A Conservative Difference Scheme for Two-Dimensional Nonlinear Schrödinger Equation with Wave Operator

Hanzhang Hu,[1,2] **Yanping Chen**[1]

[1] *School of Mathematical Science, South China Normal University, Guangzhou 520631, Guangdong, People's Republic of China*

[2] *School of Mathematics, Jiaying University, Meizhou 514015, Guangdong, P.R. China*

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A conservative difference scheme is presented for two-dimensional nonlinear Schrödinger equation with wave operator. The discrete energy method and an useful technique are used to analyze the difference scheme. It is shown, both theoretically and numerically, that the difference solution is conservative, unconditionally stable and convergent with second order in maximum norm. A numerical experiment indicates that the scheme is very effective. Numer Methods Partial Differential Eq 32: 862–876, 2016

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I. INTRODUCTION

The nonlinear Schrödinger (NLS) equation is one of the most important equations of mathematical physics with applications in many different fields, such as plasma physics, nonlinear optics, water waves, and bimolecular dynamics. In this article, we consider the following initial-boundary value problem of NLS equation with wave operator in two dimension:

$$\frac{\partial^2 u}{\partial t^2} + i\frac{\partial u}{\partial t} - \Delta u + |u|^2 u = 0, \quad (x, y, t) \in \Omega \times (0, T], \tag{1}$$

$$u(x, y, 0) = \varphi(x, y), \quad \frac{\partial u}{\partial t}(x, y, 0) = \phi(x, y), \quad (x, y) \in \overline{\Omega}, \tag{2}$$

Correspondence to: Yanping Chen, School of Mathematical Science, South China Normal University, Guangzhou 520631, Guangdong, People's Republic of China (e-mail: yanpingchen@scnu.edu.cn)
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$$u(x, y, t) = 0, \quad (x, y) \in \Gamma, \quad 0 \le t \le T, \tag{3}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian operator, Ω is $[x_l, x_r] \times [y_l, y_r]$, Γ is the boundary of Ω, $u(x, y, t)$ is a complex function, $\varphi(x, y)$, $\phi(x, y)$ are two prescribed smooth complex function, and $i^2 = -1$.

Computing the inner product of Eq. (1) with $\frac{\partial u}{\partial t}$, and then taking the real part, the conservative law is obtained as follows

$$E(t) = ||\frac{\partial u}{\partial t}(\cdot, t)||_{L^2}^2 + |u(\cdot, t)|_{H^1}^2 + \frac{1}{2}||u(\cdot, t)||_{L^4}^4 = E(0), \tag{4}$$

where the semi-norm $|\cdot|_{H^1}^2 = \iint\limits_{\Omega}[|\frac{\partial u}{\partial x}(x, y, t)|^2 + |\frac{\partial u}{\partial y}(x, y, t)|^2]\mathrm{d}x\mathrm{d}y$.

Zhang et al. found that the nonconservative schemes may easily show nonlinear blow-up when they study for NLS equation, so they presented a conservative difference scheme in [1]. Extensive mathematical and numerical studies have been carried out for the NLS equation with wave operator in the literature [2–7]. Christov and his coauthors presented another approach to get fully implicit conservative schemes of second order for nonlinear wave equations in [8, 9].

In recent years, much attention has been paid to the finite difference schemes for two-dimensional Schrödinger equations [10–16]. To measure computational error especially the phase error of numerical solutions, maximum norm error is preferable in practice or numerical analysis. By the standard H^1 energy analysis, it is not difficult to prove that the difference solutions for linear Schrödinger equations are convergent in the H^1 norm, see e.g. [10]. But the H^1 error estimate does not imply the maximum norm estimate. Wang proposed maximum norm error bound of a linearized difference scheme for a coupled NLS equations in [12], but the error estimate is not valid in high dimensions. Liao presented maximum norm error analysis of explicit schemes for two-dimensional NLS equations in [11], but it is conditionally convergent. To the best of our knowledge, there are few results about unconditional maximum norm convergence for two-dimensional NLS equation with wave operator. In general, the maximum norm priori estimates of the numerical solution are proved difficultly for the NLS equation. In fact, their proofs for conservative finite difference scheme rely strongly on not only the conservative property of the method but also the discrete version of the Sobolev inequality in one dimension in [2–6], however, the extension of the discrete version of the above Sobolev inequality is no longer valid in two dimensions in [10–12, 17]. Thus it must firstly estimate H^2 norm priori estimates of the numerical solution for maximum norm priori estimates of the numerical solution in two dimension.

