag{10.7}$$

$$v = |\nabla u|^2 u_x, \quad (x,y) \in \mathcal{R}^2,\ 0 < t \leqslant T, \tag{10.8}$$

$$w = |\nabla u|^2 u_y, \quad (x,y) \in \mathcal{R}^2,\ 0 < t \leqslant T. \tag{10.9}$$

定义 $\mathcal{W}_h$ 上网格函数

$$U_{ij}^k = u(x_i, y_j, t_k), \quad V_{ij}^k = v(x_i, y_j, t_k), \quad W_{ij}^k = w(x_i, y_j, t_k).$$

在点 (x_i, y_j, t_k) 处考虑方程 (10.7)–(10.9). 应用 Taylor 展开式, 有

$$\nabla_\tau U_{ij}^k + \delta\Delta_h^2 U_{ij}^k - \Delta_x V_{ij}^k - \Delta_y W_{ij}^k + \bar{\Delta}_h U_{ij}^{k-1} = P_{ij}^k,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.10}$$
$$V_{ij}^k = |\nabla_h U_{ij}^k|^2 \Delta_x U_{ij}^k + Q_{ij}^k, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.11}$$
$$W_{ij}^k = |\nabla_h U_{ij}^k|^2 \Delta_y U_{ij}^k + R_{ij}^k, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.12}$$

存在正常数 c_1 使得

$$|P_{ij}^k| \leqslant c_1(\tau + h_1^2 + h_2^2), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.13}$$
$$|Q_{ij}^k| \leqslant c_1(h_1^2 + h_2^2), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.14}$$
$$|R_{ij}^k| \leqslant c_1(h_1^2 + h_2^2), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n. \tag{10.15}$$

注意到初值条件

$$U_{ij}^0 = \varphi(x_i, y_j), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2, \tag{10.16}$$

在 (10.10)–(10.12) 中是略去小量项 P_{ij}^k, Q_{ij}^k, R_{ij}^k, 对 (10.7)–(10.9) 及 (10.2) 建立如下差分格式:

对 $0 \leqslant k \leqslant n$, 求 u^k, v^k, $w^k \in \mathcal{W}_h$ 使得

$$\nabla_\tau u_{ij}^k + \delta\Delta_h^2 u_{ij}^k - \Delta_x v_{ij}^k - \Delta_y w_{ij}^k + \bar{\Delta}_h u_{ij}^{k-1} = 0,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.17}$$
$$v_{ij}^k = |\nabla_h u_{ij}^k|^2 \Delta_x u_{ij}^k, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.18}$$
$$w_{ij}^k = |\nabla_h u_{ij}^k|^2 \Delta_y u_{ij}^k, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.19}$$
$$u_{ij}^0 = \varphi(x_i, y_j), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2. \tag{10.20}$$

将 (10.18), (10.19) 代入 (10.17) 得到: 对 $0 \leqslant k \leqslant n$, 求 $u^k \in \mathcal{W}_h$ 使得

$$\nabla_\tau u_{ij}^k + \delta\Delta_h^2 u_{ij}^k - \nabla_h \cdot \left(|\nabla_h u_{ij}^k|^2 \nabla_h u_{ij}^k\right) + \bar{\Delta}_h u_{ij}^{k-1} = 0,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.21}$$
$$u_{ij}^0 = \varphi(x_i, y_j), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2. \tag{10.22}$$

对 (10.1)–(10.2) 建立差分格式 (10.21)–(10.22).

10.3.2　差分格式解的有界性

定理 10.3　设 $\{u_{ij}^k\,|\,0\leqslant i\leqslant m_1, 0\leqslant j\leqslant m_2, 0\leqslant k\leqslant n\}\in\mathcal{W}_h$ 为差分格式 (10.21)–(10.22) 的解, 则有

$$\begin{aligned}&\|u^k\|^2+\tau\|\nabla_h u^k\|_4^4\leqslant\|u^0\|^2+\tau\|\nabla_h u^0\|_4^4+\frac{1}{2}L_1L_2t_k,\quad 1\leqslant k\leqslant n,\\&E^k\leqslant E^{k-1},\quad 1\leqslant k\leqslant n,\end{aligned}$$

其中

$$E^k=\frac{\delta}{2}\|\Delta_h u^k\|^2+\frac{1}{4}h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}(|\nabla_h u_{ij}^k|^2-1)^4.$$

证明　(I) 用 u^k 与 (10.21) 的两边作内积, 得

$$(\nabla_\tau u^k,u^k)+\delta(\Delta_h^2u^k,u^k)-(\nabla_h\cdot(|\nabla_h u^k|^2\nabla_h u^k),u^k)+(\bar{\Delta}_h u^{k-1},u^k)=0.$$

注意到

$$\begin{aligned}&(\nabla_\tau u^k,u^k)=\frac{1}{2\tau}(\|u^k\|^2-\|u^{k-1}\|^2)+\frac{\tau}{2}\|\nabla_\tau u^k\|^2,\\&(\Delta_h^2u^k,u^k)=\|\Delta_h u^k\|^2,\\&-(\nabla_h\cdot(|\nabla_h u^k|^2\nabla_h u^k),u^k)=(|\nabla_h u^k|^2\nabla_h u^k,\nabla_h u^k)=\|\nabla_h u^k\|_4^4,\\&-(\bar{\Delta}_h u^{k-1},u^k)=(\nabla_h u^{k-1},\nabla_h u^k)\leqslant h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left|\nabla_h u_{ij}^{k-1}\right|\cdot\left|\nabla_h u_{ij}^k\right|\\&\qquad\qquad\leqslant h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(\left|\nabla_h u_{ij}^{k-1}\right|^2\left|\nabla_h u_{ij}^k\right|^2+\frac{1}{4}\right)\\&\qquad\qquad\leqslant\frac{1}{2}\left(\|\nabla_h u^{k-1}\|_4^4+\|\nabla_h u^k\|_4^4\right)+\frac{1}{4}L_1L_2,\end{aligned}$$

有

$$\begin{aligned}&\frac{1}{2\tau}(\|u^k\|^2-\|u^{k-1}\|^2)+\frac{\tau}{2}\|\nabla_\tau u^k\|^2+\delta\|\Delta_h u^k\|^2+\|\nabla_h u^k\|_4^4\\\leqslant&\frac{1}{2}(\|\nabla_h u^{k-1}\|_4^4+\|\nabla_h u^k\|_4^4)+\frac{1}{4}L_1L_2.\end{aligned}$$

因而

$$\|u^k\|^2+\tau\|\nabla_h u^k\|_4^4\leqslant\|u^{k-1}\|^2+\tau\|\nabla_h u^{k-1}\|_4^4+\frac{1}{2}L_1L_2\tau,\quad 1\leqslant k\leqslant n.$$

递推得

$$\|u^k\|^2+\tau\|\nabla_h u^k\|_4^4\leqslant\|u^0\|^2+\tau\|\nabla_h u^0\|_4^4+\frac{1}{2}L_1L_2k\tau,\quad 1\leqslant k\leqslant n.$$

(II) 用 $\nabla_\tau u^k$ 与 (10.21) 的两边作内积, 得

$$\|\nabla_\tau u^k\|^2+\delta(\Delta_h^2 u^k,\nabla_\tau u^k)-\big(\nabla_h\cdot(|\nabla_h u^k|^2\nabla_h u^k),\nabla_\tau u^k\big)+(\bar{\Delta}_h u^{k-1},\nabla_\tau u^k)=0.$$

注意到

$$(\Delta_h^2 u^k,\nabla_\tau u^k)=\big(\Delta_h u^k,\nabla_\tau(\Delta_h u^k)\big)=\frac{1}{2}\nabla_\tau(\|\Delta_h u^k\|^2)+\frac{\tau}{2}\|\nabla_\tau(\Delta_h u^k)\|^2,$$

$$\begin{aligned}
&-\big(\nabla_h\cdot(|\nabla_h u^k|^2\nabla_h u^k),\nabla_\tau u^k\big)=(|\nabla_h u^k|^2\nabla_h u^k,\nabla_\tau\nabla_h u^k)\\
&=h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left|\nabla_h u_{ij}^k\right|^2(\nabla_h u_{ij}^k)\cdot\nabla_\tau(\nabla_h u_{ij}^k)\\
&=h_2h_2\sum_{j=1}^{m_1}\sum_{j=1}^{m_2}\left|\nabla_h u_{ij}^k\right|^2\left(\frac{1}{2}\nabla_\tau(|\nabla_h u_{ij}^k|^2)+\frac{\tau}{2}|\nabla_\tau(\nabla_h u_{ij}^k)|^2\right)\\
&=\frac{1}{2}h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h u_{ij}^k|^2\left(\nabla_\tau(|\nabla_h u_{ij}^k|^2)\right)+\frac{\tau}{2}h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h u_{ij}^k|^2\cdot|\nabla_\tau(\nabla_h u_{ij}^k)|^2\\
&=\frac{1}{2}h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left[\frac{1}{2}\nabla_\tau(|\nabla_h u_{ij}^k|^4)+\frac{\tau}{2}\left(\nabla_\tau(|\nabla_h u_{ij}^k|^2)\right)^2\right]\\
&\quad+\frac{\tau}{2}h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h u_{ij}^k|^2\cdot|\nabla_h\nabla_\tau u_{ij}^k|^2,
\end{aligned}$$

$$-(\bar{\Delta}_h u^{k-1},\nabla_\tau u^k)=(\nabla_h u^{k-1},\nabla_h\nabla_\tau u^k)=\frac{1}{2}\nabla_\tau(\|\nabla_h u^k\|^2)-\frac{\tau}{2}\|\nabla_h\nabla_\tau u^k\|^2,$$

有

$$\frac{\delta}{2}\nabla_\tau(\|\Delta_h u^k\|^2)+\frac{1}{4}\nabla_\tau(\|\nabla_h u^k\|_4^4)-\frac{\tau}{2}\nabla_\tau(\|\nabla_h u^k\|^2)\leqslant 0,\quad 1\leqslant k\leqslant n,$$

即

$$\nabla_\tau E^k\leqslant 0,\quad 1\leqslant k\leqslant n.$$

□

由定理 10.3 和定理 9.2 可得如下结论.

定理 10.4 设 $\{u_{ij}^k\,|\,0\leqslant i\leqslant m_1,0\leqslant j\leqslant m_2,0\leqslant k\leqslant n\}$ 为差分格式 (10.21)-(10.22) 的解, 则存在常数 c_2 使得

$$\|u^k\|_\infty\leqslant c_2,\quad 0\leqslant k\leqslant n. \tag{10.23}$$

10.3.3 差分格式解的存在性

定理 10.5 设 $\tau<8\delta$. 则差分格式 (10.21)–(10.22) 存在解.

证明　设第 $k-1$ 层的值 u^{k-1} 已知, 由 (10.21) 可得关于 u^k 的方程组

$$\frac{1}{\tau}(u_{ij}^k - u_{ij}^{k-1}) + \delta\Delta_h^2 u_{ij}^k - \nabla_h\cdot(|\nabla_h u_{ij}^k|^2\nabla_h u_{ij}^k) + \bar{\Delta}_h u_{ij}^{k-1} = 0,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2. \tag{10.24}$$

记

$$\Pi(w) = \frac{1}{\tau}(w_{ij} - u_{ij}^{k-1}) + \delta\Delta_h^2 w_{ij} - \nabla_h\cdot(|\nabla_h w_{ij}|^2\nabla_h w_{ij}) + \bar{\Delta}_h u_{ij}^{k-1},$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2.$$

用 w 与 $\Pi(w)$ 作内积, 得

$$\begin{aligned}\big(\Pi(w), w\big) =& \frac{1}{\tau}\big(\|w\|^2 - (u^{k-1}, w)\big) + \delta(\Delta_h^2 w, w)\\ &- \big(\nabla_h\cdot(|\nabla_h w|^2\nabla w), w\big) + (\bar{\Delta}_h u^{k-1}, w)\\ =& \frac{1}{\tau}\big(\|w\|^2 - (u^{k-1}, w)\big) + \delta\|\Delta_h w\|^2 + (|\nabla_h w|^2\nabla_h w, \nabla_h w)\\ &- (\nabla_h u^{k-1}, \nabla_h w).\end{aligned}$$

注意到

$$\begin{aligned}(\nabla_h u^{k-1}, \nabla_h w) &\leqslant \frac{1}{2}\|\nabla_h w\|^2 + \frac{1}{2}\|\nabla_h u^{k-1}\|^2 \leqslant \frac{1}{2}\|w\|\cdot\|\Delta_h w\| + \frac{1}{2}\|\nabla_h u^{k-1}\|^2\\ &\leqslant \delta\|\Delta_h w\|^2 + \frac{1}{16\delta}\|w\|^2 + \frac{1}{2}\|\nabla_h u^{k-1}\|^2,\end{aligned}$$

有

$$\begin{aligned}\big(\Pi(w), w\big) &\geqslant \frac{1}{\tau}\left(\|w\|^2 - \|u^{k-1}\|\cdot\|w\|\right) - \frac{1}{16\delta}\|w\|^2 - \frac{1}{2}\|\nabla_h u^{k-1}\|^2\\ &\geqslant \frac{1}{\tau}\Big(\frac{1}{2}\|w\|^2 - \frac{1}{2}\|u^{k-1}\|^2\Big) - \frac{1}{16\delta}\|w\|^2 - \frac{1}{2}\|\nabla_h u^{k-1}\|^2\\ &= \frac{1}{2\tau}\Big(\|w\|^2 - \|u^{k-1}\|^2 - \frac{\tau}{8\delta}\|w\|^2 - \tau\|\nabla_h u^{k-1}\|^2\Big)\\ &= \frac{1}{2\tau}\left[\left(1-\frac{\tau}{8\delta}\right)\|w\|^2 - \left(\|u^{k-1}\|^2 + \tau\|\nabla_h u^{k-1}\|^2\right)\right].\end{aligned}$$

当 $\dfrac{\tau}{8\delta} < 1$ 且 $\|w\|^2 = \dfrac{1}{1-\dfrac{\tau}{8\delta}}\left(\|u^{k-1}\|^2 + \tau\|\nabla_h u^{k-1}\|^2\right)$ 时, $(\Pi(w), w)\geqslant 0$. 由 Browder 定理 (定理 1.3) 存在 $w^*\in W_h$ 且 $\|w^*\|^2 \leqslant \dfrac{1}{1-\dfrac{\tau}{8\delta}}(\|u^{k-1}\|^2 + \tau\|\nabla_h u^{k-1}\|^2)$, 满足

$$\Pi(w^*) = 0.$$

即方程 (10.24) 存在解. □

下面来考虑解的唯一性. 为此先给出如下引理.

引理 10.2 设 $u \in \mathcal{W}_h$, $v \in \mathcal{W}_h$, 记 $\varepsilon_{ij} = u_{ij} - v_{ij}$, 则有

$$
\begin{aligned}
&|\nabla_h u_{ij}|^2 - |\nabla_h v_{ij}|^2 = \nabla_h u_{ij} \cdot \nabla_h \varepsilon_{ij} + \nabla_h v_{ij} \cdot \nabla_h \varepsilon_{ij},\\
&|\nabla_h u_{ij}|^2 \nabla_h u_{ij} - |\nabla_h v_{ij}|^2 \nabla_h v_{ij}\\
=& |\nabla_h v_{ij}|^2 \nabla_h \varepsilon_{ij} + (\nabla_h u_{ij} \cdot \nabla_h \varepsilon_{ij} + \nabla_h v_{ij} \cdot \nabla_h \varepsilon_{ij}) \nabla_h u_{ij}.
\end{aligned}
$$

证明 (I)

$$
\begin{aligned}
&|\nabla_h u_{ij}|^2 - |\nabla_h v_{ij}|^2\\
=&\nabla_h u_{ij} \cdot \nabla_h u_{ij} - \nabla_h v_{ij} \cdot \nabla_h v_{ij}\\
=&\nabla_h u_{ij} \cdot (\nabla_h u_{ij} - \nabla_h v_{ij}) + (\nabla_h u_{ij} - \nabla_h v_{ij}) \cdot \nabla_h v_{ij}\\
=&\nabla_h u_{ij} \cdot \nabla_h \varepsilon_{ij} + \nabla_h v_{ij} \cdot \nabla_h \varepsilon_{ij};
\end{aligned}
$$

(II)

$$
\begin{aligned}
&|\nabla_h u_{ij}|^2 \nabla_h u_{ij} - |\nabla_h v_{ij}|^2 \nabla_h v_{ij}\\
=&|\nabla_h v_{ij}|^2 (\nabla_h u_{ij} - \nabla_h v_{ij}) + (|\nabla_h u_{ij}|^2 - |\nabla_h v_{ij}|^2) \nabla_h u_{ij}\\
=&|\nabla_h v_{ij}|^2 \nabla_h \varepsilon_{ij} + (\nabla_h u_{ij} \cdot \nabla_h \varepsilon_{ij} + \nabla_h v_{ij} \cdot \nabla_h \varepsilon_{ij}) \nabla_h u_{ij}.
\end{aligned}
$$

□

定理 10.6 差分格式 (10.21)–(10.22) 的解是唯一的.

证明 设 (10.24) 另有解 $v^k \in \mathcal{W}_h$, 即有 $v^k \in \mathcal{W}_h$ 满足

$$
\frac{1}{\tau}\left(v_{ij}^k - u_{ij}^{k-1}\right) + \delta \Delta_h^2 v_{ij}^k - \nabla_h \cdot (|\nabla_h v_{ij}^k|^2 \nabla_h v_{ij}^k) + \bar{\Delta}_h u_{ij}^{k-1} = 0,
$$
$$
1 \leqslant i \leqslant m_1, \quad 1 \leqslant j \leqslant m_2. \tag{10.25}
$$

令

$$
\rho_{ij} = u_{ij}^k - v_{ij}^k.
$$

将 (10.24) 和 (10.25) 相减, 得

$$
\frac{1}{\tau}\rho_{ij} + \delta \Delta_h^2 \rho_{ij} - \nabla \cdot \left(|\nabla_h u_{ij}^k|^2 \nabla_h u_{ij}^k - |\nabla_h v_{ij}^k|^2 \nabla_h v_{ij}^k\right) = 0,
$$
$$
1 \leqslant i \leqslant m_1, \ 1 \leqslant j \leqslant m_2. \tag{10.26}
$$

用 ρ 与 (10.26) 的两边作内积, 得

$$
\frac{1}{\tau}\|\rho\|^2 + \delta \|\Delta_h \rho\|^2 + \left(|\nabla_h u^k|^2 \nabla_h u^k - |\nabla_h v^k|^2 \nabla_h v^k, \nabla_h \rho\right) = 0.
$$

由引理 10.2 得

$$
\frac{1}{\tau}\|\rho\|^2 + \delta \|\Delta_h \rho\|^2 + h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(|\nabla_h v_{ij}^k|^2 \nabla_h \rho_{ij}\right.
$$

$$+\left(\nabla_h u_{ij}^k\cdot\nabla_h\rho_{ij}+\nabla_h v_{ij}^k\cdot\nabla_h\rho_{ij}\right)\nabla_h u_{ij}^k,\nabla_h\rho_{ij}\big)=0,$$

即

$$\begin{aligned}&\frac{1}{\tau}\|\rho\|^2+\delta\|\Delta_h\rho\|^2+h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left[|\nabla_h v_{ij}^k|^2\cdot|\nabla_h\rho_{ij}|^2+\left(\nabla_h u_{ij}^k\cdot\nabla_h\rho_{ij}\right)^2\right]\\ =&-h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(\nabla_h v_{ij}^k\cdot\nabla_h\rho_{ij}\right)\left(\nabla_h u_{ij}^k\cdot\nabla_h\rho_{ij}\right)\\ \leqslant&\frac{1}{2}h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left[\left(\nabla_h v_{ij}^k\cdot\nabla_h\rho_{ij}\right)^2+\left(\nabla_h u_{ij}^k\cdot\nabla_h\rho_{ij}\right)^2\right].\end{aligned}$$

因而 $\|\rho\|=0$. □

10.3.4 差分格式解的收敛性

定理 10.7 设 $\{U_{ij}^k\,|\,0\leqslant i\leqslant m_1,0\leqslant j\leqslant m_2,0\leqslant k\leqslant n\}$ 为 (10.1)–(10.2) 的解, $\{u_{ij}^k\,|\,0\leqslant i\leqslant m_1,0\leqslant j\leqslant m_2,0\leqslant k\leqslant n\}$ 为 (10.21)–(10.22) 的解. 定义误差函数

$$\tilde{u}_{ij}^k=U_{ij}^k-u_{ij}^k,\quad 0\leqslant i\leqslant m_1,\quad 0\leqslant j\leqslant m_2,\quad 0\leqslant k\leqslant n.$$

则存在常数 c_3 使得

$$\|\tilde{u}^k\|\leqslant c_3(\tau+h_1^2+h_2^2),\quad 0\leqslant k\leqslant n. \tag{10.27}$$

证明 由 (10.18)–(10.19) 定义 v^k, $w^k\in\mathcal{W}_h$. 则差分格式 (10.21)–(10.22) 可以写为 (10.17)–(10.20). 记

$$\tilde{v}_{ij}^k=V_{ij}^k-v_{ij}^k,\quad \tilde{w}_{ij}^k=W_{ij}^k-w_{ij}^k,\quad 0\leqslant i\leqslant m_1,\quad 0\leqslant j\leqslant m_2,\quad 0\leqslant k\leqslant n.$$

将 (10.10)–(10.12), (10.16) 与 (10.17)–(10.20) 相减, 得

$$\nabla_\tau\tilde{u}_{ij}^k+\delta\Delta_h^2\tilde{u}_{ij}^k-\Delta_x\tilde{v}_{ij}^k-\Delta_y\tilde{w}_{ij}^k+\Delta_h\tilde{u}_{ij}^{k-1}=P_{ij}^k,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n, \tag{10.28}$$

$$\tilde{v}_{ij}^k=|\nabla_h U_{ij}^k|^2\Delta_x U_{ij}^k-|\nabla_h u_{ij}^k|^2\Delta_x u_{ij}^k+Q_{ij}^k,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n, \tag{10.29}$$

$$\tilde{w}_{ij}^k=|\nabla_h U_{ij}^k|^2\Delta_y U_{ij}^k-|\nabla_h u_{ij}^k|^2\Delta_y u_{ij}^k+R_{ij}^k,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,1\leqslant k\leqslant n, \tag{10.30}$$

$$\tilde{u}_{ij}^0=0,\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2. \tag{10.31}$$

由 (10.31) 知 (10.27) 对 $k=0$ 成立.