Numerical approximations for Schrödinger equation have drawn much attention. Wang proposed numerical studies on the split-step finite difference method for various case of NLS equations such as cubic NLS equation, coupled NLS equation with constant coefficients and GP equations in 1D, 2D, 3D in [18]. Dehghan presented a compact split-step finite difference method for solving the NLS equations with constant and variable coefficients in [19], but, unfortunately, it is difficult to generalize to wave equations by the split-step finite difference method. In [13, 16] introduced alternating direction implicit method for two-dimensional NLS equation, but we find it difficult to extend two-dimensional NLS equation with wave operator. To avoid time-step constraints, it is often preferable to solve (1) implicitly in time. However, as for the implicit schemes of nonlinear equations in high dimensions, the algebraic systems are very large. Thus, it is necessary to construct high effective algorithm. The objective of this article is twofold. The first one is to propose a conservative difference scheme for two-dimensional NLS equation with wave operator and prove the unconditional stability and convergence in maximum norm with order $O(\tau^2 + h^2)$ by the energy method. The other is to construct an iterative algorithm for the conservative difference scheme and demonstrate the effectiveness of the conservative difference scheme.

The remainder of this article is organized as follows. A conservative difference scheme is proposed and the discrete conservation law of the difference scheme is discussed in Section 2. In Section 3, the priori estimations for numerical solutions are made, the convergence and stability for the new scheme are proved. In the last section, an iterative algorithm and numerical results will be discussed.

II. SOME NOTATION AND USEFUL RESULTS

The domain $\{(x,y,t)|(x,y,t)\in\overline{\Omega}\times[0,T]\}$ is discretized into grids described by the set $\{(x_k,y_m,t_n)\}$ of nodes, in which

$$x_k = x_l + kh_1, \quad 0\le k\le K, \quad y_m = y_l + mh_2, \quad 0\le m\le M, \quad t_n = n\tau, \quad 0\le n\le N,$$

where $h_1=\frac{x_r-x_l}{K}$, $h_2=\frac{y_r-y_l}{M}$, and $\tau=\frac{T}{N}$, and K, M, N are three positive integers.

Let $U^n_{km}\cong u(x_k,y_m,t_n)$, and $\Omega_h=\{(x_k,y_m)|1\le k\le K-1, 1\le m\le M-1\}$, Γ_h denote the set of nodes on Γ, and $\overline{\Omega}_h=\Omega_h\cup\Gamma_h$.

For simplicity of exposition, the spatial step length is both h along the X axis and Y axis, the mesh ratio $r=\frac{\tau}{h}$. The following notations are used:

$$\delta_x U^n_{km} = \frac{U^n_{k+1,m}-U^n_{km}}{h}, \qquad \delta_{\bar{x}} U^n_{km} = \frac{U^n_{km}-U^n_{k-1,m}}{h},$$

$$\delta^2_x U^n_{km} = \frac{U^n_{k-1,m}-2U^n_{km}+U^n_{k+1,m}}{h^2}, \qquad \delta_t U^n_{km} = \frac{U^{n+1}_{km}-U^n_{km}}{\tau},$$

$$\delta_{\bar{t}} U^n_{km} = \frac{U^n_{km}-U^{n-1}_{km}}{\tau}, \qquad \delta_{\hat{t}} U^n_{km} = \frac{U^{n+1}_{km}-U^{n-1}_{km}}{2\tau}.$$

Similar notations $\delta_y U^n_{km}$, $\delta_{\bar{y}} U^n_{km}$, $\delta^2_y U^n_{km}$, $\delta^2_t U^n_{km}$ can also be defined and the discrete Laplacian operator $\triangle_h U_{km} = (\delta^2_x+\delta^2_y)U_{km}$.