用 $\tilde{u}^k$ 与 (10.28) 的两边作内积, 有

$$(\nabla_\tau \tilde{u}^k, \tilde{u}^k) + \delta(\Delta_h^2 \tilde{u}^k, \tilde{u}^k) - (\Delta_x \tilde{v}^k, \tilde{u}^k) - (\Delta_y \tilde{w}^k, \tilde{u}^k) + (\Delta_h \tilde{u}^{k-1}, \tilde{u}^k) = (P^k, \tilde{u}^k),\quad 1 \leqslant k \leqslant n. \tag{10.32}$$

易知

$$(\nabla_\tau \tilde{u}^k, \tilde{u}^k) = \frac{1}{2\tau}(\|\tilde{u}^k\|^2 - \|\tilde{u}^{k-1}\|^2) + \frac{\tau}{2}\|\nabla_\tau \tilde{u}^k\|^2, \tag{10.33}$$

$$(\Delta_h^2 \tilde{u}^k, \tilde{u}^k) = \|\Delta_h \tilde{u}^k\|^2. \tag{10.34}$$

由 (10.29)–(10.30) 得

$$\begin{aligned}
&-(\Delta_x \tilde{v}^k, \tilde{u}^k) - (\Delta_y \tilde{w}^k, \tilde{u}^k)\\
=&(\tilde{v}^k, \Delta_x \tilde{u}^k) + (\tilde{w}^k, \Delta_y \tilde{u}^k)\\
=&(|\nabla_h U^k|^2 \Delta_x U^k - |\nabla_h u^k|^2 \Delta_x u^k + Q^k, \Delta_x \tilde{u}^k)\\
&+(|\nabla_h U^k|^2 \Delta_y U^k - |\nabla_h u^k|^2 \Delta_y u^k + R^k, \Delta_y \tilde{u}^k)\\
=&(|\nabla_h u^k|^2 \Delta_x \tilde{u}^k + (|\nabla_h U^k|^2 - |\nabla_h u^k|^2)\Delta_x U^k + Q^k, \Delta_x \tilde{u}^k)\\
&+(|\nabla_h u^k|^2 \Delta_y \tilde{u}^k + (|\nabla_h U^k| - |\nabla_h u^k|^2)\Delta_y U^k + R^k, \Delta_y \tilde{u}^k)\\
=&h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} |\nabla_h u_{ij}^k|^2 \cdot |\nabla_h \tilde{u}_{ij}^k|^2\\
&+h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} (|\nabla_h U_{ij}^k|^2 - |\nabla_h u_{ij}^k|^2)\nabla_h U_{ij}^k \cdot \nabla_h \tilde{u}_{ij}^k\\
&+(Q^k, \Delta_x \tilde{u}^k) + (R^k, \Delta_y \tilde{u}^k).
\end{aligned} \tag{10.35}$$

此外,

$$(\bar{\Delta}_h \tilde{u}^{k-1}, \tilde{u}^k) = -(\nabla_h \tilde{u}^{k-1}, \nabla_h \tilde{u}^k). \tag{10.36}$$

将 (10.33)–(10.36) 代入 (10.32), 得

$$\begin{aligned}
&\frac{1}{2\tau}(\|\tilde{u}^k\|^2 - \|\tilde{u}^{k-1}\|^2) + \delta\|\Delta_h \tilde{u}^k\|^2 + h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} |\nabla_h u_{ij}^k|^2 \cdot |\nabla_h \tilde{u}_{ij}^k|^2\\
\leqslant& -h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} (|\nabla_h U_{ij}^k|^2 - |\nabla_h u_{ij}^k|^2)\nabla_h U_{ij}^k \cdot \nabla_h \tilde{u}_{ij}^k + (\nabla_h \tilde{u}^{k-1}, \nabla_h \tilde{u}^k)\\
&-(Q^k, \Delta_x \tilde{u}^k) - (R^k, \Delta_y \tilde{u}^k) + (P^k, \tilde{u}^k), \quad 1 \leqslant k \leqslant n.
\end{aligned} \tag{10.37}$$

记

$$c_4 = \max_{(x,y)\in\bar{\Omega},0\leqslant t\leqslant T}\left\{u_x^2(x,y,t)\right\} + \max_{(x,y)\in\bar{\Omega},0\leqslant t\leqslant T}\left\{u_y^2(x,y,t)\right\}, \tag{10.38}$$

则有

$$|\nabla_h U_{ij}^k|^2 \leqslant c_4. \tag{10.39}$$

由引理 10.2, 得

$$\begin{aligned}
&-h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(|\nabla_h U_{ij}^k|^2-|\nabla_h u_{ij}^k|^2\right)\nabla_h U_{ij}^k\cdot\nabla_h\tilde{u}_{ij}^k\\
=&-h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(\nabla_h U_{ij}^k\cdot\nabla_h\tilde{u}_{ij}^k+\nabla_h u_{ij}^k\cdot\nabla_h\tilde{u}_{ij}^k\right)\nabla_h U_{ij}^k\cdot\nabla_h\tilde{u}_{ij}^k\\
\leqslant&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(|\nabla_h U_{ij}^k|\cdot|\nabla_h\tilde{u}_{ij}^k|+|\nabla_h u_{ij}^k|\cdot|\nabla_h\tilde{u}_{ij}^k|\right)|\nabla_h U_{ij}^k|\cdot|\nabla_h\tilde{u}_{ij}^k|\\
=&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h U_{ij}^k|^2\cdot|\nabla_h\tilde{u}_{ij}^k|^2\\
&+h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_1}\left(|\nabla_h u_{ij}^k|\cdot|\nabla_h\tilde{u}_{ij}^k|\right)\cdot\left(|\nabla_h U_{ij}^k|\cdot|\nabla_h\tilde{u}_{ij}^k|\right)\\
\leqslant&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h U_{ij}^k|^2\cdot|\nabla_h\tilde{u}_{ij}^k|^2\\
&+h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(|\nabla_h u_{ij}^k|^2\cdot|\nabla_h\tilde{u}_{ij}^k|^2+\frac{1}{4}|\nabla_h U_{ij}^k|^2\cdot|\nabla_h\tilde{u}_{ij}^k|^2\right)\\
\leqslant&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h u_{ij}^k|^2\cdot|\nabla_h\tilde{u}_{ij}^k|^2+\frac{5}{4}h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h U_{ij}^k|^2\cdot|\nabla_h\tilde{u}_{ij}^k|^2\\
\leqslant&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h u_{ij}^k|^2\cdot|\nabla_h\tilde{u}_{ij}^k|^2+\frac{5}{4}c_4\|\nabla_h\tilde{u}^k\|^2,
\end{aligned}$$

此外,

$$\begin{aligned}
(\nabla_h\tilde{u}^{k-1},\nabla_h\tilde{u}^k)\leqslant&\frac{1}{2}\|\nabla_h\tilde{u}^{k-1}\|^2+\frac{1}{2}\|\nabla_h\tilde{u}^k\|^2,\\
&-(Q^k,\Delta_x\tilde{u}^k)-(R^k,\Delta_y\tilde{u}^k)\\
&\leqslant\|Q^k\|^2+\frac{1}{4}\|\Delta_x\tilde{u}^k\|^2+\|R^k\|^2+\frac{1}{4}\|\Delta_x\tilde{u}^k\|^2\\
&=\|Q^k\|^2+\|R^k\|^2+\frac{1}{4}\|\nabla_h\tilde{u}^k\|^2,
\end{aligned}$$

$$(P^k, \tilde{u}^k) \leqslant \|P^k\|^2 + \frac{1}{4}\|\tilde{u}^k\|^2.$$

将以上四式代入 (10.37), 得

$$\begin{aligned}
&\frac{1}{2\tau}(\|\tilde{u}^k\|^2 - \|\tilde{u}^{k-1}\|^2) + \delta\|\Delta_h u^k\|^2 \\
\leqslant&\left(\frac{1}{4} + \frac{5}{4}c_4\right)\|\nabla_h \tilde{u}^k\|^2 + \frac{1}{2}\|\nabla_h \tilde{u}^{k-1}\|^2 + \frac{1}{2}\|\nabla_h \tilde{u}^k\|^2 + \frac{1}{4}\|\tilde{u}^k\|^2 \\
&+ \|P^k\|^2 + \|Q^k\|^2 + \|R^k\|^2 \\
\leqslant&\left(\frac{3}{4} + \frac{5}{4}c_4\right)\|\nabla_h \tilde{u}^k\|^2 + \frac{1}{2}\|\nabla_h \tilde{u}^{k-1}\|^2 + \frac{1}{4}\|\tilde{u}^k\|^2 \\
&+ \|P^k\|^2 + \|Q^k\|^2 + \|R^k\|^2 \\
\leqslant&\left(\frac{3}{4} + \frac{5}{4}c_4\right)\|\Delta_h \tilde{u}^k\| \cdot \|\tilde{u}^k\| + \frac{1}{2}\|\Delta_h \tilde{u}^{k-1}\| \cdot \|\tilde{u}^{k-1}\| + \frac{1}{4}\|\tilde{u}^k\|^2 \\
&+ \|P^k\|^2 + \|Q^k\|^2 + \|R^k\|^2 \\
\leqslant&\frac{\delta}{2}\|\Delta_h \tilde{u}^k\|^2 + \frac{1}{2\delta}\Big(\frac{3}{4} + \frac{5}{4}c_4\Big)^2\|\tilde{u}^k\|^2 + \frac{\delta}{2}\|\Delta_h \tilde{u}^{k-1}\|^2 + \frac{1}{2\delta}\cdot\frac{1}{4}\|\tilde{u}^{k-1}\|^2 \\
&+ \frac{1}{4}\|\tilde{u}^k\|^2 + \|P^k\|^2 + \|Q^k\|^2 + \|R^k\|^2, \quad 1 \leqslant k \leqslant n.
\end{aligned}$$

注意到 (10.13)–(10.15), 由上式可得

$$\begin{aligned}
&\frac{1}{2\tau}\left[(\|\tilde{u}^k\|^2 + \tau\delta\|\Delta_h \tilde{u}^k\|^2) - (\|\tilde{u}^{k-1}\|^2 + \tau\delta\|\Delta_h \tilde{u}^{k-1}\|^2)\right] \\
\leqslant&\left[\frac{1}{4} + \frac{1}{2\delta}\Big(\frac{3}{4} + \frac{5}{4}c_4\Big)^2\right]\|\tilde{u}^k\|^2 + \frac{1}{8\delta}\|\tilde{u}^{k-1}\|^2 + 3L_1L_2c_1^2(\tau + h_1^2 + h_2^2)^2, \quad 1 \leqslant k \leqslant n.
\end{aligned}$$

记

$$F^k = \|\tilde{u}^k\|^2 + \tau\delta\|\Delta_h \tilde{u}^k\|^2,$$

由上式可得

$$\frac{1}{2\tau}(F^k - F^{k-1}) \leqslant \left[\frac{1}{4} + \frac{1}{2\delta}\left(\frac{3}{4} + \frac{5}{4}c_4\right)^2\right](F^k + F^{k-1}) + 3L_1L_2c_1^2(\tau + h_1^2 + h_2^2)^2, \\ 1 \leqslant k \leqslant n,$$

即

$$\begin{aligned}
&\left\{1 - \left[\frac{1}{2} + \frac{1}{\delta}\left(\frac{3}{4} + \frac{5}{4}c_4\right)^2\right]\tau\right\}F^k \\
\leqslant&\left\{1 + \left[\frac{1}{2} + \frac{1}{\delta}\left(\frac{3}{4} + \frac{5}{4}c_4\right)^2\right]\tau\right\}F^{k-1} + 6L_1L_2c_1^2\tau(\tau + h_1^2 + h_2^2)^2, \quad 1 \leqslant k \leqslant n.
\end{aligned}$$

当 $\left[\frac{1}{2} + \frac{1}{\delta}\left(\frac{3}{4} + \frac{5}{4}c_4\right)^2\right]\tau \leqslant \frac{1}{3}$ 时,

$$F^k \leqslant \left\{1+3\left[\frac{1}{2}+\frac{1}{\delta}\left(\frac{3}{4}+\frac{5}{4}c_4\right)^2\right]\tau\right\}F^{k-1}+9L_1L_2c_1^2\tau(\tau+h_1^2+h_2^2)^2, \quad 1\leqslant k\leqslant n.$$

由 Gronwall 不等式, 并注意到 $F^0=0$, 得

$$\|\tilde{u}^k\|^2 \leqslant F^k \leqslant e^{3\left[\frac{1}{2}+\frac{1}{\delta}\left(\frac{3}{4}+\frac{5}{4}c_4\right)^2\right]T}\cdot\frac{3L_1L_2}{\frac{1}{2}+\frac{1}{\delta}\left(\frac{3}{4}+\frac{5}{4}c_4\right)^2}(\tau+h_1^2+h_2^2)^2,$$
$$1\leqslant k\leqslant n.$$

两边开平方, 即得所要结果. □

10.4 二层线性化向后 Euler 差分格式

10.4.1 差分格式的建立

在点 (x_i,y_j,t_k) 处考虑方程 (10.7)–(10.9), 应用 Taylor 展开式, 得

$$\nabla_\tau U_{ij}^k+\delta\Delta_h^2U_{ij}^k-\Delta_xV_{ij}^k-\Delta_yW_{ij}^k+\bar{\Delta}_hU_{ij}^{k-1}=\hat{P}_{ij}^k,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n, \tag{10.40}$$

$$V_{ij}^k=\left|\nabla_hU_{ij}^{k-1}\right|^2\Delta_xU_{ij}^k+\hat{Q}_{ij}^k, \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n, \tag{10.41}$$

$$W_{ij}^k=\left|\nabla_hU_{ij}^{k-1}\right|^2\Delta_yU_{ij}^k+\hat{R}_{ij}^k, \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n. \tag{10.42}$$

存在常数 c_5 使得

$$\left|\hat{P}_{ij}^k\right|\leqslant c_5(\tau+h_1^2+h_2^2), \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n, \tag{10.43}$$

$$\left|\hat{Q}_{ij}^k\right|\leqslant c_5(\tau+h_1^2+h_2^2), \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n, \tag{10.44}$$

$$\left|\hat{R}_{ij}^k\right|\leqslant c_5(\tau+h_1^2+h_2^2), \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n. \tag{10.45}$$

注意到初值条件

$$U_{ij}^0=\varphi(x_i,y_j), \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{10.46}$$

在 (10.40)–(10.42) 中略去小量项, 对 (10.7)–(10.9) 及 (10.2) 建立如下差分格式:

对 $0\leqslant k\leqslant n$, 求 u^k, v^k, $w^k\in\mathcal{W}_h$ 使得

$$\nabla_\tau u_{ij}^k+\delta\Delta_h^2u_{ij}^k-\Delta_xv_{ij}^k-\Delta_yw_{ij}^k+\bar{\Delta}_hu_{ij}^{k-1}=0,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n, \tag{10.47}$$

$$v_{ij}^k = |\nabla_h u_{ij}^{k-1}|^2 \Delta_x u_{ij}^k, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.48}$$

$$w_{ij}^k = |\nabla_h u_{ij}^{k-1}|^2 \Delta_y u_{ij}^k, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.49}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2. \tag{10.50}$$

将 (10.48)–(10.49) 代入 (10.47) 中得到: 对 $0 \leqslant k \leqslant n$, 求 $u^k \in \mathcal{W}_h$ 使得

$$\nabla_\tau u_{ij}^k + \delta \Delta_h^2 u_{ij}^k - \nabla_h \cdot (|\nabla_h u_{ij}^{k-1}|^2 \nabla_h u_{ij}^k) + \bar{\Delta}_h u_{ij}^{k-1} = 0,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.51}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2. \tag{10.52}$$

对 (10.1)–(10.2) 建立差分格式 (10.51)–(10.52).

10.4.2 差分格式解的有界性

定理 10.8 设 $\{u_{ij}^k \,|\, 0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 为差分格式 (10.51)–(10.52) 的解, 则有

$$\|u^k\|^2 \leqslant \|u^0\|^2 + \frac{1}{2} L_1 L_2 t_k, \quad 0 \leqslant k \leqslant n.$$

证明 用 u^k 与 (10.51) 作内积, 得

$$(\nabla_\tau u^k, u^k) + \delta(\Delta_h^2 u^k, u^k) - (\nabla_h \cdot (|\nabla_h u^{k-1}|^2 \nabla_h u^k), u^k) + (\bar{\Delta}_h u^{k-1}, u^k) = 0.$$

注意到

$$(\nabla_\tau u^k, u^k) = \frac{1}{2\tau}(\|u^k\|^2 - \|u^{k-1}\|^2) + \frac{\tau}{2}\|\nabla_\tau u^k\|^2,$$

$$(\Delta_h^2 u^k, u^k) = \|\Delta_h u^k\|^2,$$

$$-\left(\nabla_h \cdot (|\nabla_h u^{k-1}|^2 \nabla_h u^k), u^k\right) = h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left|\nabla_h u_{ij}^{k-1}\right|^2 \cdot \left|\nabla_h u_{ij}^k\right|^2,$$

$$\begin{aligned}
-\left(\bar{\Delta}_h u^{k-1}, u^k\right) = \left(\nabla_h u^{k-1}, \nabla_h u^k\right) &= h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \nabla_h u_{ij}^{k-1} \cdot \nabla_h u_{ij}^k \\
&\leqslant h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left|\nabla_h u_{ij}^{k-1}\right| \cdot \left|\nabla_h u_{ij}^k\right| \\
&\leqslant h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(\left|\nabla_h u_{ij}^{k-1}\right|^2 \cdot \left|\nabla_h u_{ij}^k\right|^2 + \frac{1}{4}\right),
\end{aligned}$$

有

$$\frac{1}{2\tau}\left(\|u^k\|^2 - \|u^{k-1}\|^2\right) \leqslant \frac{1}{4} L_1 L_2, \quad 1 \leqslant k \leqslant n.$$

递推得

$$\|u^k\|^2 \leqslant \|u^0\|^2 + \frac{1}{2}L_1L_2k\tau = \|u^0\|^2 + \frac{1}{2}L_1L_2t_k, \quad 0 \leqslant k \leqslant n. \qquad \square$$

10.4.3 差分格式的可解性

定理 10.9 差分格式 (10.51)–(10.52) 是唯一可解的.

证明 由 (10.52) 知第 0 层的值 u^0 已给定. 设已求得第 $k-1$ 层的值 u^{k-1}. 则由 (10.51) 可得关于第 k 层值 u^k 的线性方程组. 考虑其齐次方程组

$$\frac{1}{\tau}u_{ij}^k + \delta\Delta_h^2 u_{ij}^k - \nabla_h \cdot \left(|\nabla_h u_{ij}^{k-1}|^2 \nabla_h u_{ij}^k\right) = 0,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2. \tag{10.53}$$

用 u^k 与上式两边作内积, 得

$$\frac{1}{\tau}\|u^k\|^2 + \delta\|\Delta_h u^k\|^2 + \left(|\nabla_h u^{k-1}|^2 \nabla_h u^k, \nabla_h u^k\right) = 0.$$

因而 $\|u^k\| = 0$, 即 (10.53) 只有零解. $\square$

10.4.4 差分格式解的收敛性

定理 10.10 设 $\{U_{ij}^k \,|\, 0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 为 (10.1)–(10.2) 的解, $\{u_{ij}^k \,|\, 0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 为 (10.51)–(10.52) 的解. 定义误差函数

$$\tilde{u}_{ij}^k = U_{ij}^k - u_{ij}^k, \quad 0 \leqslant i \leqslant m_1, \quad 0 \leqslant j \leqslant m_2, \quad 0 \leqslant k \leqslant n.$$

则存在常数 c_6 使得

$$\|\tilde{u}^k\| \leqslant c_6(\tau + h_1^2 + h_2^2), \quad 0 \leqslant k \leqslant n. \tag{10.54}$$

证明 由 (10.48)–(10.49) 定义 $\{v^k, w^k\}$, 则差分格式 (10.51)–(10.52) 可以写为 (10.47)–(10.50).

将 (10.40)–(10.42), (10.46) 和 (10.47)–(10.50) 相减, 得误差方程组

$$\nabla_\tau \tilde{u}_{ij}^k + \delta\Delta_h^2 \tilde{u}_{ij}^k - \Delta_x \tilde{v}_{ij}^k - \Delta_y \tilde{w}_{ij}^k + \bar{\Delta}_h \tilde{u}_{ij}^{k-1} = \hat{P}_{ij}^k,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.55}$$

$$\tilde{v}_{ij}^k = \left|\nabla_h U_{ij}^{k-1}\right|^2 \nabla_x U_{ij}^k - \left|\nabla_h u_{ij}^{k-1}\right|^2 \nabla_x u_{ij}^k + \hat{Q}_{ij}^k,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.56}$$

$$\tilde{w}_{ij}^k = \left|\nabla_h U_{ij}^{k-1}\right|^2 \nabla_y U_{ij}^k - \left|\nabla_h u_{ij}^{k-1}\right|^2 \nabla_y u_{ij}^k + \hat{R}_{ij}^k,$$
$$1 \leqslant j \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 1 \leqslant k \leqslant n, \tag{10.57}$$

$$\tilde{u}_{ij}^0 = 0, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2. \tag{10.58}$$

由 (10.58) 知 (10.54) 对 $k=0$ 成立.

将 $\tilde{u}^k$ 与 (10.55) 的两边作内积, 得

$$(\nabla_\tau \tilde{u}^k, \tilde{u}^k) + \delta(\Delta_h^2 \tilde{u}^k, \tilde{u}^k) - (\Delta_x \tilde{v}^k, \tilde{u}^k) - (\Delta_y \tilde{w}^k, \tilde{u}^k) + (\Delta_h \tilde{u}^{k-1}, \tilde{u}^k) = (\hat{P}^k, \tilde{u}^k). \tag{10.59}$$

应用 (10.56) 和 (10.57) 可得

$$\begin{aligned}
&-(\Delta_x \tilde{v}^k, \tilde{u}^k) - (\Delta_y \tilde{w}^k, \tilde{u}^k)\\
=&(\tilde{v}^k, \Delta_x \tilde{u}^k) + (\tilde{w}^k, \Delta_y \tilde{u}^k)\\
=&\Big(|\nabla_h U^{k-1}|^2 \nabla_x U^k - |\nabla_h u^{k-1}|^2 \nabla_x u^k + \tilde{Q}^k, \Delta_x \tilde{u}^k\Big)\\
&+\Big(|\nabla_h U^{k-1}|^2 \nabla_y U^k - |\nabla_h u^{k-1}|^2 \nabla_y u^k + \tilde{R}^k, \Delta_y \tilde{u}^k\Big)\\
=&\big(|\nabla_h u^{k-1}|^2 (\nabla_x U^k - \nabla_x u^k) + (|\nabla_h U^{k-1}|^2 - |\nabla_h u^{k-1}|^2) \nabla_x U^k, \Delta_x \tilde{u}^k\big)\\
&+\big(|\nabla_h u^{k-1}|^2 (\nabla_y U^k - \nabla_y u^k) + (|\nabla_h U^{k-1}|^2 - |\nabla_h u^{k-1}|^2) \nabla_y U^k, \Delta_y \tilde{u}^k\big)\\
&+(\tilde{Q}^k, \Delta_x \tilde{u}^k) + (\tilde{R}^k, \Delta_y \tilde{u}^k)\\
=&h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} |\nabla_h u_{ij}^{k-1}|^2 |\nabla_h \tilde{u}_{ij}^k|^2\\
&+h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \big(|\nabla_h U_{ij}^{k-1}|^2 - |\nabla_h u_{ij}^{k-1}|^2\big) \nabla_h U_{ij}^k \cdot \nabla_h \tilde{u}_{ij}^k\\
&+(\tilde{Q}^k, \Delta_x \tilde{u}^k) + (\tilde{R}^k, \Delta_y \tilde{u}^k).
\end{aligned}$$

将上式代入 (10.59) 可得

$$\begin{aligned}
&\frac{1}{2\tau}\big(\|\tilde{u}^k\|^2 - \|\tilde{u}^{k-1}\|^2\big) + \frac{\tau}{2}\|\nabla_\tau \tilde{u}^k\|^2\\
&+\delta\|\Delta_h \tilde{u}^k\|^2 + h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} |\nabla_h u_{ij}^{k-1}|^2 \cdot |\nabla_h \tilde{u}_{ij}^k|^2\\
=&-h_2 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \big(|\nabla_h U_{ij}^{k-1}|^2 - |\nabla_h u_{ij}^{k-1}|^2\big) \nabla_h U_{ij}^k \cdot \nabla_h \tilde{u}_{ij}^k + \big(\nabla_h \tilde{u}^{k-1}, \nabla_h \tilde{u}^k\big)\\
&-(\tilde{Q}^k, \Delta_x \tilde{u}^k) - (\tilde{R}^k, \Delta_y \tilde{u}^k) + (\hat{P}^k, \tilde{u}^k).
\end{aligned} \tag{10.60}$$

应用引理 10.2 以及 (10.38), 得到

$$
\begin{aligned}
&-h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(|\nabla_h U_{ij}^{k-1}|^2-|\nabla_h u_{ij}^{k-1}|^2\right)\nabla_h U_{ij}^k\cdot\nabla_h\tilde{u}_{ij}^k\\
=&-h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(\nabla_h U_{ij}^{k-1}\cdot\nabla_h\tilde{u}_{ij}^{k-1}+\nabla_h u_{ij}^{k-1}\cdot\nabla_h\tilde{u}_{ij}^{k-1}\right)\nabla_h U_{ij}^k\cdot\nabla_h\tilde{u}_{ij}^k\\
\leqslant&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(|\nabla_h U_{ij}^{k-1}|\cdot|\nabla_h\tilde{u}_{ij}^{k-1}|+|\nabla_h u_{ij}^{k-1}|\cdot|\nabla_h\tilde{u}_{ij}^{k-1}|\right)|\nabla_h U_{ij}^k|\cdot|\nabla_h\tilde{u}_{ij}^k|\\
\leqslant&c_4h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h\tilde{u}_{ij}^{k-1}|\cdot|\nabla_h\tilde{u}_{ij}^k|\\
&+h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(|\nabla_h u_{ij}^{k-1}|\cdot|\nabla_h\tilde{u}_{ij}^k|\right)\left(|\nabla_h\tilde{u}_{ij}^{k-1}|\cdot|\nabla_h U_{ij}^k|\right)\\
\leqslant&c_4h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h\tilde{u}_{ij}^{k-1}|\cdot|\nabla_h\tilde{u}_{ij}^k|\\
&+h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(|\nabla_h u_{ij}^{k-1}|^2\cdot|\nabla_h\tilde{u}_{ij}^k|^2+\frac{1}{4}|\nabla_h\tilde{u}_{ij}^{k-1}|^2|\nabla_h U_{ij}^k|^2\right)\\
\leqslant&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h u_{ij}^{k-1}|^2|\nabla_h\tilde{u}_{ij}^k|^2+\frac{c_4}{2}\left(\|\nabla_h\tilde{u}^{k-1}\|^2+\|\nabla_h\tilde{u}^k\|^2\right)+\frac{c_4}{4}\|\nabla_h\tilde{u}^{k-1}\|^2.
\end{aligned}
$$

将上式代入 (10.60), 得

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\|\tilde{u}^k\|^2-\|\tilde{u}^{k-1}\|^2\right)+\delta\|\Delta_h\tilde{u}^k\|^2\\
\leqslant&\frac{3c_4}{4}\|\nabla_h\tilde{u}^{k-1}\|^2+\frac{c_4}{2}\|\nabla_h\tilde{u}^k\|^2+(\nabla_h\tilde{u}^{k-1},\nabla_h\tilde{u}^k)\\
&-(\hat{Q}^k,\Delta_x\tilde{u}^k)-(\hat{R}^k,\Delta_y\tilde{u}^k)+(\hat{P}^k,\tilde{u}^k)\\
\leqslant&\frac{3c_4}{4}\|\nabla_h\tilde{u}^{k-1}\|^2+\frac{c_4}{2}\|\nabla_h\tilde{u}^k\|^2+\frac{1}{2}\|\nabla_h\tilde{u}^{k-1}\|^2+\frac{1}{2}\|\nabla_h\tilde{u}^k\|^2\\
&+\frac{1}{2}\|\Delta_x\tilde{u}^k\|^2+\frac{1}{2}\|\hat{Q}^k\|^2+\frac{1}{2}\|\Delta_y\tilde{u}^k\|^2+\frac{1}{2}\|\hat{R}^k\|^2+\frac{1}{2}\|\tilde{u}^k\|^2+\frac{1}{2}\|\hat{P}^k\|^2\\
=&\left(\frac{3c_4}{4}+\frac{1}{2}\right)\|\nabla_h\tilde{u}^{k-1}\|^2+\left(\frac{c_4}{2}+1\right)\|\nabla_h\tilde{u}^k\|^2\\
&+\frac{1}{2}\|\tilde{u}^k\|^2+\frac{1}{2}\left(\|\hat{P}^k\|^2+\|\hat{Q}^k\|^2+\|\hat{R}^k\|^2\right)\\
\leqslant&\left(\frac{3c_4}{4}+\frac{1}{2}\right)\|\Delta_h\tilde{u}^{k-1}\|\cdot\|\tilde{u}^{k-1}\|+\left(\frac{c_4}{2}+1\right)\|\Delta_h\tilde{u}^k\|\cdot\|\tilde{u}^k\|\\
&+\frac{1}{2}\|\tilde{u}^k\|^2+\frac{1}{2}\left(\|\hat{P}^k\|^2+\|\hat{Q}^k\|^2+\|\hat{R}^k\|^2\right)
\end{aligned}
$$

$$\leqslant \frac{\delta}{2}\|\Delta_h \tilde{u}^{k-1}\|^2 + \frac{1}{2\delta}\left(\frac{3c_4}{4}+\frac{1}{2}\right)^2\|\tilde{u}^{k-1}\|^2 + \frac{\delta}{2}\|\Delta_h \tilde{u}^k\|^2 + \frac{1}{2\delta}\left(\frac{c_4}{2}+1\right)^2\|\tilde{u}^k\|^2$$
$$+\frac{1}{2}\|\tilde{u}^k\|^2 + \frac{1}{2}\left(\|\hat{P}^k\|^2 + \|\hat{Q}^k\|^2 + \|\hat{R}^k\|^2\right), \quad 1 \leqslant k \leqslant n.$$

记

$$F^k = \|\tilde{u}^k\|^2 + \tau\delta\|\Delta_h \tilde{u}^k\|.$$

注意到 (10.43)–(10.45), 有

$$\frac{1}{2\tau}(F^k - F^{k-1})$$
$$\leqslant \max\left\{\frac{1}{2\delta}\left(\frac{3c_4}{4}+\frac{1}{2}\right)^2, \frac{1}{2\delta}\left(\frac{c_4}{2}+1\right)^2 + \frac{1}{2}\right\}(F^k + F^{k-1})$$
$$+\frac{3}{2}c_5^2(\tau + h_1^2 + h_2^2)^2, \quad 1 \leqslant k \leqslant n.$$

记

$$c_7 = \max\left\{\frac{1}{2\delta}\left(\frac{3c_4}{4}+\frac{1}{2}\right)^2, \ \frac{1}{2\delta}\left(\frac{c_4}{2}+1\right)^2 + \frac{1}{2}\right\},$$

则有

$$(1-2c_7\tau)F^k \leqslant (1+2c_7\tau)F^{k-1} + 3c_5^2\tau(\tau + h_1^2 + h_2^2)^2, \quad 1 \leqslant k \leqslant n.$$

当 $2c_7\tau \leqslant \frac{1}{3}$ 时,

$$F^k \leqslant (1+6c_7\tau)F^{k-1} + \frac{9}{2}c_5^2\tau(\tau + h_1^2 + h_2^2), \quad 1 \leqslant k \leqslant n.$$

由 Gronwall 不等式, 得

$$F^k \leqslant e^{6c_7k\tau}\cdot\frac{3c_5^2}{4c_7}(\tau + h_1^2 + h_2^2)^2, \quad 1 \leqslant k \leqslant n.$$

因而

$$\|\tilde{u}^k\|^2 \leqslant e^{6c_7k\tau}\cdot\frac{3c_5^2}{4c_7}(\tau + h_1^2 + h_2^2)^2, \quad 1 \leqslant k \leqslant n.$$

两边开方即得所要结果. □

10.5 三层线性化向后 Euler 型差分格式

10.5.1 差分格式的建立

记

$$\varphi_{ij} = \varphi(x_i, y_j), \quad \psi_{ij} = \psi(x_i, y_j).$$

由 (10.1)–(10.2), 得

$$u_t(x,y,0) = -\delta\Delta^2\varphi(x,y) + \nabla\cdot(|\nabla\varphi(x,y)|^2\nabla\varphi(x,y)) - \Delta\varphi(x,y) \equiv \psi(x,y).$$

在点 (x_i, y_j, t_1) 处考虑微分方程 (10.7)–(10.8).

由

$$\frac{1}{2}\left[u_t(x,y,t_1) + u_t(x,y,t_0)\right] = \frac{1}{\tau}\left[u(x,y,t_1) - u(x,y,t_0)\right] + O(\tau^2),$$

得

$$u_t(x,y,t_1) = \frac{2}{\tau}\left[u(x,y,t_1) - u(x,y,t_0)\right] - u_t(x,y,t_0) + O(\tau^2).$$

再注意到

$$u(x,y,t_1) = u(x,y,0) + \tau u_t(x,y,0) + O(\tau^2),$$

可得

$$2\nabla_\tau U_{ij}^1 - \psi_{ij} + \delta\Delta_h^2 U_{ij}^1 - \Delta_x V_{ij}^1 - \Delta_y W_{ij}^1 + \bar{\Delta}_h(U_{ij}^0 + \tau\psi_{ij}) = \breve{P}_{ij}^1,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2, \tag{10.61}$$
$$V_{ij}^1 = |\nabla_h(U_{ij}^0 + \tau\psi_{ij})|^2 \Delta_x U_{ij}^1 + \breve{Q}_{ij}^1, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2, \tag{10.62}$$
$$W_{ij}^1 = |\nabla_h(U_{ij}^0 + \tau\psi_{ij})|^2 \Delta_y U_{ij}^1 + \breve{R}_{ij}^1, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2. \tag{10.63}$$

存在常数 c_8 使得

$$|\breve{P}_{ij}^1| \leqslant c_8(\tau^2 + h_1^2 + h_2^2), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2, \tag{10.64}$$
$$|\breve{Q}_{ij}^1| \leqslant c_8(\tau^2 + h_1^2 + h_2^2), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2, \tag{10.65}$$
$$|\breve{R}_{ij}^1| \leqslant c_8(\tau^2 + h_1^2 + h_2^2), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2. \tag{10.66}$$

在点 (x_i, y_j, t_k) $(2 \leqslant k \leqslant n)$ 处考虑微分方程 (10.7)–(10.8). 应用 Taylor 展开式, 可得

$$\nabla_{2\tau} U_{ij}^k + \delta\Delta_h^2 U_{ij}^k - \Delta_x V_{ij}^k - \Delta_y W_{ij}^k + \bar{\Delta}_h\left(2U_{ij}^{k-1} - U_{ij}^{k-2}\right) = \breve{P}_{ij}^k,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 2 \leqslant k \leqslant n, \tag{10.67}$$
$$V_{ij}^k = \left|\nabla_h\left(2U_{ij}^{k-1} - U_{ij}^{k-2}\right)\right|^2 \Delta_x U_{ij}^k + \breve{Q}_{ij}^k,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 2 \leqslant k \leqslant n, \tag{10.68}$$
$$W_{ij}^k = \left|\nabla_h\left(2U_{ij}^{k-1} - U_{ij}^{k-2}\right)\right|^2 \Delta_y U_{ij}^k + \breve{R}_{ij}^k,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 2 \leqslant k \leqslant n. \tag{10.69}$$

存在常数 c_9 使得

$$|\breve{P}_{ij}^k| \leqslant c_9(\tau^2+h_1^2+h_2^2), \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 2\leqslant k\leqslant n, \tag{10.70}$$

$$|\breve{Q}_{ij}^k| \leqslant c_9(\tau^2+h_1^2+h_2^2), \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 2\leqslant k\leqslant n, \tag{10.71}$$

$$|\breve{R}_{ij}^k| \leqslant c_9(\tau^2+h_1^2+h_2^2), \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 2\leqslant k\leqslant n. \tag{10.72}$$

注意到初值条件

$$U_{ij}^0=\varphi_{ij}, \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{10.73}$$

在 (10.61)–(10.63), (10.67)–(10.69) 中略去小量项, 对 (10.7)–(10.9) 及 (10.2) 建立如下差分格式:

对 $0\leqslant k\leqslant n$, 求 u^k, v^k, $w^k\in\mathcal{W}_h$ 使得

$$2\nabla_\tau u_{ij}^1-\psi_{ij}+\delta\Delta_h^2u_{ij}^1-\Delta_xv_{ij}^1-\Delta_yw_{ij}^1+\bar{\Delta}_h(u_{ij}^0+\tau\psi_{ij})=0,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{10.74}$$

$$v_{ij}^1=|\nabla_h(u_{ij}^0+\tau\psi_{ij})|^2\Delta_xu_{ij}^1, \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{10.75}$$

$$w_{ij}^1=|\nabla_h(u_{ij}^0+\tau\psi_{ij})|^2\Delta_yu_{ij}^1, \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{10.76}$$

$$\nabla_{2\tau}u_{ij}^k+\delta\Delta_h^2u_{ij}^k-\Delta_xv_{ij}^k-\Delta_yw_{ij}^k+\bar{\Delta}_h\left(2u_{ij}^{k-1}-u_{ij}^{k-2}\right)=0,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 2\leqslant k\leqslant n, \tag{10.77}$$

$$v_{ij}^k=\left|\nabla_h\left(2u_{ij}^{k-1}-u_{ij}^{k-2}\right)\right|^2\Delta_xu_{ij}^k,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 2\leqslant k\leqslant n, \tag{10.78}$$

$$w_{ij}^k=\left|\nabla_h\left(2u_{ij}^{k-1}-u_{ij}^{k-2}\right)\right|^2\Delta_yu_{ij}^k,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 2\leqslant k\leqslant n, \tag{10.79}$$

$$u_{ij}^0=\varphi_{ij}, \quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2. \tag{10.80}$$

将 (10.75)–(10.76) 代入 (10.74), 将 (10.78)–(10.79) 代入 (10.77), 可得如下差分格式:

求 $u^k\in\mathcal{W}_h$ 满足

$$2\nabla_\tau u_{ij}^1-\psi_{ij}+\delta\Delta_h^2u_{ij}^1-\nabla_h\cdot(|\nabla_h(u_{ij}^0+\tau\psi_{ij})|^2\nabla_hu_{ij}^1)+\bar{\Delta}_h(u_{ij}^0+\tau\psi_{ij})=0,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{10.81}$$

$$\nabla_{2\tau}u_{ij}^k+\delta\Delta_h^2u_{ij}^k-\nabla_h\cdot(|\nabla_h(2u_{ij}^{k-1}-u_{ij}^{k-2})|^2\nabla_hu_{ij}^k)+\bar{\Delta}_h(2u_{ij}^{k-1}-u_{ij}^{k-2})=0,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 2\leqslant k\leqslant n, \tag{10.82}$$

$$u_{ij}^0=\varphi_{ij},\ 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2. \tag{10.83}$$

对 (10.1)–(10.2) 建立差分格式 (10.81)–(10.83).

10.5.2 差分格式解的有界性

定理 10.11　设 $\{u_{ij}^k \mid 1 \leqslant i \leqslant m_1, 1 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n\}$ 是差分格式 (10.81)–(10.83) 的解. 记

$$E^k = \|u^k\|^2 + \|2u^k - u^{k-1}\|^2, \quad 1 \leqslant k \leqslant n,$$

则有

$$E^k \leqslant 6\|u^0 + \frac{\tau}{2}\psi\|^2 + 5\|u^0\|^2 + \frac{5}{4}L_1L_2t_k, \quad 2 \leqslant k \leqslant n, \tag{10.84}$$

$$\|u^1\|^2 \leqslant \left\|u^0 + \frac{\tau}{2}\psi\right\|^2 + \frac{1}{4}L_1L_2t_1. \tag{10.85}$$

证明　(I) 用 u^1 与 (10.81) 的两边作内积, 得

$$\begin{aligned}&2(\nabla_\tau u^1, u^1) - (\psi, u^1) + \delta(\Delta_h^2 u^1, u^1) - (\nabla_h \cdot (|\nabla_h(u^0 + \tau\psi)|^2 \nabla_h u^1), u^1)\\ &+ (\bar{\Delta}_h(u^0 + \tau\psi), u^1) = 0.\end{aligned}$$

注意到

$$2(\nabla_\tau u^1, u^1) - (\psi, u^1) = \frac{2}{\tau}\|u^1\|^2 - \left(\frac{2}{\tau}u^0 + \psi, u^1\right),$$

$$(\Delta_h^2 u^1, u^1) = \|\Delta_h u^1\|^2,$$

$$\begin{aligned}&-(\nabla_h \cdot (|\nabla_h(u^0 + \tau\psi)|^2 \nabla_h u^1), u^1)\\ =&(|\nabla_h(u^0 + \tau\psi)|^2 \nabla_h u^1, \nabla_h u^1)\\ =&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left|\nabla_h(u_{ij}^0 + \tau\psi_{ij})\right|^2 \cdot \left|\nabla_h u_{ij}^1\right|^2,\end{aligned}$$

$$\begin{aligned}&-(\bar{\Delta}_h(u^0 + \tau\psi), u^1)\\ =&\big(\nabla_h(u^0 + \tau\psi), \nabla_h u^1\big)\\ \leqslant&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left|\nabla_h(u_{ij}^0 + \tau\psi_{ij})\right| \cdot \left|\nabla_h u_{ij}^1\right|\\ \leqslant&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(|\nabla_h(U_{ij}^0 + \tau\psi_{ij})|^2 \cdot |\nabla_h u_{ij}^1|^2 + \frac{1}{4}\right)\\ =&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left|\nabla_h(u_{ij}^0 + \tau\psi_{ij})\right|^2 \left|\nabla_h u_{ij}^1\right|^2 + \frac{1}{4}L_1L_2,\end{aligned}$$

有

$$\frac{2}{\tau}\|u^1\|^2 + \delta\|\Delta_h u^1\|^2$$

$$
\begin{aligned}
&\leqslant \left(\frac{2}{\tau}u^0 + \psi, u^1\right) + \frac{1}{4}L_1L_2 \\
&= \frac{2}{\tau}\left(u^0 + \frac{\tau}{2}\psi, u^1\right) + \frac{1}{4}L_1L_2 \\
&\leqslant \frac{1}{\tau}\|u^1\|^2 + \frac{1}{\tau}\left\|u^0 + \frac{\tau}{2}\psi\right\|^2 + \frac{1}{4}L_1L_2.
\end{aligned}
$$

因而

$$\|u^1\|^2 \leqslant \left\|u^0 + \frac{\tau}{2}\psi\right\|^2 + \frac{1}{4}L_1L_2t_1.$$

(II) 用 u^k 与 (10.82) 的两边作内积, 得到

$$
\begin{aligned}
&(\nabla_{2\tau}u^k, u^k) + \delta(\Delta_h^2 u^k, u^k) - (\nabla_h \cdot (|\nabla_h(2u^{k-1} - u^{k-2})|^2 \nabla_h u^k), u^k) \\
&+ (\bar{\Delta}_h(2u^{k-1} - u^{k-2}), u^k) = 0.
\end{aligned}
$$

注意到

$$
\begin{aligned}
&(\nabla_{2\tau}u^k, u^k) \\
=&\frac{1}{4\tau}\left(\|u^k\|^2 - \|u^{k-1}\|^2 + \|2u^k - u^{k-1}\|^2\right. \\
&\left.-\|2u^{k-1} - u^{k-2}\|^2 + \|u^k - 2u^{k-1} + u^{k-2}\|^2\right),
\end{aligned}
$$

$$(\Delta_h^2 u^k, u^k) = \|\Delta_h u^k\|^2,$$

$$
\begin{aligned}
&-\left(\nabla_h \cdot \left(|\nabla_h(2u^{k-1} - u^{k-2})|^2 \nabla_h u^k\right), u^k\right) \\
=&\left(|\nabla_h(2u^{k-1} - u^{k-2})|^2 \nabla_h u^k, \nabla_h u^k\right) \\
=&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left|\nabla_h(2u_{ij}^{k-1} - u_{ij}^{k-2})\right|^2\left|\nabla_h u_{ij}^k\right|^2,
\end{aligned}
$$

$$
\begin{aligned}
&\left(-\bar{\Delta}_h(2u^{k-1} - u^{k-2}), u^k\right) \\
=&\left(\nabla_h(2u^{k-1} - u^{k-2}), \nabla_h u^k\right) \\
=&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\nabla_h\left(2u_{ij}^{k-1} - u_{ij}^{k-2}\right)\cdot\nabla_h u_{ij}^k \\
\leqslant&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(|\nabla_h(2u_{ij}^{k-1} - u_{ij}^{k-2})|^2\cdot|\nabla_h u_{ij}^k|^2 + \frac{1}{4}\right) \\
=&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}|\nabla_h(2u_{ij}^{k-1} - u_{ij}^{k-2})|^2\cdot|\nabla_h u_{ij}^k|^2 + \frac{1}{4}L_1L_2,
\end{aligned}
$$

有

$$\frac{1}{4\tau}(E^k - E^{k-1}) + \delta\|\Delta_h u^k\|^2 \leqslant \frac{1}{4}L_1L_2, \quad 2 \leqslant k \leqslant n,$$

因而

$$E^k \leqslant E^{k-1} + \tau L_1 L_2 \leqslant \cdots \leqslant E^1 + (k-1)\tau L_1 L_2, \quad 1 \leqslant k \leqslant n.$$

注意到

$$\begin{aligned} E^1 &= \|u^1\|^2 + \|2u^1 - u^0\|^2 \\ &= 5\|u^1\|^2 - 4(u^0, u^1) + \|u^0\|^2 \\ &\leqslant 6\|u^1\|^2 + 5\|u^0\|^2 \\ &\leqslant 6\left(\|u^0 + \frac{\tau}{2}\psi\|^2 + \frac{1}{4}L_1 L_2 t_1\right) + 5\|u^0\|^2, \end{aligned}$$

有

$$\begin{aligned} E^k &\leqslant 6\left\|u^0 + \frac{\tau}{2}\psi\right\|^2 + 5\|u^0\|^2 + \left[\frac{3}{2}\tau + (k-1)\tau\right] L_1 L_2 \\ &\leqslant 6\left\|u^0 + \frac{\tau}{2}\psi\right\|^2 + 5\|u^0\|^2 + \frac{5}{4}k\tau L_1 L_2, \quad 2 \leqslant k \leqslant n. \end{aligned}$$

□

注 10.1　综合 (10.84) 和 (10.85) 可得

$$\|u^k\|^2 \leqslant 6\left\|u^0 + \frac{\tau}{2}\psi\right\|^2 + 5\|u^0\|^2 + \frac{5}{4}L_1 L_2 t_k, \quad 1 \leqslant k \leqslant n.$$

注 10.2

$$E^k = (\sqrt{2}-1)^2\left(\|u^k\|^2 + \|u^{k-1}\|^2\right) + 2\left\|\sqrt{2}+1u^k - \sqrt{2}-1u^{k-1}\right\|^2.$$

10.5.3　差分格式的可解性

定理 10.12　差分格式 (10.81)–(10.83) 是唯一可解的.