Let $V_h=\{\boldsymbol{v}|\boldsymbol{v}=\{v_{km}|(x_k,y_m)\in\overline{\Omega}_h$, and $v_{km}=0,(x_k,y_m)\in\Gamma\}$, for any $v,w\in V_h$, we define the discrete inner product as

$$(\boldsymbol{v},\boldsymbol{w}) = h^2\sum_{k=1}^{K-1}\sum_{m=1}^{M-1} v_{km}\bar{w}_{km}.$$

Accordingly, L^2-norm $||v^n||=\sqrt{(v^n,v^n)}$, similarly, we define discrete Sobolev norms (seminorm) as follows:

$$||\delta_x v^n|| = \sqrt{h^2\sum_{k=1}^{K-1}\sum_{m=1}^{M-1}|\delta_x v^n_{km}|^2}, \qquad ||\delta_y v^n|| = \sqrt{h^2\sum_{k=1}^{K-1}\sum_{m=1}^{M-1}|\delta_y v^n_{km}|^2},$$

$$||\delta^2_x v^n|| = \sqrt{h^2\sum_{k=1}^{K-1}\sum_{m=1}^{M-1}|\delta^2_x v^n_{km}|^2}, \qquad ||\delta^2_y v^n|| = \sqrt{h^2\sum_{k=1}^{K-1}\sum_{m=1}^{M-1}|\delta^2_y v^n_{km}|^2},$$

$$||\delta_x\delta_y v^n|| = \sqrt{h^2\sum_{k=1}^{K-1}\sum_{m=1}^{M-1}|\delta_x\delta_y v_{km}^n|^2}, \qquad |v^n|_1 = \sqrt{||\delta_x v^n||^2 + ||\delta_y v^n||^2},$$

$$||v^n||_\infty = \max_{0\le k,m\le M.}|v_{km}^n|, \qquad ||\Delta_h v^n|| = \sqrt{||\delta_x^2 v^n||^2 + ||\delta_y^2 v^n||^2 + 2||\delta_x\delta_y v^n||^2}.$$

In this article, C denotes a general positive constant which may have different values in different places.

Now, we present the following conservative difference scheme for the problem (1) − (3):

$$\delta_t^2 U_{km}^n + i\delta_{\hat{t}} U_{km}^n - \frac{1}{2}\Delta_h(U_{km}^{n+1} + U_{km}^{n-1}) + \frac{1}{4}(|U_{km}^{n+1}|^2 + |U_{km}^{n-1}|^2)(U_{km}^{n+1} + U_{km}^{n-1}) = 0,$$
$$k, m = 1, 2, \cdots, M-1, \quad n = 1, 2, \cdots, N-1, \tag{5}$$

$$U_{km}^n = 0, \quad (x_k, y_m) \in \Gamma, \quad n = 0, 1, 2, \cdots, N, \tag{6}$$

$$U_{km}^0 = \varphi(x_k, y_m), \quad U_{km}^{-1} = U_{km}^1 - 2\tau\phi(x_k, y_m), \quad k, m = 0, 1, 2, \cdots, M. \tag{7}$$

Theorem 2.1. *The difference scheme (5)–(7) is conservative in the sence:*

$$E^n = ||\delta_t U^n||^2 + \frac{1}{2}(|U^{n+1}|_1^2 + |U^n|_1^2) + \frac{1}{4}(||U^{n+1}||_4^4 + ||U^n||_4^4) = E^{n-1} = \cdots = E^0, \tag{8}$$

where $||U^n||_4^4 = h^2\sum_{k=1}^{K-1}\sum_{m=1}^{M-1}|U_{km}^n|^4$.