证明　由 (10.83) 知 u^0 已给定.

(I) (10.81) 是关于 u^1 的线性方程组. 考虑其齐次方程组

$$\frac{2}{\tau}u_{ij}^1 + \delta\Delta_h^2 u_{ij}^1 - \nabla_h \cdot \left(|\nabla_h(u_{ij}^0 + \tau\psi_{ij})|^2 \nabla_h u_{ij}^1\right) = 0, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2.$$

用 u^1 与上式的两边作内积, 得到

$$\frac{2}{\tau}\|u^1\|^2 + \delta\|\Delta_h u^1\|^2 + \left(|\nabla(u^0 + \tau\psi)|^2 \nabla_h u^1, \nabla_h u^1\right) = 0.$$

易知 $\|u^1\| = 0$. 因而 (10.81) 唯一确定的 u^1.

(II) 设 u^{k-2}, u^{k-1} 已知. 则由 (10.82) 可得关于 u^k 的线性方程组. 考虑其齐次方程组

$$\frac{3}{2\tau}u_{ij}^k + \delta\Delta_h^2 u_{ij}^k - \nabla_h\left(|\nabla_h(2u_{ij}^{k-1} - u_{ij}^{k-2})|^2\nabla_h u_{ij}^k\right) = 0, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2.$$

用 u^k 与上式的两边作内积, 可得

$$\frac{3}{2\tau}\|u^k\|^2 + \delta\|\Delta_h u^k\|^2 + \left(|\nabla_h(2u^{k-1} - u^{k-2})|^2\nabla_h u^k, \nabla_h u^k\right) = 0.$$

易知 $\|u^k\| = 0$. 因而 (10.82) 唯一确定 u^k. □

10.5.4 差分格式解的收敛性

定理 10.13 设 $\{U_{ij}^k\,|\,0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 为 (10.1)–(10.2) 的解, $\{u_{ij}^k\,|\,0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 为 (10.81)–(10.83) 的解. 定义误差函数

$$\tilde{u}_{ij}^k = U_{ij}^k - u_{ij}^k, \quad 0 \leqslant i \leqslant m_1, \quad 0 \leqslant j \leqslant m_2, \quad 0 \leqslant k \leqslant n.$$

则存在常数 c_{10} 使得

$$\|\tilde{u}^k\| \leqslant c_{10}(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant k \leqslant n. \tag{10.86}$$

证明 利用 (10.75), (10.76), (10.78), (10.79) 定义 $\{v^k, w^k\}$. 则差分格式 (10.81)–(10.83) 可写为 (10.74)–(10.80).

定义

$$\tilde{v}_{ij}^k = V_{ij}^k - v_{ij}^k, \quad \tilde{w}_{ij}^k = W_{ij}^k - v_{ij}^k.$$

将 (10.61)–(10.63), (10.67)–(10.69), (10.73) 和 (10.74)–(10.80) 依次相减, 得到误差方程组

$$2\nabla_\tau \tilde{u}_{ij}^1 + \delta\Delta_h^2\tilde{u}_{ij}^1 - \Delta_x\tilde{v}_{ij}^1 - \Delta_y\tilde{w}_{ij}^1 = \breve{P}_{ij}^1, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2, \tag{10.87}$$

$$\tilde{v}_{ij}^1 = \left|\nabla_h\left(U_{ij}^0 + \tau\psi_{ij}\right)\right|^2\Delta_x\tilde{u}_{ij}^1 + \breve{Q}_{ij}^1, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2, \tag{10.88}$$

$$\tilde{w}_{ij}^1 = \left|\nabla_h\left(U_{ij}^0 + \tau\psi_{ij}\right)\right|^2\Delta_y\tilde{u}_{ij}^1 + \breve{R}_{ij}^1, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2, \tag{10.89}$$

$$\nabla_{2\tau}u_{ij}^k + \delta\Delta_h^2\tilde{u}_{ij}^k - \Delta_x\tilde{v}_{ij}^k - \Delta_y\tilde{w}_{ij}^k + \bar{\Delta}_h\left(2\tilde{u}_{ij}^{k-1} - \tilde{u}_{ij}^{k-2}\right) = \breve{P}_{ij}^k,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 2 \leqslant k \leqslant n, \tag{10.90}$$

$$\tilde{v}_{ij}^k = \left|\nabla_h\left(2U_{ij}^{k-1} - U_{ij}^{k-2}\right)\right|^2\Delta_x U_{ij}^k - \left|\nabla_h\left(2u_{ij}^{k-1} - u_{ij}^{k-2}\right)\right|^2\Delta_x u_{ij}^k + \breve{Q}_{ij}^k,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 2 \leqslant k \leqslant n, \tag{10.91}$$

$$\tilde{w}_{ij}^k = \left|\nabla_h\left(2U_{ij}^{k-1} - U_{ij}^{k-2}\right)\right|^2\Delta_y U_{ij}^k - \left|\nabla_h\left(2u_{ij}^{k-1} - u_{ij}^{k-2}\right)\right|^2\Delta_y u_{ij}^k + \breve{R}_{ij}^k,$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 2 \leqslant k \leqslant n, \tag{10.92}$$

$$\tilde{u}_{ij}^0 = 0, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2, \tag{10.93}$$

由 (10.93) 易知

$$\|\tilde{u}^0\| = 0. \tag{10.94}$$

(I) 将 (10.87) 的两边与 $\tilde{u}^1$ 作内积, 得

$$2(\nabla_\tau \tilde{u}^1, \tilde{u}^1) + \delta(\Delta_h^2 \tilde{u}^1, \tilde{u}^1) - (\Delta_x \tilde{v}^1, \tilde{u}^1) - (\Delta_y \tilde{w}^1, \tilde{u}^1) = (\breve{P}^1, \tilde{u}^1).$$

注意到

$$\begin{aligned}
&-(\Delta_x \tilde{v}^1, \tilde{u}^1) - (\Delta_y \tilde{w}^1, \tilde{u}^1)\\
=&(\tilde{v}^1, \Delta_x u^1) + (\tilde{w}^1, \Delta_y \tilde{u}^1)\\
=&\left(\left|\nabla_h\left(U^0 + \tau\psi\right)\right|^2 \Delta_x \tilde{u}^1 + \breve{Q}^1, \Delta_x \tilde{u}^1\right) + \left(\left|\nabla_h\left(U^0 + \tau\psi\right)\right|^2 \Delta_y \tilde{u}^1 + \breve{R}^1, \Delta_y \tilde{u}^1\right)\\
=&\left(\left|\nabla_h\left(U^0 + \tau\psi\right)\right|^2 \nabla_h \tilde{u}^1, \nabla_h \tilde{u}^1\right) + \left(\breve{Q}^1, \Delta_x \tilde{u}^1\right) + \left(\breve{R}^1, \Delta_y \tilde{u}^1\right),
\end{aligned}$$

可得

$$\begin{aligned}
&\frac{2}{\tau}\|\tilde{u}^1\|^2 + \delta\|\Delta_h \tilde{u}^1\|^2\\
\leqslant&-(\breve{Q}^1, \Delta_x \tilde{u}^1) - (\breve{R}^1, \Delta_y \tilde{u}^1) + (\breve{P}^1, u^1)\\
\leqslant&\frac{1}{2}\|\nabla_h \tilde{u}^1\|^2 + \frac{1}{2}\|\tilde{u}^1\|^2 + \frac{1}{2}\left(\|\breve{Q}^1\|^2 + \|\breve{R}^1\|^2 + \|\breve{P}^1\|^2\right)\\
\leqslant&\frac{1}{2}\|\tilde{u}^1\|\|\Delta_h \tilde{u}^1\| + \frac{1}{2}\|\tilde{u}^1\|^2 + \frac{3}{2}c_8^2(\tau^2 + h_1^2 + h_2^2)^2\\
\leqslant&\frac{1}{2}\delta\|\Delta_h \tilde{u}^1\|^2 + \frac{1}{2\delta}\cdot\frac{1}{4}\|\tilde{u}^1\|^2 + \frac{1}{2}\|\tilde{u}^1\|^2 + \frac{3}{2}c_8^2(\tau^2 + h_1^2 + h_2^2)^2.
\end{aligned} \tag{10.95}$$

因而

$$\left[1 - \frac{1}{4}\left(\frac{1}{4\delta} + 1\right)\tau\right]\|\tilde{u}^1\|^2 \leqslant \frac{3}{4}c_8^2\tau(\tau^2 + h_1^2 + h_2^2)^2.$$

当 $\left(\frac{1}{4\delta} + 1\right)\tau \leqslant 2$ 时,

$$\|\tilde{u}^1\|^2 \leqslant \frac{3}{2}c_8^2\tau(\tau^2 + h_1^2 + h_2^2)^2. \tag{10.96}$$

再由 (10.95) 可得

$$\begin{aligned}
\frac{\delta}{2}\|\Delta_h \tilde{u}^1\|^2 \leqslant& \left(\frac{1}{8\delta} + \frac{1}{2}\right)\|\tilde{u}^1\|^2 + \frac{3}{2}c_8^2(\tau^2 + h_1^2 + h_2^2)^2\\
\leqslant& \frac{3}{4}\left(\frac{1}{4\delta} + 1\right)c_8^2\tau(\tau^2 + h_1^2 + h_2^2)^2 + \frac{3}{2}c_8^2(\tau^2 + h_1^2 + h_2^2)^2,
\end{aligned}$$

因而,

$$\|\Delta_h \tilde{u}^1\|^2 \leqslant \frac{1}{\delta}\left[\frac{3}{2}\left(\frac{1}{4\delta} + 1\right)\tau + 3\right]c_8^2(\tau^2 + h_1^2 + h_2^2)^2. \tag{10.97}$$

(II) 用 $\tilde{u}^k$ 与 (10.90) 的两边作内积, 得

$$(\nabla_{2\tau}u^k, u^k) + \delta(\Delta_h^2\tilde{u}^k, u^k) - (\Delta_x\tilde{v}^k, \tilde{u}^k) - (\Delta_y\tilde{w}^k, \tilde{u}^k) \tag{10.98}$$

$$+ \left(\bar{\Delta}_h\left(2\tilde{u}^{k-1} - \tilde{u}^{k-2}\right), \tilde{u}^k\right) = (\breve{P}^k, \tilde{u}^k). \tag{10.99}$$

对上式中的诸项分析可得

$$\begin{aligned}(\nabla_{2\tau}\tilde{u}^k, \tilde{u}^k) = \frac{1}{4\tau}\big[\,(\|\tilde{u}^k\|^2 + \|2\tilde{u}^k - \tilde{u}^{k-1}\|^2) - (\|\tilde{u}^{k-1}\|^2 + \|2\tilde{u}^{k-1} - \tilde{u}^{k-2}\|^2)\\ +\|\tilde{u}^k - 2\tilde{u}^{k-1} + \tilde{u}^{k-2}\|^2\big],\end{aligned} \tag{10.100}$$

$$(\Delta_h^2\tilde{u}^k, \tilde{u}^k) = \|\Delta_h\tilde{u}^k\|^2, \tag{10.101}$$

$$\begin{aligned}&-(\Delta_x\tilde{v}^k, \tilde{u}^k) - (\Delta_y\tilde{w}^k, \tilde{u}^k)\\
=&(\tilde{v}^k, \Delta_x\tilde{u}^k) + (\tilde{w}^k, \Delta_y\tilde{u}^k)\\
=&\big(|\nabla_h(2U^{k-1} - U^{k-2})|^2\Delta_xU^k - |\nabla_h(2u^{k-1} - u^{k-2})|^2\Delta_yu^k + \breve{Q}^k, \Delta_x\tilde{u}^k\big)\\
&+\big(|\nabla_h(2U^{k-1} - U^{k-2})|^2\Delta_yU^k - |\nabla_h(2u^{k-1} - u^{k-2})|^2\Delta_yu^k + \breve{R}^k, \Delta_y\tilde{u}^k\big)\\
=&\big(|\nabla_h(2u^{k-1} - u^{k-2})|^2(\Delta_xU^k - \Delta_xu^k)\\
&+\big(|\nabla_h(2U^{k-1} - U^{k-2})|^2 - |\nabla_h(2u^{k-1} - u^{k-2})|^2\big)\Delta_xU^k, \Delta_x\tilde{u}^k\big)\\
&+\big(|\nabla_h(2u^{k-1} - u^{k-2})|^2(\Delta_yU^k - \Delta_yu^k)\\
&+\big(|\nabla_h(2U^{k-1} - U^{k-2})|^2 - |\nabla_h(2u^{k-1} - u^{k-2})|^2\big)\Delta_yU^k, \Delta_y\tilde{u}^k\big)\\
&+(\breve{Q}^k, \Delta_x\tilde{u}^k) + (\breve{R}^k, \Delta_y\tilde{u}^k)\\
=&\big(|\nabla_h(2u^{k-1} - u^{k-2})|^2\nabla_h\tilde{u}^k, \nabla_h\tilde{u}^k\big)\\
&+\big((|\nabla_h(2U^{k-1} - U^{k-2})|^2 - |\nabla_h(2u^{k-1} - u^{k-2})|^2)\nabla_hU^k, \nabla_h\tilde{u}^k\big)\\
&+(\breve{Q}^k, \Delta_x\tilde{u}^k) + (\breve{R}^k, \Delta_y\tilde{u}^k),\end{aligned} \tag{10.102}$$

$$\begin{aligned}&-\big(\bar{\Delta}_h(2\tilde{u}^{k-1} - \tilde{u}^{k-2}), \tilde{u}^k\big)\\
=&\big(\nabla_h(2\tilde{u}^{k-1} - \tilde{u}^{k-2}), \nabla_h\tilde{u}^k\big)\\
\leqslant&\|\nabla_h(2\tilde{u}^{k-1} - \tilde{u}^{k-2})\|\cdot\|\nabla_h\tilde{u}^k\|.\end{aligned} \tag{10.103}$$

将 (10.100)–(10.103) 代入 (10.99), 得到

$$\begin{aligned}&\frac{1}{4\tau}\left[(\|\tilde{u}^k\|^2 + \|2\tilde{u}^k - \tilde{u}^{k-1}\|^2) - (\|\tilde{u}^{k-1}\| + \|2\tilde{u}^{k-1} - \tilde{u}^{k-2}\|^2)\right] + \delta\|\Delta_h\tilde{u}^k\|^2\\
&+\big(|\nabla_h(2u^{k-1} - u^{k-2})|^2\nabla_h\tilde{u}^k, \nabla_h\tilde{u}^k\big)\\
=&-\big((|\nabla_h(2U^{k-1} - U^{k-2})|^2 - |\nabla_h(2u^{k-1} - u^{k-2})|^2)\nabla_hU^k, \nabla_h\tilde{u}^k\big)\end{aligned}$$

$$+\|\nabla_h(2\tilde{u}^{k-1}-\tilde{u}^{k-2})\|\cdot\|\nabla_h\tilde{u}^k\|$$
$$+(\breve{P}^k,\tilde{u}^k)-(\breve{Q}^k,\Delta_x\tilde{u}^k)-(\breve{R}^k,\Delta_y\tilde{u}^k). \tag{10.104}$$

注意到

$$\|\nabla_h U^k\|^2\leqslant c_4,$$

有

$$\begin{aligned}
&-\left((|\nabla_h(2U^{k-1}-U^{k-2})|^2-|\nabla_h(2u^{k-1}-u^{k-2})|^2)\nabla_h U^k,\nabla_h\tilde{u}^k\right)\\
=&-\Big(\big(\nabla_h(2U^{k-1}-U^{k-2})\cdot\nabla_h(2\tilde{u}^{k-1}-\tilde{u}^{k-2})\\
&+\nabla_h(2u^{k-1}-u^{k-2})\cdot\nabla_h(2\tilde{u}^{k-1}-\tilde{u}^{k-2})\big)\nabla_h U^k,\ \nabla_h\tilde{u}^k\Big)\\
\leqslant& h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left|\nabla_h\left(2U_{ij}^{k-1}-U_{ij}^{k-2}\right)\right|\cdot\left|\nabla_h\left(2\tilde{u}_{ij}^{k-1}-\tilde{u}_{ij}^{k-2}\right)\right|\cdot\left|\nabla_h U_{ij}^k\right|\cdot\left|\nabla_h\tilde{u}_{ij}^k\right|\\
&+h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left|\nabla_h\left(2u_{ij}^{k-1}-u_{ij}^{k-2}\right)\right|\cdot\left|\nabla_h\left(2\tilde{u}_{ij}^{k-1}-\tilde{u}_{ij}^{k-2}\right)\right|\cdot\left|\nabla_h U_{ij}^k\right|\cdot\left|\nabla_h\tilde{u}_{ij}^k\right|\\
\leqslant& 3c_4h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left|\nabla_h\left(2\tilde{u}_{ij}^{k-1}-\tilde{u}_{ij}^{k-2}\right)\right|\cdot\left|\nabla_h\tilde{u}_{ij}^k\right|\\
&+h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\Big(\left|\nabla_h\left(2u_{ij}^{k-1}-u_{ij}^{k-2}\right)\right|^2\left|\nabla_h\tilde{u}_{ij}^k\right|^2\\
&+\frac{1}{4}\left|\nabla_h\left(2\tilde{u}_{ij}^{k-1}-\tilde{u}_{ij}^{k-2}\right)\right|^2\left|\nabla_h U_{ij}^k\right|^2\Big)\\
\leqslant&\left(\left|\nabla_h\left(2u^{k-1}-u^{k-2}\right)\right|^2\nabla_h\tilde{u}^k,\nabla_h\tilde{u}^k\right)\\
&+3c_4\|\nabla_h(2\tilde{u}^{k-1}-\tilde{u}^{k-2})\|\cdot\|\nabla_h\tilde{u}^k\|+\frac{1}{4}c_4\|\nabla_h(2\tilde{u}^{k-1}-\tilde{u}^{k-2})\|^2.
\end{aligned}$$

将上式代入 (10.104) 得

$$\begin{aligned}
&\frac{1}{4\tau}\left[(\|\tilde{u}^k\|^2+\|2\tilde{u}^k-\tilde{u}^{k-1}\|^2)-(\|\tilde{u}^{k-1}\|^2+\|2\tilde{u}^{k-1}-\tilde{u}^{k-2}\|^2)\right]+\delta\|\Delta_h\tilde{u}^k\|^2\\
\leqslant&\ 3c_4\|\nabla_h\left(2\tilde{u}^{k-1}-\tilde{u}^{k-2}\right)\|\cdot\|\nabla_h\tilde{u}^k\|+\frac{1}{4}c_4\|\nabla_h(2\tilde{u}^{k-1}-\tilde{u}^{k-2})\|^2\\
&+\|\nabla_h(2\tilde{u}^{k-1}-\tilde{u}^{k-2})\|\cdot\|\nabla_h\tilde{u}^k\|+\frac{1}{2}(\|\tilde{u}^k\|+\|\breve{P}^k\|^2)+\frac{1}{2}(\|\Delta_x\tilde{u}^k\|^2+\|\hat{Q}^k\|^2)\\
&+\frac{1}{2}\left(\|\Delta_y\tilde{u}^k\|^2+\|\breve{R}^k\|^2\right)\\
\leqslant&(1+3c_4)\cdot\frac{1}{2}\left(\left\|\nabla_h\left(2\tilde{u}^{k-1}-\tilde{u}^{k-2}\right)\right\|^2+\left\|\nabla_h\tilde{u}^k\right\|^2\right)+\frac{1}{4}c_4\left\|\nabla_h\left(2\tilde{u}^{k-1}-\tilde{u}^{k-2}\right)\right\|^2\\
&+\frac{1}{2}\|\nabla_h\tilde{u}^k\|^2+\frac{1}{2}\|\tilde{u}^k\|^2+\frac{1}{2}\left(\|\breve{P}^k\|^2+\|\breve{Q}^k\|^2+\|\breve{R}^k\|^2\right)
\end{aligned}$$

$$= \left(\frac{1}{2}+\frac{7c_4}{4}\right)\|\nabla_h(2\tilde{u}^{k-1}-\tilde{u}^{k-2})\|^2+\left(1+\frac{3}{2}c_4\right)\|\nabla_h\tilde{u}^k\|^2+\frac{1}{2}\|\tilde{u}^k\|^2 + \frac{1}{2}\left(\|\breve{P}^k\|^2+\|\breve{Q}^k\|^2+\|\breve{R}^k\|^2\right)$$

$$\leqslant \left(\frac{1}{2}+\frac{7c_4}{4}\right)\|\Delta_h(2\tilde{u}^{k-1}-\tilde{u}^{k-2})\|\cdot\|2\tilde{u}^{k-1}-\tilde{u}^{k-2}\|+\left(1+\frac{3}{2}c_4\right)\|\Delta_h\tilde{u}^k\|\cdot\|\tilde{u}^k\| + \frac{1}{2}\|\tilde{u}^k\|^2+\frac{1}{2}(\|\breve{P}^k\|^2+\|\breve{Q}^k\|^2+\|\breve{R}^k\|^2)$$

$$\leqslant \varepsilon\|\Delta_h(2\tilde{u}^{k-1}-\tilde{u}^{k-2})\|^2+\frac{1}{4\varepsilon}\left(\frac{1}{2}+\frac{7c_4}{4}\right)^2\|2\tilde{u}^{k-1}-\tilde{u}^{k-2}\|^2+\frac{\delta}{3}\|\Delta_h\tilde{u}^k\|^2 + \left[\frac{1}{2}+\frac{3}{4\delta}\left(1+\frac{3}{2}c_4\right)^2\right]\|\tilde{u}^k\|^2+\frac{1}{2}(\|\breve{p}^k\|^2+\|\breve{Q}^k\|^2+\|\breve{R}^k\|^2).$$

注意到

$$\begin{aligned}
&\|2\tilde{u}^{k-1}-\tilde{u}^{k-2}\|^2\\
=&4\|\tilde{u}^{k-1}\|^2-4(\tilde{u}^{k-1},\tilde{u}^{k-2})+\|\tilde{u}^{k-2}\|^2\\
\leqslant& 4\|\tilde{u}^{k-1}\|^2+\left(\|\tilde{u}^{k-1}\|^2+4\|\tilde{u}^{k-2}\|^2\right)+\|\tilde{u}^{k-2}\|^2\\
=&5\left(\|\tilde{u}^{k-1}\|^2+\|\tilde{u}^{k-2}\|^2\right),
\end{aligned}$$

$$\left\|\Delta_h\left(2\tilde{u}^{k-1}-\tilde{u}^{k-2}\right)\right\|^2\leqslant 5\left(\|\Delta_h\tilde{u}^{k-1}\|^2+\|\Delta_h\tilde{u}^{k-2}\|^2\right),$$

得到

$$\frac{1}{4\tau}\left[(\|\tilde{u}^k\|^2+\|2\tilde{u}^k-\tilde{u}^{k-1}\|^2)-(\|\tilde{u}^{k-1}\|^2+\|2\tilde{u}^{k-1}-\tilde{u}^{k-2}\|^2)\right]+\frac{2}{3}\delta\|\Delta_h\tilde{u}^k\|^2$$
$$\leqslant 5\varepsilon\left(\|\Delta_h\tilde{u}^{k-1}\|^2+\|\Delta_h\tilde{u}^{k-2}\|^2\right)+\frac{5}{4\varepsilon}\left(\frac{1}{2}+\frac{7c_4}{4}\right)^2\left(\|\tilde{u}^{k-1}\|^2+\|\tilde{u}^{k-2}\|^2\right) + \left[\frac{1}{2}+\frac{3}{4\delta}\left(1+\frac{3}{2}c_4\right)^2\right]\|\tilde{u}^k\|^2+\frac{1}{2}\left(\|\breve{P}^k\|^2+\|\breve{Q}^k\|^2+\|\breve{R}^k\|^2\right).$$

取 $\varepsilon=\frac{\delta}{15}$, 并注意到 (10.70)–(10.72) 得到

$$\frac{1}{4\tau}\left[(\|\tilde{u}^k\|^2+\|2\tilde{u}^k-\tilde{u}^{k-1}\|^2)-(\|\tilde{u}^{k-1}\|^2+\|2\tilde{u}^{k-1}-\tilde{u}^{k-2}\|^2)\right] + \frac{\delta}{3}\left(2\|\Delta_h\tilde{u}^k\|^2-\|\Delta_h\tilde{u}^{k-1}\|^2-\|\Delta_h\tilde{u}^{k-2}\|^2\right)$$
$$\leqslant\left[\frac{1}{2}+\frac{3}{4\delta}\left(1+\frac{3}{2}c_4\right)^2\right]\|\tilde{u}^k\|^2+\frac{75}{4\delta}\left(\frac{1}{2}+\frac{7c_4}{4}\right)^2\left(\|\tilde{u}^{k-1}\|^2+\|\tilde{u}^{k-2}\|^2\right)$$

$$+\frac{3}{2}c_9^2\left(\tau^2+h_1^2+h_2^2\right)^2,\quad 2\leqslant k\leqslant n.$$

将上式中的 k 换为 l, 并对 l 从 2 到 k 求和, 得

$$\begin{aligned}&\frac{1}{4\tau}\left[\left(\|\tilde{u}^k\|^2+\|2\tilde{u}^k-\tilde{u}^{k-1}\|^2\right)-\left(\|\tilde{u}^1\|^2+\|2\tilde{u}^1-\tilde{u}^0\|^2\right)\right]\\&+\frac{\delta}{3}\left[2\sum_{l=2}^{k}\|\Delta_h\tilde{u}^l\|^2-\sum_{l=2}^{k}\|\Delta_h\tilde{u}^{l-1}\|^2-\sum_{l=2}^{k}\|\Delta_h\tilde{u}^{l-2}\|^2\right]\\\leqslant&\left[\frac{1}{2}+\frac{3}{4\delta}\left(1+\frac{3}{2}c_4\right)^2\right]\sum_{l=2}^{k}\|\tilde{u}^l\|^2+\frac{75}{4\delta}\left(\frac{1}{2}+\frac{7c_4}{4}\right)^2\left(\sum_{l=2}^{k}\|\tilde{u}^{l-1}\|^2+\sum_{l=2}^{k}\|\tilde{u}^{l-2}\|^2\right)\\&+\frac{3}{2}(k-1)c_0^2(\tau^2+h_1^2+h_2^2)^2,\quad 2\leqslant k\leqslant n.\end{aligned}$$

上式变形, 可得

$$\begin{aligned}&\frac{1}{4\tau}\left[\left(\|\tilde{u}^k\|^2+\|2\tilde{u}^k-\tilde{u}^{k-1}\|^2\right)-\left(\|\tilde{u}^1\|^2+\|2\tilde{u}^1-\tilde{u}^0\|^2\right)\right]\\&+\frac{\delta}{3}\left[2\|\Delta_h\tilde{u}^k\|^2+\|\Delta_h\tilde{u}^{k-1}\|^2-2\|\Delta_h\tilde{u}^1\|^2-\|\Delta_h\tilde{u}^0\|^2\right]\\\leqslant&\left[\frac{1}{2}+\frac{3}{4\delta}\left(1+\frac{3}{2}c_4\right)^2\right]\sum_{l=2}^{k}\|\tilde{u}^l\|^2+\frac{75}{4\delta}\left(\frac{1}{2}+\frac{7c_4}{4}\right)^2\left(\sum_{l=1}^{k-1}\|\tilde{u}^l\|^2+\sum_{l=0}^{k-2}\|\tilde{u}^l\|^2\right)\\&+\frac{3}{2}(k-1)c_9^2(\tau^2+h_1^2+h_2^2)^2,\quad 2\leqslant k\leqslant n.\end{aligned}$$

注意到 (10.93), (10.96), (10.97), 存在常数 c_{10} 使得

$$\|\tilde{u}^k\|^2\leqslant c_{10}\tau\sum_{l=2}^{k-1}\|\tilde{u}^l\|^2+c_{10}(\tau^2+h_1^2+h_2^2)^2,\quad 2\leqslant k\leqslant n.$$

由 Gronwall 不等式, 得到

$$\|\tilde{u}^k\|^2\leqslant e^{c_{10}T}\cdot c_{10}(\tau^2+h_1^2+h_2^2)^2,\quad 2\leqslant k\leqslant n. \tag{10.105}$$

综合 (10.94), (10.96) 及 (10.105), 定理得证. □

10.6 小结与延拓

本章考虑了外延增长模型 (10.1)–(10.2) 周期边界值问题的数值求解. 建立了三个差分格式, 分别为二层非线性向后 Euler 格式、二层线性化向后 Euler 差分格式、三层线性化向后 Euler 型差分格式, 分析了差分格式解有界性、存在性、唯一性和收敛性. 二层线性化向后 Euler 格式取自 [23], 三层线性化向后 Euler 型差分格式取自 [14].

对问题 (10.1)–(10.2) 还可以建立如下四层线性化 Crank-Nicolson 型差分格式 [23]:

求 $u^k \in \mathcal{W}_h$

$$\delta_t u_{ij}^{k+\frac{1}{2}} + \delta\Delta_h^2 u_{ij}^{k+\frac{1}{2}} - \nabla_h \cdot \left(|\nabla_h \hat{u}_{ij}^k|^2 \nabla_h u_{ij}^{k+\frac{1}{2}}\right) + \bar{\Delta}_h u_{ij}^{\hat{k}} = 0,$$

$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 2 \leqslant k \leqslant n-1, \tag{10.106}$$

$$\delta_t u_{ij}^{k+\frac{1}{2}} + \delta\Delta_h^2 u_{ij}^{k+\frac{1}{2}} - \nabla_h \left(|\nabla_h \hat{\phi}_{ij}^k|^2 \nabla_h u_{ij}^{k+\frac{1}{2}}\right) + \bar{\Delta}_h \hat{\phi}^k = 0,$$

$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ k = 0, 1 \tag{10.107}$$

$$u_{ij}^0 = \varphi(x_i, y_j), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2, \tag{10.108}$$

其中

$$\hat{\phi}_{ij}^k = u(x_i, y_j, 0) + \left(k + \frac{1}{2}\right) \tau u_t(x_i, y_j, 0), \quad k = 0, 1,$$

$$\hat{u}_{ij}^k = 2u_{ij}^{k-\frac{1}{2}} - u_{ij}^{k-\frac{3}{2}}, \quad 2 \leqslant k \leqslant n-1.$$

可以证明差分格式 (10.106)–(10.108) 解的有界性、存在性、唯一性和收敛性.

考虑形如

$$\phi_t = -\nabla \cdot \left(\frac{\nabla\phi}{1 + |\nabla\phi|^2}\right) - \delta\Delta^2\phi \tag{10.109}$$

的梯度流.

上式可以进一步写为

$$\phi_t = \nabla \cdot \left(\frac{|\nabla\phi|^2}{1 + |\nabla\phi|^2} \nabla\phi\right) - \Delta\phi - \delta\Delta^2\phi. \tag{10.110}$$

当 $|\nabla\phi|^2 \ll 1$ 时, 可用下式近似代替 (10.110):

$$\phi_t = \nabla \cdot (|\nabla\phi|^2 \nabla\phi) - \Delta\phi - \delta\Delta^2\phi. \tag{10.111}$$

称 (10.111) 为有斜率选择的外延增长模型方程, 而称 (10.109) 为无斜率选择的外延增长模型方程 [38]. 关于无斜率选择的方程 (10.109) 的数值求解可以参考 [24] 和 [38].

第 11 章　相场晶体模型方程的差分方法

11.1 引　　言

本章考虑二维相场晶体模型问题

$$\phi_t = \nabla\cdot(M(\phi)\nabla\mu), \quad (x,y)\in\mathcal{R}^2,\ 0<t\leqslant T, \tag{11.1}$$

$$\mu = \Delta^2\phi + 2\Delta\phi + \phi^3 + (1-\gamma)\phi, \quad (x,y)\in\mathcal{R}^2,\ 0<t\leqslant T, \tag{11.2}$$

$$\phi(x,y,0) = \psi(x,y), \quad (x,y)\in\mathcal{R}^2 \tag{11.3}$$

的差分方法, 其中 γ 是一个小于 1 的正常数, ∇ 为梯度算子, Δ 为 Laplace 算子, μ 是化学势, $M(\phi)$ 为迁移率, ϕ 在 R^2 上关于盒子 $\Omega=(0,L_1)\times(0,L_2)$ 是周期的.

定理 11.1　设 $\{\phi(x,y,t),\mu(x,y,t)\}$ 为 (11.1)–(11.3) 的解. 记

$$E(\phi(\cdot,\cdot,t)) = \iint\limits_{\Omega}\left[\frac{1}{2}(\Delta\phi)^2 - |\nabla\phi|^2 + \frac{1}{4}\phi^4 + \frac{1-\gamma}{2}\phi^2\right]\mathrm{d}x\mathrm{d}y,$$

则有

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\phi(\cdot,\cdot,t)) + \iint\limits_{\Omega} M(\phi)|\nabla\mu|^2\mathrm{d}x\mathrm{d}y = 0, \quad 0<t\leqslant T. \tag{11.4}$$

证明　用 μ 与 (11.1) 相乘, 并关于 (x,y) 在 Ω 上积分, 得

$$\iint\limits_{\Omega}\phi_t\mu\mathrm{d}x\mathrm{d}y = \iint\limits_{\Omega}\left[\nabla\cdot(M(\phi)\nabla\mu)\right]\mu\mathrm{d}x\mathrm{d}y.$$

将 (11.2) 代入上式, 得

$$\iint\limits_{\Omega}[(\Delta^2\phi) + 2(\Delta\phi) + \phi^3 + (1-\gamma)\phi]\phi_t\mathrm{d}x\mathrm{d}y = \iint\limits_{\Omega}[\nabla\cdot(M(\phi)\nabla\mu)]\mu\mathrm{d}x\mathrm{d}y.$$

应用分部求积公式及周期边界条件, 得

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint\limits_{\Omega}\left[\frac{1}{2}(\Delta\phi)^2 - |\nabla\phi|^2 + \frac{1}{4}\phi^4 + \frac{1-\gamma}{2}\phi^2\right]\mathrm{d}x\mathrm{d}y + \iint\limits_{\Omega}M(\phi)|\nabla\mu|^2\mathrm{d}x\mathrm{d}y = 0,$$

即

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\phi(\cdot,\cdot,t)) + \iint\limits_{\Omega}M(\phi)|\nabla\mu|^2 dxdy = 0, \quad 0<t\leqslant T.$$ □

由定理 11.1 易得

$$E(\phi(\cdot,\cdot,t)) \leqslant E(\phi(\cdot,\cdot,0)), \quad 0 < t \leqslant T.$$

为简单起见, 以下仅考虑 $M(\phi) \equiv 1$ 的情形.

11.2 记号与基本引理

取正整数 m_1, m_2, n. 记

$$\begin{aligned}
&h_1 = L_1/m_1, \quad h_2 = L_2/m_2, \quad \tau = T/n;\\
&x_i = ih_1, \quad 0 \leqslant i \leqslant m_1; \quad y_j = jh_2, \quad 0 \leqslant j \leqslant m_2;\\
&t_k = k\tau, \quad 0 \leqslant k \leqslant n; \quad t_{k+\frac{1}{2}} = \frac{1}{2}(t_k + t_{k+1}), \quad 0 \leqslant k \leqslant n-1.
\end{aligned}$$

假设存在正常数 β 使得 $\beta^{-1} \leqslant \dfrac{h_1}{h_2} \leqslant \beta$. 记

$$\begin{aligned}
&\Omega_{h_1,h_2} = \{(x_i, y_j)\,|\,0 \leqslant i \leqslant m_1, 0 \leqslant j \leqslant m_2\}, \quad \Omega_\tau = \{t_k\,|\,0 \leqslant k \leqslant n\},\\
&\mathcal{W}_h = \{u\,|\,u = \{u_{ij}\}, u_{i+m_1,j} = u_{i,j}, u_{i,j} = u_{i,j+m_2}\}.
\end{aligned}$$

对于 $u, v \in \mathcal{W}_h$, 记

$$\begin{aligned}
&\delta_x u_{i+\frac{1}{2},j} = \frac{1}{h_1}(u_{i+1,j} - u_{ij}), \quad \delta_y u_{i,j+\frac{1}{2}} = \frac{1}{h_2}(u_{i,j+1} - u_{i,j}),\\
&\delta_x^2 u_{ij} = \frac{1}{h_1}(\delta_x u_{i+\frac{1}{2},j} - \delta_x u_{i-\frac{1}{2}}, j), \quad \delta_y^2 u_{ij} = \frac{1}{h_2}(\delta_y u_{i,j+\frac{1}{2}} - \delta_y u_{i,j-\frac{1}{2}}),\\
&\Delta_h u_{ij} = \delta_x^2 u_{ij} + \delta_y^2 u_{ij},\\
&(u, v) = h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} u_{ij} v_{ij}, \quad \|u\| = \sqrt{(u, u)},\\
&\langle \delta_x u, \delta_x v\rangle = h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} (\delta_x u_{i-\frac{1}{2},j})(\delta_x v_{i-\frac{1}{2},j}), \quad \|\delta_x u\| = \sqrt{\langle \delta_x u, \delta_x u\rangle},\\
&\langle \delta_y u, \delta_y v\rangle = h_1 h_2 \sum_{j=1}^{m_1}\sum_{j=1}^{m_2} (\delta_y u_{i,j-\frac{1}{2}})(\delta_y v_{i,j-\frac{1}{2}}), \quad \|\delta_y u\| = \sqrt{\langle \delta_y u, \delta_y u\rangle},\\
&|u|_1 = \sqrt{\|\delta_x u\|^2 + \|\delta_y u\|^2},\\
&(\Delta_h u, \Delta_h v) = h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} (\Delta_h u_{ij})(\Delta_h v_{ij}), \quad \|\Delta_h u\| = \sqrt{(\Delta_h u, \Delta_h u)},
\end{aligned}$$

$$|\Delta u|_1 = \sqrt{h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left\{[\delta_x(\Delta_h u)_{i-\frac{1}{2},j}]^2 + [\delta_y(\Delta_h u)_{i,j-\frac{1}{2}}]^2\right\}},$$

$$\|u\|_\infty = \max_{1\leqslant i\leqslant m_1, 1\leqslant j\leqslant m_2} |u_{ij}|, \quad \|u\|_4 = \sqrt[4]{h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}(u_{ij})^4}.$$

引理 11.1　设 $u \in \mathcal{W}_h$, 则有

$$-(\Delta_h u, u) = |u|_1^2. \tag{11.5}$$

证明　由分部求和公式及周期性可得

$$\begin{aligned}
&-(\Delta_h u, u)\\
&= -h_1 h_2 \sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left(\delta_x^2 u_{i,j} + \delta_y^2 u_{ij}\right) u_{ij}\\
&= h_2 \sum_{j=1}^{m_2}\left[-h_1 \sum_{i=1}^{m_1}(\delta_x^2 u_{i,j}) u_{ij}\right] + h_1 \sum_{i=1}^{m_1}\left[-h_2 \sum_{j=1}^{m_2}(\delta_y^2 u_{ij}) u_{ij}\right]\\
&= h_2 \sum_{j=1}^{m_2}\left[h_1 \sum_{i=1}^{m_1}(\delta_x u_{i-\frac{1}{2},j})^2\right] + h_1 \sum_{i=1}^{m_1}\left[h_2 \sum_{j=1}^{m_2}(\delta_y u_{i,j-\frac{1}{2}})^2\right]\\
&= |u|_1^2.
\end{aligned}$$

□

引理 11.2　设 $u \in \mathcal{W}_h$, 对任意的 $\varepsilon > 0$ 有

$$\|u\|_\infty \leqslant \varepsilon\|\Delta_h u\| + \sqrt{3}\left[\frac{1}{\varepsilon} + \frac{1}{2}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)\right]\|u\|.$$

证明　设

$$\|u\|_\infty = |u_{i_0,j_0}|.$$

由引理 1.1 (e), 有

$$\begin{aligned}
u_{i_0,j_0}^2 \leqslant & \varepsilon h_1 \sum_{i=1}^{m_1} (\delta_x u_{i-\frac{1}{2},j_0})^2 + \left(\frac{1}{\varepsilon} + \frac{1}{L_1}\right) h_1 \sum_{i=1}^{m_1} u_{i,j_0}^2 \\
\leqslant & \varepsilon h_1 \sum_{i=1}^{m_1} \left\{ \varepsilon h_2 \sum_{j=1}^{m_2} (\delta_y \delta_x u_{i-\frac{1}{2},j-\frac{1}{2}})^2 + \left(\frac{1}{\varepsilon} + \frac{1}{L_2}\right) h_2 \sum_{j=1}^{m_2} (\delta_x u_{i-\frac{1}{2},j})^2 \right\} \\
& + \left(\frac{1}{\varepsilon} + \frac{1}{L_1}\right) h_1 \sum_{i=1}^{m_1} \left\{ \varepsilon h_2 \sum_{j=1}^{m_2} (\delta_y u_{i,j-\frac{1}{2}})^2 + \left(\frac{1}{\varepsilon} + \frac{1}{L_2}\right) h_2 \sum_{j=1}^{m_2} u_{i,j}^2 \right\} \\
\leqslant & \varepsilon^2 \|\delta_y \delta_x u\|^2 + \left[\varepsilon \left(\frac{1}{L_1} + \frac{1}{L_2}\right) + 2\right] |u|_1^2 \\
& + \left(\frac{1}{\varepsilon} + \frac{1}{L_1}\right)\left(\frac{1}{\varepsilon} + \frac{1}{L_2}\right) \|u\|^2.
\end{aligned} \tag{11.6}$$

注意到

$$\begin{aligned}
& \|\Delta_h u\|^2 \\
= & h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(\delta_x^2 u_{ij} + \delta_y^2 u_{ij}\right)^2 \\
= & h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(\delta_x^2 u_{ij}\right)^2 + 2h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(\delta_x^2 u_{ij}\right)\left(\delta_y^2 u_{ij}\right) + h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(\delta_y^2 u_{ij}\right)^2 \\
= & h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(\delta_x^2 u_{ij}\right)^2 + 2h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(\delta_x \delta_y u_{i-\frac{1}{2},j-\frac{1}{2}}\right)^2 + h_1 h_2 \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \left(\delta_y^2 u_{ij}\right)^2 \\
= & \|\delta_x^2 u\|^2 + 2\|\delta_x \delta_y u\|^2 + \|\delta_y^2 u\|^2,
\end{aligned}$$

可得

$$\|\delta_x \delta_y u\|^2 \leqslant \frac{1}{2} \|\Delta_h u\|^2. \tag{11.7}$$

应用引理 11.1, 对任意的 $\varepsilon_1 > 0$, 可得

$$|u|_1^2 = -(\Delta_h u, u) \leqslant \varepsilon_1 \|\Delta_h u\|^2 + \frac{1}{4\varepsilon_1} \|u\|^2. \tag{11.8}$$

将 (11.7) 和 (11.8) 代入 (11.6) 得到

$$\begin{aligned}
u_{i_0,j_0}^2 \leqslant & \frac{1}{2}\varepsilon^2 \|\Delta_h u\|^2 + \left[\varepsilon \left(\frac{1}{L_1} + \frac{1}{L_2}\right) + 2\right] \left(\varepsilon_1 \|\Delta_h u\|^2 + \frac{1}{4\varepsilon_1} \|u\|^2\right) \\
& + \left(\frac{1}{\varepsilon} + \frac{1}{L_1}\right)\left(\frac{1}{\varepsilon} + \frac{1}{L_2}\right) \|u\|^2.
\end{aligned}$$

取 ε_1 使得 $\left[\varepsilon\left(\frac{1}{L_1}+\frac{1}{L_2}\right)+2\right]\varepsilon_1=\frac{1}{2}\varepsilon^2$, 并注意到

$$\frac{4}{L_1L_2}\leqslant\left(\frac{1}{L_1}+\frac{1}{L_2}\right)^2,$$

可得

$$\begin{aligned}u_{i_0,j_0}^2\leqslant&\varepsilon^2\|\Delta_hu\|^2+\left\{\frac{\left[\varepsilon\left(\frac{1}{L_1}+\frac{1}{L_2}\right)+2\right]^2}{2\varepsilon^2}+\left(\frac{1}{\varepsilon}+\frac{1}{L_1}\right)\left(\frac{1}{\varepsilon}+\frac{1}{L_2}\right)\right\}\|u\|^2\\=&\varepsilon^2\|\Delta_hu\|^2+\left[\frac{3}{\varepsilon^2}+\frac{3}{\varepsilon}\left(\frac{1}{L_1}+\frac{1}{L_2}\right)+\frac{1}{2}\left(\frac{1}{L_1}+\frac{1}{L_2}\right)^2+\frac{1}{L_1L_2}\right]\|u\|^2\\\leqslant&\varepsilon^2\|\Delta_hu\|^2+3\left[\frac{1}{\varepsilon}+\frac{1}{2}\left(\frac{1}{L_1}+\frac{1}{L_2}\right)\right]^2\|u\|^2.\end{aligned}$$

两边开平方得到

$$\|u\|_\infty\leqslant\varepsilon\|\Delta_hu\|+\sqrt{3}\left[\frac{1}{\varepsilon}+\frac{1}{2}\left(\frac{1}{L_1}+\frac{1}{L_2}\right)\right]\|u\|.$$

□

引理 11.3 ([37])　设 $u\in\mathcal{W}_h$, 则对任意 $\varepsilon>0$, 有

$$\|\Delta_hu\|^2\leqslant\frac{2\varepsilon}{3}|\Delta_hu|_1^2+\frac{1}{3\varepsilon^2}\|u\|^2.$$

证明　由 $u\in\mathcal{W}_h$ 可知 $\Delta_hu\in\mathcal{W}_h$.