Proof. Computing the inner product of (5) with $\delta_{\hat{t}} U^n$, and then taking the real part, we obtain

$$M_1 + M_2 - M_3 + M_4 = 0,$$

where

$$M_1 = Re(\delta_t^2 U^n, \delta_{\hat{t}} U^n) = \frac{1}{2}\delta_{\bar{t}}(||\delta_t U^n||^2), \tag{9}$$

$$M_2 = Re(i\delta_{\hat{t}} U^n, \delta_{\hat{t}} U^n) = 0, \tag{10}$$

$$M_3 = \frac{1}{2}Re(\Delta_h(U^{n+1} + U^{n-1}), \delta_{\hat{t}} U^n) = -\frac{1}{4\tau}(|U^{n+1}|_1^2 - |U^{n-1}|_1^2), \tag{11}$$

$$M_4 = \frac{1}{4}Re((|U^{n+1}|^2 + |U^{n-1}|^2)(U^{n+1} + U^{n-1}), \delta_{\hat{t}} U^n) = \frac{1}{8\tau}(||U^{n+1}||_4^4 - ||U^{n-1}||_4^4). \tag{12}$$

Let

$$E^n = ||\delta_t U^n||^2 + \frac{1}{2}(|U^{n+1}|_1^2 + |U^n|_1^2) + \frac{1}{4}(||U^{n+1}||_4^4 + ||U^n||_4^4).$$

We can get

$$E^n = E^{n-1} = \cdots = E^0,$$

where

$$E^0 = ||\delta_t U^0||^2 + \frac{1}{2}(|U^1|_1^2 + |U^0|_1^2) + \frac{1}{4}(||U^1||_4^4 + ||U^0||_4^4).$$

∎

To obtain the error estimate in the maximum norm, we need the following lemmas.

Lemma 2.1 (Discrete Sobolev's inequality [11]). *For any mesh function $U_{km} \in V_h$, there exists a constant C_Ω dependent on the domain such that*

$$||U||_\infty \le C_\Omega(||U|| + ||\Delta_h U||). \tag{13}$$

Lemma 2.2 (Gronwall's inequality [20]). *Suppose that the discrete mesh function $\{w_n, n = 1, 2, \cdots, N\}$, $N\tau = T$, satisfies the inequality*

$$w_n \le A + \sum_{k=1}^{n} B_k w_k \tau, \quad 0 \le n\tau \le T,$$

where A and B_k $(k = 1, 2, \cdots, N)$ are nonnegative constants. Then,

$$||w_n||_\infty \le Ae^{2\sum_{k=1}^{N} B_k \tau}, \tag{14}$$

where τ is sufficiently small, such that $\tau(\max_{k=1,2,\cdots,N} B_k) \le \frac{1}{2}$.

Lemma 2.3. ([20]). *Suppose that $F(w) = F(w_1, \cdots, w_m)$ is $n(\ge 1)$ times continuously differentiable with respect to the variables $w = (w_1, \cdots, w_m)$. The norms $||w_{kh}||$ and $||\delta w_{kh}||$ $(k = 1, 2, \cdots, m)$ of the discrete functions $w_{kh} = \{w_{kj} | j = 0, 1, 2, \cdots, J\}$ $(k = 1, 2, \cdots, m)$ defined on the grid points $x_j = jh$ $(1, 2, \cdots, J)$ with $Jh = l$ are bounded. For the compound discrete function $F_h = \{F_j = F(w_j) | j = 0, 1, 2, \cdots, J\}$, there are estimates*

$$||\delta^n F_h|| \le C_1 ||\delta^n w|| + C_2,$$

for $n \ge 1$, where the constants C_1, C_2 depend on the norms $||w_{kh}||$ and $||w_{kh}||_1$ $(k = 1, 2, \cdots, m)$ and the compound function F(w), but they are in dependent of the steplength $h \ge 0$.

Lemma 2.4 ([4]). *For any mesh functions $\{U^n\}$, there is*

$$||U^{n+1}||^2 - ||U^{n-1}||^2 \le 2\tau \left[||\delta_{\hat{t}} U^n||^2 + \frac{1}{2}(|U^{n+1}|^2 + |U^{n-1}|^2) \right].$$

Lemma 2.5 ([21]). *For any mesh functions $\{v\} \in V_h$, there is*

$$\left[\frac{6}{x_r - x_l} + \frac{6}{y_r - y_l} \right] ||v||^2 \le |v|_1^2.$$

III. NUMERICAL ANALYSIS

A. A Priori Estimate

First, we will estimate the difference solution by some important lemmas.

Theorem 3.1. *Suppose that the solution of the Eqs. (1)–(3) satisfies the initial value $\varphi(x, y) \in C^{(1,1)}$, $\phi(x, y) \in C^{(0,0)}$, then the solution of the difference scheme $(5) - (7)$ satisfies the following estimates:*

$$||U^n|| \le C, \quad |U^n|_1 \le C, \quad ||\delta_t U^n|| \le C, \quad ||U^n||_\infty \le C.$$

Proof. From (8), we obtain

$$E^n = ||\delta_t U^n||^2 + \frac{1}{2}(|U^{n+1}|_1^2 + |U^n|_1^2) + \frac{1}{4}(||U^{n+1}||_4^4 + ||U^n||_4^4) = E^0 = C.$$

So we have

$$|U^n|_1 \le C, \quad ||\delta_t U^n|| \le C, \quad ||U^n||_4^4 \le C.$$

Let

$$f_{km}^n = \frac{1}{4}(|U_{km}^{n+1}|^2 + |U_{km}^{n-1}|^2)(U_{km}^{n+1} + U_{km}^{n-1}).$$

Thus, we have

$$Re(f^n, \delta_x^2 \delta_{\hat t} U^n) = Re(\delta_x^2 f^n, \delta_{\hat t} U^n) \le C_3(||\delta_x^2 f^n||^2 + ||\delta_t U^n||^2 + ||\delta_t U^{n-1}||^2).$$

Computing the inner product of (5) with $\delta_x^2 \delta_{\hat t} U^n$, then taking the real part, we get

$$\begin{aligned}
&\frac{1}{2\tau}(||\delta_x \delta_t U^n||^2 - ||\delta_x \delta_t U^{n-1}||^2)\\
&\quad + \frac{1}{4\tau}[(||\delta_x^2 U^{n+1}||^2 + ||\delta_x \delta_y U^{n+1}||^2) - (||\delta_x^2 U^{n-1}||^2 + ||\delta_x \delta_y U^{n-1}||^2)]\\
&\quad \le C_3(||\delta_x^2 f^n||^2 + ||\delta_t U^n||^2 + ||\delta_t U^{n-1}||^2)\\
&\quad \le C_3(||U^{n+1}||^2 + ||U^{n-1}||^2 + ||\delta_t U^n||^2 + ||\delta_t U^{n-1}||^2 + ||\delta_x^2 U^{n+1}||^2 + ||\delta_x^2 U^{n-1}||^2), \quad (15)
\end{aligned}$$

where the last inequality is obtained by Lemma 2.3. Computing the inner product of (5) with $\delta_y^2 \delta_{\hat t} U^n$, then taking the real part, we also get

$$\begin{aligned}
&\frac{1}{2\tau}(||\delta_y \delta_t U^n||^2 - ||\delta_y \delta_t U^{n-1}||^2)\\
&\quad + \frac{1}{4\tau}[(||\delta_y^2 U^{n+1}||^2 + ||\delta_x \delta_y U^{n+1}||^2) - (||\delta_y^2 U^{n-1}||^2 + ||\delta_x \delta_y U^{n-1}||^2)]\\
&\quad \le C_3(||\delta_y^2 f^n||^2 + ||\delta_t U^n||^2 + ||\delta_t U^{n-1}||^2)\\
&\quad \le C_3(||U^{n+1}||^2 + ||U^{n-1}||^2 + ||\delta_t U^n||^2 + ||\delta_t U^{n-1}||^2 + ||\delta_y^2 U^{n+1}||^2 + ||\delta_y^2 U^{n-1}||^2). \quad (16)
\end{aligned}$$