由分部求和公式及周期性可得

$$\begin{aligned}\|\Delta_hu\|^2=&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}(\Delta_hu_{ij})(\Delta_hu_{ij})\\=&h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}(\delta_x^2u_{ij}+\delta_y^2u_{ij})(\Delta_hu_{ij})\\=&-h_1h_2\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}\left[(\delta_xu_{i-\frac{1}{2},j})\delta_x(\Delta_hu_{i-\frac{1}{2},j})+(\delta_yu_{i,j-\frac{1}{2}})\delta_y(\Delta_hu_{i,j-\frac{1}{2}})\right]\\\leqslant&\frac{\varepsilon}{2}|\Delta_hu|_1^2+\frac{1}{2\varepsilon}|u|_1^2.\end{aligned}\tag{11.9}$$

由引理 11.1 可得

$$|u|_1^2=-(\Delta_hu,u)\leqslant\frac{\varepsilon}{2}\|\Delta_hu\|^2+\frac{1}{2\varepsilon}\|u\|^2.\tag{11.10}$$

将 (11.10) 代入 (11.9), 得到

$$\|\Delta_h u\|^2 \leqslant \frac{\varepsilon}{2}|\Delta_h u|_1^2 + \frac{1}{2\varepsilon}\left(\frac{\varepsilon}{2}\|\Delta_h u\|^2 + \frac{1}{2\varepsilon}\|u\|^2\right),$$

即

$$\|\Delta_h u\|^2 \leqslant \frac{2\varepsilon}{3}|\Delta_h u|_1^2 + \frac{1}{3\varepsilon^2}\|u\|^2.$$

□

11.3　二层非线性差分格式

定义 $\mathcal{W}_h$ 上的网格函数 Φ^n, U^n：

$$\Phi_{ij}^n = \phi(x_i, y_j, t_n), \quad U_{ij}^n = \mu(x_i, y_j, t_n).$$

记

$$c_0 = \max_{(x,y)\in\Omega, 0\leqslant t\leqslant T} |\phi(x, y, t)|.$$

11.3.1　差分格式的建立

在点 $(x_i, y_j, t_{k+\frac{1}{2}})$ 处考虑方程 (11.1) 和 (11.2) 可得

$$\begin{aligned}\phi_t(x_i, y_j, t_{k+\frac{1}{2}}) &= \Delta\mu(x_i, y_j, t_{k+\frac{1}{2}}),\\ \mu(x_i, y_j, t_{k+\frac{1}{2}}) &= \Delta_h^2\phi(x_i, y_j, t_{k+\frac{1}{2}}) + 2\Delta_h\phi(x_i, y_j, t_{k+\frac{1}{2}}) + \phi^3(x_i, y_j, t_{k+\frac{1}{2}})\\ &\quad + (1-\gamma)\phi(x_i, y_j, t_{k+\frac{1}{2}}).\end{aligned}$$

应用 Taylor 展开式可得

$$\delta_t\Phi_{ij}^{k+\frac{1}{2}} = \Delta_h U_{ij}^{k+\frac{1}{2}} + P_{ij}^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{11.11}$$

$$U_{ij}^{k+\frac{1}{2}} = \Delta_h^2\Phi_{ij}^{k+\frac{1}{2}} + 2\Delta_h\Phi_{ij}^{k+\frac{1}{2}} + (\Phi_{ij}^{k+\frac{1}{2}})\frac{(\Phi_{ij}^k)^2 + (\Phi_{ij}^{k+1})^2}{2} + (1-\gamma)\Phi_{ij}^{k+\frac{1}{2}} + Q_{ij}^{k+\frac{1}{2}},$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{11.12}$$

且存在常数 c_1 使得

$$|P_{ij}^{k+\frac{1}{2}}| \leqslant c_1(\tau^2 + h_1^2 + h_2^2), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{11.13}$$

$$|Q_{ij}^{k+\frac{1}{2}}| \leqslant c_1(\tau^2 + h_1^2 + h_2^2), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1. \tag{11.14}$$

注意到初值条件

$$\Phi_{ij}^0 = \psi(x_i, y_j), \quad 1 \leqslant i \leqslant m_1, 1 \leqslant j \leqslant m_2. \tag{11.15}$$

我们对 (11.1)–(11.3) 建立如下差分格式:
对 $0 \leqslant k \leqslant n$, 求 $\phi^k, \mu^k \in \mathcal{W}_h$ 满足

$$\delta_t \phi_{ij}^{k+\frac{1}{2}} = \Delta_h \mu_{ij}^{k+\frac{1}{2}}, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{11.16}$$

$$\mu_{ij}^{k+\frac{1}{2}} = \Delta_h^2 \phi_{ij}^{k+\frac{1}{2}} + 2\Delta_h \phi_{ij}^{k+\frac{1}{2}} + \phi_{ij}^{k+\frac{1}{2}} \frac{(\phi_{ij}^k)^2 + (\phi_{ij}^{k+1})^2}{2} + (1-\gamma)\phi_{ij}^{k+\frac{1}{2}},$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{11.17}$$

$$\phi_{ij}^0 = \psi(x_i, y_j), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2. \tag{11.18}$$

将 (11.17) 代入 (11.16) 得到如下差分格式:
求 $\phi^k \in \mathcal{W}_h$ 满足

$$\delta_t \phi_{ij}^{k+\frac{1}{2}} = \Delta_h \left[\Delta_h^2 \phi_{ij}^{k+\frac{1}{2}} + 2\Delta_h \phi_{ij}^{k+\frac{1}{2}} + \phi_{ij}^{k+\frac{1}{2}} \frac{(\phi_{ij}^k)^2 + (\phi_{ij}^{k+1})^2}{2} + (1-\gamma)\phi_{ij}^{k+\frac{1}{2}} \right],$$
$$1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2,\ 0 \leqslant k \leqslant n-1, \tag{11.19}$$

$$\phi_{ij}^0 = \psi(x_1, y_j), \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2. \tag{11.20}$$

11.3.2 差分格式解的有界性

定理 11.2　设 $\{\phi_{ij}^k \,|\, 1 \leqslant i \leqslant m_1, 1 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 为差分格式 (11.16)–(11.18) 的解. 记

$$F^k = \frac{1}{2}\|\Delta_h \phi^k\|^2 - |\phi^k|_1^2 + \frac{1}{4}\|\phi^k\|_4^4 + \frac{1-\gamma}{2}\|\phi^k\|^2, \quad 0 \leqslant k \leqslant n,$$

则有

$$F^k \leqslant F^0, \quad 1 \leqslant k \leqslant n.$$

证明　用 $\mu^{k+\frac{1}{2}}$ 与 (11.16) 的两边作内积, 注意到 $\mu^{k+\frac{1}{2}}$ 的周期性, 可得

$$(\delta_t \phi^{k+\frac{1}{2}}, \mu^{k+\frac{1}{2}}) = (\Delta_h \mu^{k+\frac{1}{2}}, \mu^{k+\frac{1}{2}}) = -|\mu^{k+\frac{1}{2}}|_1^2 \leqslant 0.$$

将 (11.17) 代入上式, 得到

$$\left(\delta_t \phi^{k+\frac{1}{2}}, \Delta_h^2 \phi^{k+\frac{1}{2}} + 2\Delta_h \phi^{k+\frac{1}{2}} + \phi^{k+\frac{1}{2}} \frac{(\phi^k)^2 + (\phi^{k+1})^2}{2} + (1-\gamma)\phi^{k+\frac{1}{2}}\right) \leqslant 0.$$

应用分部求和公式并注意到 $\phi^{k+\frac{1}{2}}$ 和 $\Delta_h \phi^{k+\frac{1}{2}}$ 的周期性, 得到

$$\frac{1}{2\tau}(\|\Delta_h \phi^{k+1}\|^2 - \|\Delta_h \phi^k\|^2) - \frac{1}{\tau}(|\phi^{k+1}|_1^2 - |\phi^k|_1^2)$$
$$+ \frac{1}{4\tau}(\|\phi^{k+1}\|_4^4 - \|\phi^k\|_4^4) + (1-\gamma)\frac{1}{2\tau} \cdot (\|\phi^{k+1}\|^2 - \|\phi^k\|^2) \leqslant 0, \quad 0 \leqslant k \leqslant n.$$

易知

$$\frac{1}{2\tau}(F^{k+1}-F^k)\leqslant 0,\quad 0\leqslant k\leqslant n-1.$$

因而

$$F^k\leqslant F^0,\quad 1\leqslant k\leqslant n.$$

□

定理 11.3 设 $\{\phi_{ij}^k\,|\,1\leqslant i\leqslant m_1, 1\leqslant j\leqslant m_2, 0\leqslant k\leqslant n\}$ 为差分格式 (11.16)-(11.18) 的解, 则存在常数 c_2 使得

$$\|\phi^k\|_\infty\leqslant c_2,\quad 0\leqslant k\leqslant n.$$

证明 注意到 $(\varepsilon^2-1)^2\geqslant 0$, 有

$$\frac{1}{4}(\phi_{ij}^k)^4\geqslant\frac{1}{2}(\phi_{ij}^k)^2-\frac{1}{4},\quad 1\leqslant i\leqslant m_1,\quad 1\leqslant j\leqslant m_2.$$

将上式乘以 h_1h_2, 并对 i,j 求和, 得到

$$\frac{1}{4}\|\phi^k\|_4^4\geqslant\frac{1}{2}\|\phi\|^2-\frac{L_1L_2}{4}.$$

对任意 $\varepsilon>0$, 由 Cauchy-Schwarz 不等式和

$$(\Delta_h\phi^k,\phi^k)=-|\phi^k|_1^2,$$

得

$$|\phi^k|_1^2\leqslant\frac{\varepsilon}{2}\|\Delta_h\phi^k\|^2+\frac{1}{2\varepsilon}\|\phi^k\|^2.$$

记 $a=1-\gamma>0$. 有

$$\begin{aligned}F(\phi^k)=&\frac{1}{2}\|\Delta_h\phi^k\|^2-|\phi^k|_1^2+\frac{1}{4}\|\phi^k\|_4^4+\frac{a}{2}\|\phi^k\|^2\\ \geqslant&\frac{1}{2}\|\Delta_h\phi^k\|^2-\left(\frac{\varepsilon}{2}\|\Delta_h\phi^k\|^2+\frac{1}{2\varepsilon}\|\phi^k\|^2\right)+\left(\frac{1}{2}\|\phi^k\|^2-\frac{L_1L_2}{4}\right)+\frac{a}{2}\|\phi^k\|^2\\ =&\frac{1-\varepsilon}{2}\|\Delta_h\phi^k\|^2+\frac{1}{2}\left(1+a-\frac{1}{\varepsilon}\right)\|\phi^k\|^2-\frac{L_1L_2}{4}.\end{aligned}$$

选择 ε 使得 $\dfrac{1-\varepsilon}{2}>0$ 和 $\dfrac{1}{2}\left(1+a-\dfrac{1}{\varepsilon}\right)>0$. 这等价于 $\dfrac{1}{1+a}<\varepsilon<1$. 令

$$\frac{1-\varepsilon}{2}=\frac{1}{2}\left(1+a-\frac{1}{\varepsilon}\right),$$

可得唯一正根 $\varepsilon^*=\dfrac{\sqrt{a^2+4}-a}{2}\in\left(\dfrac{1}{1+a},1\right)$. 于是

$$F(\phi^k)\geqslant\frac{1-\varepsilon^*}{2}(\|\Delta_h\phi^k\|^2+\|\phi^k\|^2)-\frac{L_1L_2}{4}.$$

由定理 11.2 知

$$\frac{1-\varepsilon^*}{2}(\|\Delta_h\phi^k\|^2+\|\phi^k\|^2)\leqslant F^0+\frac{L_1L_2}{4},\quad 0\leqslant k\leqslant n.$$

再由引理 11.2 知存在常数 c_2 使得

$$\|\phi^k\|_\infty\leqslant c_2,\quad 0\leqslant k\leqslant n.$$ □

11.3.3 差分格式解的存在性和唯一性

定理 11.4 当 $\left(2+\frac{25}{8}c_2^4\right)\tau<1$ 时, 差分格式 (11.19)–(11.20) 存在解.

证明 由 (11.20) 知第 0 层值 ϕ^0 已知. 设第 k 层值 ϕ^k 已知, 由 (11.19) 可得关于第 $k+1$ 层值 ϕ^{k+1} 的方程组. 注意到$\delta_t\phi_{ij}^{k+\frac{1}{2}}=\frac{2}{\tau}(\phi_{ij}^{k+\frac{1}{2}}-\phi_{ij}^k)$, 可先求出 $\phi_{ij}^{k+\frac{1}{2}}$, 再令 $\phi_{ij}^{k+1}=2\phi_{ij}^{k+\frac{1}{2}}-\phi_{ij}^k$. 记 $w_{ij}=\phi_{ij}^{k+\frac{1}{2}}$, 则由 (11.19) 可得

$$\frac{2}{\tau}(w_{ij}-\phi_{ij}^k)-\Delta_h\left[\Delta_h^2w_{ij}+2\Delta_hw_{ij}+w_{ij}\cdot\frac{(2w_{ij}-\phi_{ij}^k)^2+(\phi_{ij}^k)^2}{2}+(1-\gamma)w_{ij}\right]=0,$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2. \tag{11.21}$$

定义 $\Pi:\mathcal{W}_h\to\mathcal{W}_h$ 如下

$$\begin{aligned}\Pi(w)_{ij}=&\frac{2}{\tau}(w_{ij}-\phi_{ij}^k)-\Delta_h\Big[\Delta_h^2w_{ij}\\&+2\Delta_hw_{ij}+w_{ij}\frac{(2w_{ij}-\phi_{ij}^k)^2+(\phi_{ij}^k)^2}{2}+(1-\gamma)w_{ij}\Big],\end{aligned}$$

则有

$$\begin{aligned}&\big(\Pi(w),w\big)\\=&\frac{2}{\tau}[\|w\|^2-(w,\phi^k)]\\&-\left(\Delta_h\left(\Delta_h^2w+2\Delta_hw+w\frac{(2w-\phi^k)^2+(\phi^k)^2}{2}+(1-\gamma)w\right),w\right)\\=&\frac{2}{\tau}[\|w\|^2-(w,\phi^k)]\\&-\left(\Delta_h^2w+2\Delta_hw+w\frac{(2w-\phi^k)^2+(\phi^k)^2}{2}+(1-\gamma)w,\Delta_hw\right).\end{aligned}\tag{11.22}$$

由定理 11.3 知

$$\|w\|_\infty\leqslant c_2,\quad \|\phi^k\|_\infty\leqslant c_2.$$

于是

$$\begin{aligned}&\left\|\left(w\frac{(2w-\phi^k)^2+(\phi^k)^2}{2},\Delta_h w\right)\right\|\\&\leqslant\frac{\|2w-\phi^k\|_\infty^2+\|\phi^k\|_\infty^2}{2}\|w\|\cdot\|\Delta_h w\|\\&\leqslant 5c_2^2\|w\|\cdot\|\Delta_h w\|;\end{aligned}\tag{11.23}$$

此外有

$$-(\Delta_h^2 w,\Delta_h w)=|\Delta_h w|_1^2.\tag{11.24}$$

将 (11.23) 和 (11.24) 代入 (11.22), 并应用引理 11.3 得

$$\begin{aligned}(\Pi(w),w)\geqslant&\frac{2}{\tau}(\|w\|^2-(w,\phi^k))+|\Delta_h w|_1^2-2\|\Delta_h w\|^2\\&-5c_2^2\|w\|\cdot\|\Delta_h w\|+(1-\gamma)|w|_1^2\\\geqslant&\frac{2}{\tau}(\|w\|^2-\|w\|\cdot\|\phi^k\|)+\frac{3}{2\varepsilon}\left(\|\Delta_h w\|^2-\frac{1}{3\varepsilon^2}\|w\|^2\right)\\&-2\|\Delta_h w\|^2-5c_2^2\|w\|\cdot\|\Delta_h w\|.\end{aligned}$$

取 $\varepsilon=\dfrac{1}{2}$, 得

$$\begin{aligned}(\Pi(w),w)\geqslant&\frac{2}{\tau}(\|w\|^2-\|w\|\cdot\|\phi^k\|)+3\left(\|\Delta_h w\|^2-\frac{4}{3}\|w\|^2\right)\\&-2\|\Delta_h w\|^2-\left(\|\Delta_h w\|^2+\frac{25}{4}c_2^4\|w\|^2\right)\\=&\frac{2}{\tau}(\|w\|^2-\|w\|\cdot\|\phi^k\|)-\left(4+\frac{25}{4}c_2^4\right)\|w\|^2\\=&\frac{2}{\tau}\left\{\left[1-\left(2+\frac{25}{8}c_2^4\right)\tau\right]\|w\|-\|\phi^k\|\right\}\|w\|,\end{aligned}$$

当

$$\tau<\frac{1}{2+\frac{25}{8}c_2^4},\qquad\|w\|=\frac{\|\phi^k\|}{1-\left(2+\frac{25}{8}c_2^4\right)\tau}$$

时,

$$(\Pi(w),w)\geqslant 0.$$

由 Browder 定理 (定理 1.3) 知存在 $w^*\in\mathcal{W}_h$, $\|w^*\|\leqslant\dfrac{\|\phi^k\|}{1-\left(2+\frac{25}{8}c_2^4\right)\tau}$, 使得

$$\Pi(w^*)=0.$$

□

定理 11.5　当 $\tau < \dfrac{8}{16+121c_2^4}$ 时, 差分格式 (11.19)-(11.20) 的解是唯一的.

证明　设 (11.21) 另有解 $z \in \mathcal{W}_h$ 满足

$$\frac{2}{\tau}(z_{ij}-\phi_{ij}^k) - \Delta_h\left[\Delta_h^2 z_{ij} + 2\Delta_h z_{ij} + z_{ij}\frac{(2z_{ij}-\phi_{ij}^k)^2+(\phi_{ij}^k)^2}{2}\right.$$
$$\left.+(1-\gamma)z_{ij}\right] = 0, \quad 1 \leqslant i \leqslant m_1, 1 \leqslant j \leqslant m_2. \tag{11.25}$$

令

$$\rho_{ij} = w_{ij} - z_{ij}.$$

将 (11.21) 和 (11.25) 相减, 得

$$\frac{2}{\tau}\rho_{ij} - \Delta_h\left[\Delta_h^2\rho_{ij} + 2\Delta_h\rho_{ij} + w_{ij}\frac{(2w_{ij}-\phi_{ij}^k)^2+(\phi_{ij}^k)^2}{2}\right.$$
$$\left.- z_{ij}\frac{(2z_{ij}-\phi_{ij}^k)^2+(\phi_{ij}^k)^2}{2} + (1-\gamma)\rho_{ij}\right] = 0, \quad 1 \leqslant i \leqslant m_1,\ 1 \leqslant j \leqslant m_2.$$

用 ρ 与上式两端作内积, 得

$$\frac{2}{\tau}\|\rho\|^2 - (\Delta_h(\Delta_h^2\rho),\rho) - 2(\Delta_h^2\rho,\rho)$$
$$-\left(\Delta_h\left(w\frac{(2w-\phi^k)^2+(\phi^k)^2}{2} - z\frac{(2z+\phi^k)^2+(\phi^k)^2}{2}\right),\rho\right) - (1-\gamma)(\Delta_h\rho,\rho) = 0.$$

由分部求和公式, 可得

$$\frac{2}{\tau}\|\rho\|^2 + |\Delta_h\rho|_1^2 - 2\|\Delta_h\rho\|^2 - \frac{1}{2}\Big(w(2w-\phi^k)^2 - z(2z-\phi^k)^2 - \rho(\phi^k)^2, \Delta_h\rho\Big)$$
$$+(1-\gamma)|\rho|_1^2 = 0. \tag{11.26}$$

注意到

$$\begin{aligned}
&w(2w-\phi^k)^2 - z(2z-\phi^k)^2 - \rho(\phi^k)^2\\
=&(w-z)(2w-\phi^k)^2 + z[(2w-\phi^k)^2 - (2z-\phi^k)^2] - \rho(\phi^k)^2\\
=&\rho(2w-\phi^k)^2 + z(2w+2z-2\phi^k)2(w-z) - \rho(\phi^k)^2
\end{aligned}$$

及

$$\|w\|_\infty \leqslant c_2, \quad \|\phi^k\|_\infty \leqslant c_2, \quad \|z\|_\infty \leqslant c_2,$$

有

$$\|w(2w-\phi^k)^2 - z(2z-\phi^k)^2 - \rho(\phi^k)^2\| \leqslant 9c_2^2\|\rho\| + 12c_2^2\|\rho\| + c_2^2\|\rho\| = 22c_2^2\|\rho\|.$$

由 (11.26) 可得

$$\frac{2}{\tau}\|\rho\|^2+|\Delta_h\rho|_1^2\leqslant 2\|\Delta_h\rho\|^2+11c_2^2\|\rho\|\cdot\|\Delta_h\rho\|\leqslant 3\|\Delta_h\rho\|^2+\frac{121}{4}c_2^4\|\rho\|^2.$$