Adding (15) to (16), we can obtain

$$\begin{aligned}
&[(||\delta_x \delta_t U^n||^2 + ||\delta_y \delta_t U^n||^2) - (||\delta_x \delta_t U^{n-1}||^2 + ||\delta_y \delta_t U^{n-1}||^2)] + \frac{1}{2}(||\Delta_h U^{n+1}||^2 - ||\Delta_h U^{n-1}||^2)\\
&\quad \le C_4 \tau(||\Delta_h U^{n+1}||^2 + ||\Delta_h U^{n-1}||^2 + ||\delta_t U^n||^2 + ||\delta_t U^{n-1}||^2 + ||U^{n+1}||^2 + ||U^{n-1}||^2), \quad (17)
\end{aligned}$$

and from (17) and Lemma 2.4, we get

$$\begin{aligned}
&[(||\delta_x\delta_t U^n||^2 + ||\delta_y\delta_t U^n||^2) - (||\delta_x\delta_t U^{n-1}||^2 + ||\delta_y\delta_t U^{n-1}||^2)] \\
&\quad + \frac{1}{2}(||\Delta_h U^{n+1}||^2 - ||\Delta_h U^{n-1}||^2) + (||U^{n+1}||^2 - ||U^{n-1}||^2) \\
&\qquad \le C_4\tau(||\Delta_h U^{n+1}||^2 + ||\Delta_h U^{n-1}||^2 + ||\delta_t U^n||^2 + ||\delta_t U^{n-1}||^2 + ||U^{n+1}||^2 + ||U^{n-1}||^2).
\end{aligned} \tag{18}$$

Let

$$A_n = ||\delta_x\delta_t U^n||^2 + ||\delta_y\delta_t U^n||^2 + \frac{1}{2}(||\Delta_h U^{n+1}||^2 + ||\Delta_h U^n||^2) + (||U^{n+1}||^2 + ||U^n||^2).$$

Then (18) can be written as

$$\begin{aligned}
A_n \le A_0 + C_4\tau\sum_{k=1}^{n}(&||\Delta_h U^{k+1}||^2 + ||\Delta_h U^{k-1}||^2 + ||\delta_t U^k||^2 + ||\delta_t U^{k-1}||^2 \\
&+ ||U^{k+1}||^2 + ||U^{k-1}||^2).
\end{aligned} \tag{19}$$

From Lemma 2.2 and Lemma 2.5, we have

$$||U^n||^2 \le C, \quad ||\Delta_h U^n||^2 \le C.$$

From Lemma 2.1, we have

$$||U^n||_\infty^2 \le C.$$

■

B. Convergence and Stability

Now, we will prove convergence and stability in the maximum norm.

Assume that $u_{km}^n = u(kh, mh, n\tau)$ is the solution of the Eqs. (1)–(3) on the grid points, then the truncation error of the scheme as

$$r_{km}^n = \delta_t^2 u_{km}^n + i\delta_{\hat t} u_{km}^n - \frac{1}{2}[\Delta_h u_{km}^{n+1} + \Delta_h u_{km}^{n-1}] + \frac{1}{4}(|u_{km}^{n+1}|^2 + |u_{km}^{n-1}|^2)(u_{km}^{n+1} + u_{km}^{n-1}). \tag{20}$$

According to Taylor' expansion, it can be easily obtained that

Lemma 3.1. *Suppose that the solution of the Eqs. (1)–(3) satisfies $u(x, y, t) \in C^{(4,4,4)}$, then the truncation error of the scheme (5)–(7) is of order*

$$r_{km}^n = O(\tau^2 + h^2). \tag{21}$$

Theorem 3.2. *Suppose that the solution of the Eqs. (1)–(3) satisfies $u(x, y, t) \in C^{(5,5,4)}$, the numerical solution of the scheme (5)–(7) converges to the solution of the Eqs. (1)–(3) with order $O(\tau^2 + h^2)$ by the L_∞ norm.*

Proof. Let

$$e_{km}^n = u_{km}^n - U_{km}^n.$$

Subtracting (5) from (20), we get

$$r_{km}^n = \delta_t^2 e_{km}^n + i\delta_{\hat{t}} e_{km}^n - \frac{1}{2}[\Delta_h e_{km}^{n+1} + \Delta_h e_{km}^{n-1}]$$
$$+ \frac{1}{4}[(|u_{km}^{n+1}|^2 + |u_{km}^{n-1}|^2)(u_{km}^{n+1} + u_{km}^{n-1}) - (|U_{km}^{n+1}|^2 + |U_{km}^{n-1}|^2)(U_{km}^{n+1} + U_{km}^{n-1})]. \tag{22}$$