应用引理 11.3 得

$$\frac{2}{\tau}\|\rho\|^2+|\Delta_h\rho|_1^2\leqslant 3\left[\frac{2\varepsilon}{3}|\Delta_h\rho|_1^2+\frac{1}{3\varepsilon^2}\|\rho\|^2\right]+\frac{121}{4}c_2^4\|\rho\|^2.$$

取 $\varepsilon=\dfrac{1}{2}$ 得

$$\frac{2}{\tau}\|\rho\|^2\leqslant\left(4+\frac{121}{4}c_2^4\right)\|\rho\|^2,$$

当 $\tau<\dfrac{8}{16+121c_2^4}$ 时, $\|\rho\|=0$. □

11.3.4 差分格式解的收敛性

定理 11.6 设 $\{\Phi_{ij}^k,U_{ij}^k\,|\,1\leqslant i\leqslant m_1,1\leqslant j\leqslant m_2,0\leqslant k\leqslant n\}$ 是问题 (11.1)–(11.3) 的解, $\{\phi_{ij}^k,u_{ij}^k\,|\,1\leqslant i\leqslant m_1,1\leqslant j\leqslant m_2,0\leqslant k\leqslant n\}$ 是差分格式 (11.16)–(11.18) 的解. 定义网格函数

$$\tilde{\phi}_{ij}^k=\Phi_{ij}^k-\phi_{ij}^k,\quad \tilde{\mu}_{ij}^k=U_{ij}^k-\mu_{ij}^k,\quad 1\leqslant i\leqslant m_1,\quad 1\leqslant j\leqslant m_2,\quad 0\leqslant k\leqslant n.$$

则存在与步长 τ,h_1,h_2 的无关的常数 c_3 使得

$$\|\tilde{\phi}^k\|\leqslant c_3(\tau^2+h_1^2+h_2^2),\quad 0\leqslant k\leqslant n. \tag{11.27}$$

证明 将 (11.11)–(11.12), (11.15) 与 (11.16)–(11.18) 相减, 得

$$\delta_t\tilde{\phi}_{ij}^{k+\frac12}=\Delta_h\tilde{\mu}_{ij}^{k+\frac12}+P_{ij}^{k+\frac12},$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 0\leqslant k\leqslant n-1, \tag{11.28}$$

$$\tilde{\mu}_{ij}^{k+\frac12}=\Delta_h^2\tilde{\phi}_{ij}^{k+\frac12}+2\Delta_h\tilde{\phi}_{ij}^{k+\frac12}+\Phi_{ij}^{k+\frac12}\left[\frac{(\Phi_{ij}^k)^2+(\Phi_{ij}^{k+1})^2}{2}\right.$$
$$\left.-\phi_{ij}^{k+\frac12}\frac{(\phi_{ij}^k)^2+(\phi_{ij}^{k+1})^2}{2}\right]+(1-\gamma)\tilde{\phi}_{ij}^{k+\frac12}+Q_{ij}^{k+\frac12},$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 0\leqslant k\leqslant n-1, \tag{11.29}$$

$$\tilde{\mu}_{ij}^0=0,\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2. \tag{11.30}$$

用 $\tilde{\phi}^{k+\frac12}$ 与 (11.28) 的两边作内积, 得

$$\begin{aligned}(\delta_t\tilde{\phi}^{k+\frac12},\tilde{\phi}^{k+\frac12})&=(\Delta_h\tilde{\mu}^{k+\frac12},\tilde{\phi}^{k+\frac12})+(P^{k+\frac12},\tilde{\phi}^{k+\frac12})\\&=(\tilde{\mu}^{k+\frac12},\Delta_h\tilde{\phi}^{k+\frac12})+(P^{k+\frac12},\tilde{\phi}^{k+\frac12}),\quad 0\leqslant k\leqslant n-1.\end{aligned} \tag{11.31}$$

将 (11.29) 代入上式, 得

$$
\begin{aligned}
&\frac{1}{2\tau}(\|\tilde{\phi}^{k+1}\|^2-\|\tilde{\phi}^k\|^2)\\
=&\left(\Delta_h^2\tilde{\phi}^{k+\frac12},\Delta_h\tilde{\phi}^{k+\frac12}\right)+2\|\Delta_h\tilde{\phi}^{k+\frac12}\|^2\\
&+\left(\Phi^{k+\frac12}\frac{(\Phi^k)^2+(\Phi^{k+1})^2}{2}-\phi^{k+\frac12}\frac{(\phi^k)^2+(\phi^{k+1})^2}{2},\Delta_h\tilde{\phi}^{k+\frac12}\right)\\
&+(1-\varepsilon)\left(\tilde{\phi}^{k+\frac12},\Delta_h\tilde{\phi}^{k+\frac12}\right)+(Q^{k+\frac12},\Delta_h\tilde{\phi}^{k+\frac12})+(P^{k+\frac12},\tilde{\phi}^{k+\frac12}).
\end{aligned}
\tag{11.32}
$$

由 $\|\Phi^k\|_\infty=c_0$, $\|\phi^k\|_\infty\leqslant c_2$, 得

$$
\begin{aligned}
&\left|\Phi_{ij}^{k+\frac12}\frac{(\Phi_{ij}^k)^2+(\Phi_{ij}^{k+1})^2}{2}-\phi_{ij}^{k+\frac12}\frac{(\phi_{ij}^k)^2+(\phi_{ij}^{k+1})^2}{2}\right|\\
=&\left|\Phi_{ij}^{k+\frac12}\left[\frac{(\Phi_{ij}^k)^2+(\Phi_{ij}^{k+1})^2}{2}-\frac{(\phi_{ij}^k)^2+(\phi_{ij}^{k+1})^2}{2}\right]\right.\\
&\left.+(\Phi_{ij}^{k+\frac12}-\phi_{ij}^{k+\frac12})\frac{(\phi_{ij}^k)^2+(\phi_{ij}^{k+1})^2}{2}\right|\\
=&\left|\frac12\Phi_{ij}^{k+\frac12}\left[(\Phi_{ij}^k+\phi_{ij}^k)(\Phi_{ij}^k-\phi_{ij}^k)+(\Phi_{ij}^{k+1}+\phi_{ij}^{k+1})(\Phi_{ij}^{k+1}-\phi_{ij}^{k+1})\right]\right.\\
&\left.+\frac12\left[(\phi_{ij}^k)^2+(\phi_{ij}^{k+1})^2\right]\left(\Phi_{ij}^{k+\frac12}-\phi_{ij}^{k+\frac12}\right)\right|\\
\leqslant&\frac12c_0(c_0+c_2)(|\tilde{\phi}_{ij}^k|+|\tilde{\phi}_{ij}^{k+1}|)+c_2^2|\tilde{\phi}_{ij}^{k+\frac12}|\\
\leqslant&\frac12[c_0(c_0+c_2)+c_2^2]\left(|\tilde{\phi}_{ij}^k|+|\tilde{\phi}_{ij}^{k+1}|\right).
\end{aligned}
\tag{11.33}
$$

因而

$$
\begin{aligned}
&\frac{1}{2\tau}(\|\tilde{\phi}^{k+1}\|^2-\|\tilde{\phi}^k\|^2)\\
\leqslant&-|\Delta_h\tilde{\phi}^{k+\frac12}|_1^2+2\|\Delta_h\tilde{\phi}^{k+\frac12}\|^2\\
&+\frac12(c_0^2+c_0c_2+c_2^2)(\|\tilde{\phi}^k\|+\|\tilde{\phi}^{k+1}\|)\|\Delta_h\tilde{\phi}^{k+\frac12}\|\\
&-(1-\gamma)|\tilde{\phi}^{k+\frac12}|_1^2+\|Q^{k+\frac12}\|\cdot\|\Delta_h\tilde{\phi}^{k+\frac12}\|+\|P^{k+\frac12}\|\cdot\|\tilde{\phi}^{k+\frac12}\|\\
\leqslant&-|\Delta_h\tilde{\phi}^{k+\frac12}|_1^2+2\|\Delta_h\tilde{\phi}^{k+\frac12}\|^2\\
&+\frac12\|\Delta_h\tilde{\phi}^{k+\frac12}\|^2+\frac18(c_0^2+c_0c_2+c_2^2)^2(\|\tilde{\phi}^k\|+\|\tilde{\phi}^{k+1}\|)^2\\
&+\left(\frac12\|\Delta_h\tilde{\phi}^{k+\frac12}\|^2+\frac12\|Q^{k+\frac12}\|^2\right)+\left(\frac12\|\tilde{\phi}^{k+\frac12}\|^2+\frac12\|P^{k+\frac12}\|^2\right)\\
=&-|\Delta_h\tilde{\phi}^{k+\frac12}|_1^2+3\|\Delta_h\tilde{\phi}^{k+\frac12}\|^2+\frac14\left[(c_0^2+c_0c_2+c_2^2)^2+1\right]\left(\|\tilde{\phi}^k\|^2+\|\tilde{\phi}^{k+1}\|^2\right)\\
&+\frac12\left(\|P^{k+\frac12}\|^2+\|Q^{k+\frac12}\|^2\right).
\end{aligned}
\tag{11.34}
$$

在引理 11.3 中取 $\varepsilon = \dfrac{1}{2}$, 得

$$\|\Delta_h \tilde{\phi}^{k+\frac{1}{2}}\|^2 \leqslant \frac{1}{3}|\Delta_h \tilde{\phi}^{k+\frac{1}{2}}|_1^2 + \frac{4}{3}\|\tilde{\phi}^{k+\frac{1}{2}}\|^2, \tag{11.35}$$

将 (11.35) 代入 (11.34) 得

$$\begin{aligned}&\frac{1}{2\tau}(\|\tilde{\phi}^{k+1}\|^2 - \|\tilde{\phi}^k\|^2)\\ \leqslant\ &\frac{1}{4}[(c_0^2 + c_0c_2 + c_2^2) + 9]\big(\|\tilde{\phi}^k\|^2 + \|\tilde{\phi}^{k+1}\|^2\big) + \frac{1}{2}(\|P^{k+\frac{1}{2}}\|^2 + \|Q^{k+\frac{1}{2}}\|^2).\end{aligned}$$

注意到 (11.13)–(11.14) 得

$$\begin{aligned}&\left[1 - \frac{1}{2}(c_0^2 + c_0c_2 + c_2^2 + 9)\tau\right]\|\tilde{\phi}^{k+1}\|^2\\ \leqslant\ &\left[1 + \frac{1}{2}(c_0^2 + c_0c_2 + c_2^2 + 9)\tau\right]\|\tilde{\phi}^k\|^2 + 2L_1L_2c_1^2(\tau^2 + h_1^2 + h_2^2)^2\tau, \quad 0 \leqslant k \leqslant n-1.\end{aligned}$$

当 $(c_0^2 + c_0c_2 + c_2^2 + 9)\tau \leqslant \dfrac{2}{3}$ 时, 由上式得

$$\|\tilde{\phi}^{k+1}\|^2 \leqslant \left[1 + \frac{3}{2}(c_0^2 + c_0c_2 + c_2^2 + 9)\tau\right] + 3L_1L_2c_1^2(\tau^2 + h_1^2 + h_2^2)^2\tau,\ 0 \leqslant k \leqslant n-1. \tag{11.36}$$

再由 Gronwall 不等式得到

$$\|\tilde{\phi}^{k+1}\|^2 \leqslant e^{\frac{3}{2}(c_0^2+c_0c_2+c_2^2+9)T}\frac{2L_1L_2c_1^2}{c_0^2 + c_0c_2 + c_2^2 + 9}(\tau^2 + h_1^2 + h_2^2)^2, \quad 0 \leqslant k \leqslant n-1.$$

□

11.4 三层线性化差分格式

11.4.1 差分格式的建立

由 Taylor 展开式, 有

$$\phi(x_i, y_j, t_{\frac{1}{2}}) = \phi(x_i, y_j, 0) + \frac{\tau}{2}\phi_t(x_i, y_j, 0) + O(\tau^2).$$

记

$$\hat{\phi}_{ij} = \phi(x_i, y_j, 0) + \frac{\tau}{2}\phi_t(x_i, y_j, 0), \qquad c_4 = \max_{(x,y)\in\bar{\Omega}, 0\leqslant t\leqslant T}|\phi_t(x, y, t)|.$$

当 $\dfrac{1}{2}c_4\tau \leqslant 1$ 时,

$$|\hat{\phi}_{ij}| \leqslant c_0 + \frac{1}{2}c_4\tau \leqslant c_0 + 1. \tag{11.37}$$

在点 $(x_i,y_j,t_{\frac{1}{2}})$ 处和点 (x_i,y_j,t_k) 处分别考虑 (11.1)–(11.2), 由 Taylor 展开式, 有

$$\delta_t\Phi_{ij}^{\frac{1}{2}}=\Delta_h U_{ij}^{\frac{1}{2}}+\hat{P}_{ij}^{0},\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{11.38}$$

$$U_{ij}^{\frac{1}{2}}=\Delta_h^2\Phi_{ij}^{\frac{1}{2}}+2\Delta_h\Phi_{ij}^{\frac{1}{2}}+(\hat{\phi}_{ij})^2\Phi_{ij}^{\frac{1}{2}}+(1-\gamma)\Phi_{ij}^{\frac{1}{2}}+\hat{Q}_{ij}^{0},\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2 \tag{11.39}$$

和

$$\Delta_t\Phi_{ij}^{k}=\Delta_h U_{ij}^{\bar{k}}+\hat{P}_{ij}^{k},\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, 1\leqslant k\leqslant n-1, \tag{11.40}$$

$$U_{ij}^{\bar{k}}=\Delta_h^2\Phi_{ij}^{\bar{k}}+2\Delta_h\Phi_{ij}^{\bar{k}}+(\Phi_{ij}^{k})^2\Phi_{ij}^{\bar{k}}+(1-\gamma)\Phi_{ij}^{\bar{k}}+\hat{Q}_{ij}^{k},\quad 1\leqslant i\leqslant m_1,\quad 1\leqslant j\leqslant m_2,\quad 1\leqslant k\leqslant n-1. \tag{11.41}$$

存在常数 c_5 使得

$$|\hat{P}_{ij}^{k}|\leqslant c_5(\tau^2+h_1^2+h_2^2),\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 0\leqslant k\leqslant n-1, \tag{11.42}$$

$$|\hat{Q}_{ij}^{k}|\leqslant c_5(\tau^2+h_1^2+h_2^2),\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 0\leqslant k\leqslant n-1. \tag{11.43}$$

注意到初值条件

$$\Phi_{ij}^{0}=\psi(x_i,y_j),\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{11.44}$$

在 (11.38)–(11.41) 中略去小量项, 对 (11.1)–(11.3) 建立如下线性化差分格式:

求 $\phi^k\in\mathcal{W}_h, k=0,1,\cdots,n$ 使得

$$\delta_t\phi_{ij}^{\frac{1}{2}}=\Delta_h\mu_{ij}^{\frac{1}{2}},\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{11.45}$$

$$\mu_{ij}^{\frac{1}{2}}=\Delta_h^2\phi_{ij}^{\frac{1}{2}}+2\Delta_h\phi_{ij}^{\frac{1}{2}}+(\hat{\phi}_{ij})^2\phi_{ij}^{\frac{1}{2}}+(1-\gamma)\phi_{ij}^{\frac{1}{2}},\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{11.46}$$

$$\Delta_t\phi_{ij}^{k}=\Delta_h\mu_{ij}^{\bar{k}},\quad 1\leqslant i\leqslant m_1, 1\leqslant j\leqslant m_2, 1\leqslant k\leqslant n-1, \tag{11.47}$$

$$\mu_{ij}^{\bar{k}}=\Delta_h^2\phi_{ij}^{\bar{k}}+2\Delta_h\phi_{ij}^{\bar{k}}+(\phi_{ij}^{k})^2\phi_{ij}^{\bar{k}}+(1-\gamma)\phi_{ij}^{\bar{k}},\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n-1, \tag{11.48}$$

$$\phi_{ij}^{0}=\psi(x_i,y_j),\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2. \tag{11.49}$$

将 (11.46) 代入 (11.45), (11.48) 代入 (11.47), 可得

求 $\phi^k \in \mathcal{W}_h,\ k=0,1,2,\cdots,n$, 使得

$$\delta_t\phi_{ij}^{\frac{1}{2}}=\Delta_h\big(\Delta_h^2\phi_{ij}^{\frac{1}{2}}+2\Delta_h\phi_{ij}^{\frac{1}{2}}+(\hat{\phi}_{ij})^2\phi_{ij}^{\frac{1}{2}}+(1-\gamma)\phi_{ij}^{\frac{1}{2}}\big),$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2, \tag{11.50}$$
$$\Delta_t\phi_{ij}^{k}=\Delta_h\big(\Delta_h^2\phi_{ij}^{\bar{k}}+2\Delta_h\phi_{ij}^{\bar{k}}+(\phi_{ij}^{k})^2\phi_{ij}^{\bar{k}}+(1-\gamma)\phi_{ij}^{\bar{k}}\big),$$
$$1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2,\ 1\leqslant k\leqslant n-1, \tag{11.51}$$
$$\phi_{ij}^0=\psi(x_i,y_j),\quad 1\leqslant i\leqslant m_1,\ 1\leqslant j\leqslant m_2. \tag{11.52}$$

由 (11.52) 知 ϕ^0 已给定. 由 (11.50) 可得关于 ϕ^1 的线性方程组. 当 ϕ^{k-1},ϕ^k 已求得时, (11.51) 为关于 ϕ^{k+1} 的线性方程组. 可用迭代法求这些线性方程组.

11.4.2 差分格式解的能量稳定性

定理 11.7 设 $\{\phi_{ij}^k\,|\,1\leqslant i\leqslant m_1,1\leqslant j\leqslant m_2,0\leqslant k\leqslant n\}$ 为 (11.45)-(11.49) 的解. 定义

$$\begin{aligned}G(\phi^k,\phi^{k+1})=&\frac{1}{4}(\|\Delta_h\phi^k\|^2+\|\Delta_h\phi^{k+1}\|^2)-\frac{1}{2}\big(|\phi^k|_1^2+|\phi^{k+1}|_1^2\big)\\&+\frac{1}{4}((\phi^k)^2,(\phi^{k+1})^2)+\frac{1-\gamma}{4}(\|\phi^k\|^2+\|\phi^{k+1}\|^2).\end{aligned}$$

则有

$$\begin{aligned}&\frac{1}{4}(\|\Delta_h\phi^0\|^2+\|\nabla_h\phi^1\|^2)-\frac{1}{2}\big(|\phi^0|_1^2+|\phi^1|_1^2\big)\\&+\frac{1}{4}((\hat{\phi})^2,(\phi^1)^2)+\frac{1-\gamma}{4}(\|\phi^0\|^2+\|\phi^1\|^2)\\\leqslant&\frac{1}{2}\|\Delta_h\phi^0\|^2-|\phi^0|_1^2+\frac{1}{4}((\hat{\phi})^2,(\phi^0)^2)+\frac{1-\gamma}{2}\|\phi^0\|^2,\end{aligned} \tag{11.53}$$

$$G(\phi^k,\phi^{k+1})\leqslant G(\phi^{k-1},\phi^k),\quad 1\leqslant k\leqslant n-1. \tag{11.54}$$

证明 (I) 用 $\mu^{\frac{1}{2}}$ 与 (11.45) 的两边作内积, 并利用 $\mu^{\frac{1}{2}}$ 的周期性, 有

$$(\delta_t\phi^{\frac{1}{2}},\mu^{\frac{1}{2}})=(\Delta_h\mu^{\frac{1}{2}},\mu^{\frac{1}{2}})=-|\mu^{\frac{1}{2}}|_1^2\leqslant 0.$$

将 (11.46) 代入, 得

$$(\delta_t\phi^{\frac{1}{2}},\Delta_h^2\phi^{\frac{1}{2}}+2\Delta_h\phi^{\frac{1}{2}}+(\hat{\phi})^2\phi^{\frac{1}{2}}+(1-\gamma)\phi^{\frac{1}{2}})\leqslant 0.$$

利用分部求和公式, $\phi^{\frac{1}{2}}$ 及 $\Delta_h\phi^{\frac{1}{2}}$ 的周期性, 得

$$\begin{aligned}&\frac{1}{2\tau}(\|\Delta_h\phi^1\|^2-\|\Delta_h\phi^0\|^2)-\frac{1}{\tau}(|\phi^1|_1^2-|\phi^0|_1^2)\\&+\frac{1}{2\tau}[((\hat{\phi})^2,(\phi^1)^2)-((\hat{\phi})^2,(\phi^0)^2)]+\frac{1-\gamma}{2\tau}(\|\phi^1\|^2-\|\phi^0\|^2)\leqslant 0.\end{aligned}$$

变形即得 (11.53).

(II) 用 $\mu^{\bar{k}}$ 与 (11.9) 的两边作内积, 利用 $\mu^{\bar{k}}$ 的周期性, 得到

$$\left(\Delta_t\phi^k, \mu^{\bar{k}}\right) = \left(\Delta_h\mu^{\bar{k}}, \mu^{\bar{k}}\right) = -\|\nabla_h\mu^k\|^2 \leqslant 0.$$

将 (11.48) 代入上式, 得

$$\left(\Delta_t\phi^k, \Delta_h^2\phi^{\bar{k}} + 2\Delta_h\phi^{\bar{k}} + (\phi^k)^2\phi^{\bar{k}} + (1-\gamma)\phi^{\bar{k}}\right) \leqslant 0.$$

利用分部求和公式, $\phi^{\bar{k}}$ 及 $\Delta_h\phi^{\bar{k}}$ 的周期性, 得

$$\begin{aligned}&\frac{1}{4\tau}\left(\|\Delta_h\phi^{k+1}\|^2 - \|\Delta_h\phi^{k-1}\|^2\right) - \frac{1}{2\tau}\left(|\phi^{k+1}|_1^2 - |\phi^{k-1}|_1^2\right)\\&+\frac{1}{4\tau}\Big[\left((\phi^k)^2, (\phi^{k+1})^2\right) - \left((\phi^{k-1})^2, (\phi^k)^2\right)\Big] + \frac{1-\gamma}{4\tau}\left(\|\phi^{k+1}\|^2 - \|\phi^{k-1}\|^2\right) \leqslant 0.\end{aligned}$$

变形可得 (11.54). □

11.4.3 差分格式解的收敛性

定理 11.8 设 $\{\Phi_{ij}^k, U_{ij}^k \,|\, 1 \leqslant i \leqslant m_1, 1 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 是问题 (11.1)-(11.3) 的解, $\{\phi_{ij}^k, u_{ij}^k \,|\, 1 \leqslant i \leqslant m_1, 1 \leqslant j \leqslant m_2, 0 \leqslant k \leqslant n\}$ 是差分格式 (11.45)-(11.49) 的解. 记

$$\tilde{\phi}_{ij}^k = \Phi_{ij}^k - \phi_{ij}^k, \quad \tilde{\mu}_{ij}^k = U_{ij}^k - \mu_{ij}^k, \quad 1 \leqslant i \leqslant m_1, \quad 1 \leqslant j \leqslant m_2, \quad 0 \leqslant k \leqslant n,$$

$$c_6 = \max\left\{(2c_0+1)^2c_0^2, \frac{9}{2} + (c_0+1)^4\right\}, \quad c_7 = e^{3c_6T}\sqrt{8 + \frac{1}{c_6}L_1L_2c_5},$$

则当步长 τ, h_1, h_2 满足 $\dfrac{\tau^2 + h_1^2 + h_2^2}{h_1^{\frac{1}{2}}h_2^{\frac{1}{2}}} \leqslant \dfrac{1}{c_7}$ 且 τ 适当小时, 差分格式 (11.45)–(11.49) 存在唯一解, 且

$$\|\tilde{\phi}^k\| \leqslant c_7(\tau^2 + h_1^2 + h_2^2), \quad 0 \leqslant k \leqslant n. \tag{11.55}$$

证明 将 (11.38)–(11.41), (11.44) 与 (11.45)–(11.49) 相减, 得误差方程组

$$\delta_t\tilde{\phi}_{ij}^{\frac{1}{2}} = \Delta_h\tilde{\mu}_{ij}^{\frac{1}{2}} + \hat{P}_{ij}^0, \ \ 1 \leqslant i \leqslant m_1, 1 \leqslant j \leqslant m_2, \tag{11.56}$$

$$\begin{aligned}\tilde{\mu}_{ij}^{\frac{1}{2}} = \Delta_h^2\tilde{\phi}_{ij}^{\frac{1}{2}} + 2\Delta_h\tilde{\phi}_{ij}^{\frac{1}{2}} + (\hat{\phi}_{ij})^2\tilde{\phi}_{ij}^{\frac{1}{2}} + (1-\gamma)\tilde{\phi}_{ij}^{\frac{1}{2}} + \hat{Q}_{ij}^0,&\\ 1 \leqslant i \leqslant m_1, \ 1 \leqslant j \leqslant m_2,&\end{aligned} \tag{11.57}$$

$$\Delta_t\tilde{\phi}_{ij}^k = \Delta_h\tilde{\mu}_{ij}^{\bar{k}} + \hat{P}_{ij}^k, \ \ 1 \leqslant i \leqslant m_1, \ 1 \leqslant j \leqslant m_2, \ 1 \leqslant k \leqslant n-1, \tag{11.58}$$

$$\begin{aligned}\tilde{\mu}_{ij}^{\bar{k}} = \Delta_h^2\tilde{\phi}_{ij}^{\bar{k}} + 2\Delta_h\tilde{\phi}_{ij}^{\bar{k}} + (\Phi_{ij}^k)^2\Phi_{ij}^{\bar{k}} - (\phi_{ij}^k)^2\phi_{ij}^{\bar{k}} + (1-\gamma)\tilde{\phi}_{ij}^{\bar{k}} + \hat{Q}_{ij}^k,&\\ 1 \leqslant i \leqslant m_1, \ 1 \leqslant j \leqslant m_2, \ 1 \leqslant k \leqslant n-1,&\end{aligned} \tag{11.59}$$

$$\tilde{\phi}_{ij}^0 = 0, \ \ 1 \leqslant i \leqslant m_1, \quad 1 \leqslant j \leqslant m_2. \tag{11.60}$$

由 (11.49) 知 ϕ^0 已给定.

由 (11.60) 知 (11.55) 对 $k=0$ 成立.

(I) ϕ^1 的唯一可解性和收敛性.

(a) (11.50) 为关于 ϕ^1 的线性方程组. 考虑其齐次方程组

$$\frac{1}{\tau}\phi_{ij}^1=\Delta_h\left(\frac{1}{2}\Delta_h^2\phi_{ij}^1+\Delta_h\phi_{ij}^1+\frac{1}{2}(\hat{\phi}_{ij})^2\phi_{ij}^1+\frac{1-\gamma}{2}\phi_{ij}^1\right),\ 1\leqslant i\leqslant m_1,1\leqslant j\leqslant m_2.$$

用 $2\phi^1$ 与上式的两边作内积, 得到

$$\begin{aligned}\frac{2}{\tau}\|\phi^1\|^2=&(\Delta_h(\Delta_h^2\phi^1+2\Delta_h\phi^1+(\hat{\phi})^2\phi^1+(1-\gamma)\phi^1),\phi^1)\\=&(\Delta_h(\Delta_h^2\phi^1),\phi^1)+2\|\Delta_h\phi^1\|^2+((\hat{\phi})^2\phi^1,\Delta_h\phi^1)+(1-\gamma)(\phi^1,\Delta_h\phi^1)\\\leqslant&-|\Delta_h\phi^1|_1^2+2\|\Delta_h\phi^1\|^2+(c_0+1)^2\|\phi^1\|\cdot\|\Delta_h\phi^1\|-(1-\gamma)|\phi^1|_1^2\\\leqslant&-|\Delta_h\phi^1|_1^2+2\|\Delta_h\phi^1\|^2+\|\Delta_h\phi^1\|^2+\frac{1}{4}(c_0+1)^4\|\phi^1\|^2.\end{aligned}$$

应用引理 11.3, 并取 $\varepsilon=\dfrac{1}{2}$, 得到

$$\frac{2}{\tau}\|\phi^1\|^2\leqslant\left[4+\frac{1}{4}(c_0+1)^4\right]\|\phi^1\|^2,$$

当$\tau<\dfrac{8}{16+(c_0+1)^4}$ 时 $\|\phi^1\|^2=0$. 因而 (11.50) 关于 ϕ^1 是唯一可解的.

(b) 用 $\tilde{\phi}^{\frac{1}{2}}$ 与 (11.56) 的两边作内积, 得

$$\begin{aligned}(\delta_t\tilde{\phi}^{\frac{1}{2}},\tilde{\phi}^{\frac{1}{2}})=&(\Delta_h\tilde{\mu}^{\frac{1}{2}},\tilde{\phi}^{\frac{1}{2}})+(\hat{P}^0,\tilde{\phi}^{\frac{1}{2}})\\=&(\tilde{\mu}^{\frac{1}{2}},\Delta_h\tilde{\phi}^{\frac{1}{2}})+(\hat{P}^0,\tilde{\phi}^{\frac{1}{2}}).\end{aligned}$$

将 (11.57) 代入上式, 得

$$\begin{aligned}&\frac{1}{2\tau}(\|\tilde{\phi}^1\|^2-\|\tilde{\phi}^0\|^2)\\=&\left(\Delta_h^2\tilde{\phi}^{\frac{1}{2}}+2\Delta_h\tilde{\phi}^{\frac{1}{2}}+(\hat{\phi})^2\tilde{\phi}^{\frac{1}{2}}+(1-\gamma)\tilde{\phi}^{\frac{1}{2}}+\hat{Q}^0,\Delta_h\tilde{\phi}^{\frac{1}{2}}\right)+(\hat{P}^0,\tilde{\phi}^{\frac{1}{2}})\\=&-|\Delta_h\tilde{\phi}^{\frac{1}{2}}|_1^2+2\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|^2+((\hat{\phi}^0)^2\tilde{\phi}^{\frac{1}{2}},\Delta_h\tilde{\phi}^{\frac{1}{2}})\\&-(1-\gamma)|\tilde{\phi}^{\frac{1}{2}}|_1^2+(\hat{Q}^0,\Delta_h\tilde{\phi}^{\frac{1}{2}})+(\hat{P}^0,\tilde{\phi}^{\frac{1}{2}})\\\leqslant&-|\Delta_h\tilde{\phi}^{\frac{1}{2}}|_1^2+2\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|^2+(c_0+1)^2\|\tilde{\phi}^{\frac{1}{2}}\|\cdot\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|\\&+\|\hat{Q}^0\|\cdot\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|+\|\hat{P}^0\|\cdot\|\tilde{\phi}^{\frac{1}{2}}\|\\\leqslant&-|\Delta_h\tilde{\phi}^{\frac{1}{2}}|_1^2+2\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|^2+\frac{1}{2}\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|^2+\frac{1}{2}(c_0+1)^4\|\tilde{\phi}^{\frac{1}{2}}\|^2\\&+\frac{1}{2}\|\Delta_h\tilde{\phi}^{\frac{1}{2}}\|^2+\frac{1}{2}\|\hat{Q}^0\|^2+\frac{1}{2}\|\tilde{\phi}^{\frac{1}{2}}\|^2+\frac{1}{2}\|\hat{P}^0\|^2.\end{aligned}$$

应用引理 11.3, 并取 $\varepsilon=\dfrac{1}{2}$, 得

$$\begin{aligned}&\frac{1}{2\tau}(\|\tilde{\phi}^1\|^2-\|\tilde{\phi}^0\|^2)\\ \leqslant\,&4\|\tilde{\phi}^{\frac12}\|^2+\left[\frac12+\frac12(c_0+1)^4\right]\|\tilde{\phi}^{\frac12}\|^2+\frac12(\|\hat{P}^0\|^2+\|\hat{Q}^0\|^2).\end{aligned}$$

注意到 (11.60) 和 (11.42)–(11.43), 有

$$\frac{1}{2\tau}\|\tilde{\phi}^1\|^2\leqslant\left[\frac92+\frac12(c_0+1)^4\right]\cdot\frac14\|\tilde{\phi}^1\|^2+L_1L_2c_5^2(\tau^2+h_1^2+h_2^2)^2,$$

即

$$\left\{1-\left[\frac94+\frac14(c_0+1)^4\right]\tau\right\}\|\tilde{\phi}^1\|^2\leqslant 2L_1L_2c_5^2\tau(\tau^2+h_1^2+h_2^2)^2.$$

当 $\left[\dfrac94+\dfrac14(c_0+1)^4\right]\tau\leqslant\dfrac12$ 时,

$$\|\tilde{\phi}^1\|^2\leqslant 4L_1L_2c_5^2\tau(\tau^2+h_1^2+h_2^2).\tag{11.61}$$

(II) 设 $\phi^0,\phi^1,\cdots,\phi^l$ 已确定, 且 (11.55) 式当 $k=0,1,\cdots,l$ 时均成立. 则当 $\dfrac{\tau^2+h_1^2+h_2^2}{(h_1h_2)^{1/2}}\leqslant 1/c_7$ 时,

$$\|\tilde{\phi}^k\|_\infty\leqslant(h_1h_2)^{-\frac12}\|\tilde{\phi}^k\|\leqslant 1,\quad 0\leqslant k\leqslant l.$$

因而

$$\|\phi^k\|_\infty\leqslant\|\Phi^k\|_\infty+\|\tilde{\phi}^k\|_\infty\leqslant c_0+1,\quad 0\leqslant k\leqslant l.\tag{11.62}$$

(a) ϕ^{l+1} 的可解性.

由 (11.51) 可得关于 ϕ^{l+1} 的线性方程组. 考虑其齐次方程组

$$\frac{1}{2\tau}\phi_{ij}^{l+1}=\frac12\Delta_h\big(\Delta_h^2\phi_{ij}^{l+1}+2\Delta_h\phi_{ij}^{l+1}+(\phi_{ij}^l)^2\phi_{ij}^{l+1}+(1-\gamma)\phi_{ij}^{l+1}\big),\quad 1\leqslant i\leqslant m_1,1\leqslant j\leqslant m_2.$$

用 $2\phi^{l+1}$ 与上式两边作内积, 得

$$\begin{aligned}&\frac1\tau\|\phi^{l+1}\|^2\\ =\,&\Big(\Delta_h\big(\Delta_h^2\phi^{l+1}+2\Delta_h\phi^{l+1}+(\phi^l)^2\phi^{l+1}+(1-\gamma)\phi^{l+1}\big),\phi^{l+1}\Big)\\ =\,&-|\Delta_h\phi^{l+1}|_1^2+2\|\Delta_h\phi^{l+1}\|^2+\big((\phi^l)^2\phi^{l+1},\Delta_h\phi^{l+1}\big)-(1-\gamma)|\phi^{l+1}|_1^2\\ \leqslant\,&-|\Delta_h\phi^{l+1}|_1^2+2\|\Delta_h\phi^{l+1}\|^2+(c_0+1)^2\|\phi^{l+1}\|\cdot\|\Delta_h\phi^{l+1}\|\\ \leqslant\,&-|\Delta_h\phi^{l+1}|_1^2+2\|\Delta_h\phi^{l+1}\|^2+\|\Delta_h\phi^{l+1}\|^2+\frac14(c_0+1)^4\|\phi^{k+1}\|^2.\end{aligned}$$

应用引理 11.3, 并取 $\varepsilon=\dfrac{1}{2}$, 得

$$\frac{1}{\tau}\|\phi^{l+1}\|^2\leqslant\left[4+\frac{1}{4}(c_0+1)^4\right]\|\phi^{l+1}\|^2.$$

当 $\tau<\dfrac{4}{16+(c_0+1)^4}$ 时,

$$\|\phi^{l+1}\|^2=0.$$

因而 (11.51) 唯一确定 ϕ^{l+1}.

(b) ϕ^{l+1} 的收敛性.

注意到

$$\begin{aligned}&(\Phi_{ij}^k)^2\Phi_{ij}^{\bar{k}}-(\phi_{ij}^k)^2\phi_{ij}^{\bar{k}}\\=&[(\Phi_{ij}^k)^2-(\phi_{ij}^k)^2]\Phi_{ij}^{\bar{k}}+(\phi_{ij}^k)^2(\Phi_{ij}^{\bar{k}}-\phi_{ij}^{\bar{k}})\\=&(\Phi_{ij}^k+\phi_{ij}^k)\Phi_{ij}^{\bar{k}}(\Phi_{ij}^k-\phi_{ij}^k)+(\phi_{ij}^k)^2(\Phi_{ij}^{\bar{k}}-\phi_{ij}^{\bar{k}})\\=&(\Phi_{ij}^k+\phi_{ij}^k)\Phi_{ij}^{\bar{k}}\tilde{\phi}_{ij}^k+(\phi_{ij}^k)^2\tilde{\phi}_{ij}^{\bar{k}},\end{aligned}$$

以及 (11.62) 有

$$|(\Phi_{ij}^k)^2\Phi_{ij}^{\bar{k}}-(\phi_{ij}^k)^2\phi_{ij}^{\bar{k}}|\leqslant(2c_0+1)c_0|\tilde{\phi}_{ij}^k|+(c_0+1)^2|\tilde{\phi}_{ij}^{\bar{k}}|,\quad 0\leqslant k\leqslant l.\tag{11.63}$$

用 $\tilde{\phi}^{\bar{k}}$ 与 (11.58) 的两边作内积, 得

$$(\Delta_t\tilde{\phi}^k,\tilde{\phi}^{\bar{k}})=(\Delta_h\tilde{\mu}^{\bar{k}},\tilde{\phi}^{\bar{k}})+(\hat{P}^k,\tilde{\phi}^{\bar{k}})=(\tilde{\mu}^{\bar{k}},\Delta_h\tilde{\phi}^{\bar{k}})+(\hat{P}^k,\tilde{\phi}^{\bar{k}}).\tag{11.64}$$

将 (11.59) 代入 (11.64), 并应用 (11.63), 得到

$$\begin{aligned}&\frac{1}{4\tau}(\|\tilde{\phi}^{k+1}\|^2-\|\tilde{\phi}^{k-1}\|^2)\\=&\left(\Delta_h^2\tilde{\phi}^{\bar{k}}+2\Delta_h\tilde{\phi}^{\bar{k}}+(\Phi^k)^2\Phi^{\bar{k}}-(\phi^k)^2\phi^{\bar{k}}+(1-\gamma)\tilde{\phi}^{\bar{k}}+\hat{Q}^k,\Delta_h\tilde{\phi}^{\bar{k}}\right)+(\hat{P}^k,\tilde{\phi}^{\bar{k}})\\=&(\Delta_h^2\tilde{\phi}^{\bar{k}},\Delta_h\tilde{\phi}^{\bar{k}})+2(\Delta_h\tilde{\phi}^{\bar{k}},\Delta_h\tilde{\phi}^{\bar{k}})+((\Phi^k)^2\Phi^{\bar{k}}-(\phi^k)^2\phi^{\bar{k}},\Delta_h\tilde{\phi}^{\bar{k}})\\&-(1-\gamma)|\tilde{\phi}^{\bar{k}}|_1^2+(\hat{Q}^k,\Delta_h\tilde{\phi}^{\bar{k}})+(\hat{P}^k,\tilde{\phi}^{\bar{k}})\\\leqslant&-|\Delta_h\tilde{\phi}^{\bar{k}}|_1^2+2\|\Delta_h\tilde{\phi}^{\bar{k}}\|^2+((2c_0+1)c_0\|\tilde{\phi}^k\|+(c_0+1)^2\|\tilde{\phi}^{\bar{k}}\|)\|\Delta_h\tilde{\phi}^{\bar{k}}\|\\&+\|\hat{Q}^k\|\cdot\|\Delta_h\tilde{\phi}^{\bar{k}}\|+(\hat{P}^k,\tilde{\phi}^{\bar{k}})\\\leqslant&-|\Delta_h\tilde{\phi}^{\bar{k}}|_1^2+2\|\Delta_h\tilde{\phi}^{\bar{k}}\|^2+\frac{1}{4}\|\Delta_h\tilde{\phi}^{\bar{k}}\|^2+(2c_0+1)^2c_0^2\|\tilde{\phi}^k\|^2\\&+\frac{1}{4}\|\Delta_h\tilde{\phi}^{\bar{k}}\|^2+(c_0+1)^4\|\tilde{\phi}^{\bar{k}}\|^2+\frac{1}{2}\|\Delta_h\tilde{\phi}^{\bar{k}}\|^2+\frac{1}{2}\|\hat{Q}^k\|^2\\&+\frac{1}{2}\|\tilde{\phi}^{\bar{k}}\|^2+\frac{1}{2}\|\hat{P}^k\|^2,\quad 1\leqslant k\leqslant l.\end{aligned}$$

应用引理 11.3, 并取 $\varepsilon=\dfrac{1}{2}$, 得

$$
\begin{aligned}
&\frac{1}{4\tau}(\|\tilde{\phi}^{k+1}\|^2-\|\tilde{\phi}^{k-1}\|^2)\\
\leqslant&4\|\tilde{\phi}^{\bar{k}}\|^2+(2c_0+1)^2c_0^2\|\tilde{\phi}^k\|^2+\left[(c_0+1)^4+\frac{1}{2}\right]\|\tilde{\phi}^{\bar{k}}\|^2+\frac{1}{2}(\|\hat{P}^k\|^2+\|\hat{Q}^k\|^2)\\
\leqslant&(2c_0+1)^2c_0^2\|\tilde{\phi}^k\|^2+\left[\frac{9}{2}+(c_0+1)^4\right]\cdot\frac{\|\tilde{\phi}^{k+1}\|^2+\|\tilde{\phi}^{k-1}\|^2}{2}+\frac{1}{2}(\|\tilde{P}^k\|^2+\|\tilde{Q}^k\|^2)\\
\leqslant&\max\left\{(2c_0+1)^2c_0^2,\frac{9}{2}+(c_0+1)^4\right\}\left(\|\tilde{\phi}^k\|^2+\frac{\|\tilde{\phi}^{k+1}\|^2+\|\tilde{\phi}^{k-1}\|^2}{2}\right)\\
&+L_1L_2c_5^2(\tau^2+h_1^2+h_2^2)^2.
\end{aligned}
$$

记

$$
c_6=\max\left\{(2c_0+1)^2c_0^2,\frac{9}{2}(c_0+1)^4\right\},\quad F^k=\frac{\|\tilde{\phi}^{k+1}\|^2+\|\tilde{\phi}^k\|^2}{2},
$$

则有

$$
\frac{1}{2\tau}(F^{k+1}-F^k)\leqslant c_6(F^{k+1}+F^k)+L_1L_2c_5^2(\tau^2+h_1^2+h_2^2)^2,\quad 1\leqslant k\leqslant l,
$$

即

$$
(1-2c_6\tau)F^{k+1}\leqslant(1+2c_6\tau)F^k+2L_1L_2c_5^2\tau(\tau^2+h_1^2+h_2^2)^2,\quad 1\leqslant k\leqslant l.
$$

当 $2c_6\tau\leqslant\dfrac{1}{3}$ 时,

$$
F^{k+1}\leqslant(1+6c_6\tau)F^k+3L_1L_2c_5^2\tau(\tau^2+h_1^2+h_2^2)^2,\quad 1\leqslant k\leqslant l.
$$

由 Gronwall 不等式, 得

$$
F^{l+1}\leqslant e^{6c_6T}\left[F^1+\frac{3L_1L_2c_5^2}{6c_6}(\tau^2+h_1^2+h_2^2)^2\right].
$$

注意到 (11.61), 有

$$
\frac{1}{2}(\|\tilde{\phi}^l\|^2+\|\hat{\phi}^{l+1}\|^2)\leqslant e^{6c_6T}\left[4L_1L_2c_5^2\tau(\tau^2+h_1^2+h_2^2)^2+\frac{L_1L_2c_5^2}{2c_6}(\tau^2+h_1^2+h_2^2)^2\right].
$$

因而

$$
\|\tilde{\phi}^{l+1}\|\leqslant e^{3c_6T}\sqrt{8+\frac{1}{c_6}L_1L_2}c_5(\tau^2+h_1^2+h_2^2),
$$

即 (11.55) 对 $k=l+1$ 成立.

由归纳原理, 定理证毕. □

11.5 小结与延拓

相场晶体模型方程 (11.1) 是六阶的非线性发展方程.

文 [41] 对其建立了时间二阶的二层非线性差分格式 (11.19)–(11.20). 证明了定理 11.2. 本章中我们进一步用嵌入定理 11.3 证明了差分格式的解在无穷模下的有界性, 用 Browder 定理证明了差分格式解的存在性, 还证明了差分格式解的唯一性和在 L_2 模下的收敛性.

在文 [11] 中, 我们对相场晶体模型方程构造了三层线性化格式. 证明了能量稳定性. 证明了差分格式在 L_2 模下是条件收敛的. 该条件是为理论上保证差分格式解在无穷模下是有界的.

Hu, Wise, Wang [18] 建立了时间二阶的三层非线性差分格式

$$\delta_t\phi_{ij}^{k+\frac{1}{2}} = \Delta_h\mu_{ij}^{k+\frac{1}{2}},$$

$$\mu_{ij}^{k+\frac{1}{2}} = \Delta_h^2\phi_{ij}^{k+\frac{1}{2}} + 3\Delta_h\phi_{ij}^k - \Delta_h\phi^{k-1} + \phi_{ij}^{k+\frac{1}{2}}\frac{(\phi_{ij}^k)^2 + (\phi_{ij}^{k+1})^2}{2} + (1-\gamma)\phi_{ij}^{k+\frac{1}{2}},$$

证明了能量有界和唯一可解.

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