Computing the inner product of (22) with $\delta_{\hat{t}} e_{km}^n$, and taking the real part, for the left term of the above equality (22), we get

$$Re(r^n, \delta_{\hat{t}} e^n) \leq [O(\tau^2 + h^2)]^2 + ||\delta_t e^n||^2 + ||\delta_t e^{n-1}||^2.$$

For the right part of the above equality (22), where the nonlinear term is calculated as follows:

$$\begin{aligned}
&\frac{1}{4} Re((|u^{n+1}|^2 + |u^{n-1}|^2)(u^{n+1} + u^{n-1}) - (|U^{n+1}|^2 + |U^{n-1}|^2)(U^{n+1} + U^{n-1}), \delta_{\hat{t}} e^n) \\
&= \frac{1}{4} Re((|u^{n+1}|^2 + |u^{n-1}|^2)(e^{n+1} + e^{n-1}) \\
&\quad + [(|u^{n+1}|^2 + |u^{n-1}|^2) - (|U^{n+1}|^2 + |U^{n-1}|^2)](U^{n+1} + U^{n-1}), \delta_{\hat{t}} e^n) \\
&= \frac{1}{4} Re((|u^{n+1}|^2 + |u^{n-1}|^2)(e^{n+1} + e^{n-1}), \delta_{\hat{t}} e^n) \\
&\quad + \frac{1}{4} Re([(|u^{n+1}|^2 + |u^{n-1}|^2) - (|U^{n+1}|^2 + |U^{n-1}|^2)](U^{n+1} + U^{n-1}), \delta_{\hat{t}} e^n) \\
&= \frac{1}{8} Re((|u^{n+1}|^2 + |u^{n-1}|^2)(e^{n+1} + e^{n-1}), \delta_t e^n + \delta_t e^{n-1}) + \frac{1}{8} Re([u^{n+1}\bar{e}^{n+1} \\
&\quad + e^{n+1}\bar{U}^{n+1} + u^{n-1}\bar{e}^{n-1} + e^{n-1}\bar{U}^{n-1}](U^{n+1} + U^{n-1}), \delta_t e^n + \delta_t e^{n-1}) \\
&\leq C(||e^{n+1}||^2 + ||e^{n-1}||^2 + ||\delta_t e^n||^2 + ||\delta_t e^{n-1}||^2).
\end{aligned} \tag{23}$$

From Lemma 2.4, we obtain

$$||e^{n+1}||^2 - ||e^{n-1}||^2 \leq 2\tau \left[\frac{(||\delta_t e^n||^2 + ||\delta_t e^{n-1}||^2)}{2} + \frac{(|e^{n+1}|^2 + |e^{n-1}|^2)}{2} \right]. \tag{24}$$

For the rest terms of the above equality (22), in a similar way as proved in Theorem 2.1, and from (23) − (24), we have

$$\begin{aligned}
&\frac{1}{2\tau}(||\delta_t e^n||^2 - ||\delta_t e^{n-1}||^2) + \frac{1}{4\tau}(||e^{n+1}||_1^2 - ||e^{n-1}||_1^2) + \frac{1}{2\tau}(||e^{n+1}||^2 - ||e^{n-1}||^2) \\
&\quad \leq C(|\tau^2 + h^2|^2 + ||\delta_t e^n||^2 + ||\delta_t e^{n-1}||^2 + ||e^{n+1}||^2 + ||e^{n-1}||^2).
\end{aligned} \tag{25}$$

Summing (25) up for n, we have

$$||\delta_t e^n||^2 + \frac{1}{2}(||e^{n+1}||_1^2 + ||e^n||_1^2) + (||e^{n+1}||^2 + ||e^n||^2)$$
$$\leq C|\tau^2 + h^2|^2 + C\tau \sum_{k=1}^{n}[||\delta_t e^k||^2 + \frac{1}{2}(|e^{k+1}|_1^2 + |e^k|_1^2) + (||e^{k+1}||^2 + ||e^k||^2)]. \tag{26}$$

According to Lemma 2.2, when τ is small enough, it follows that

$$||\delta_t e^n||^2 + \frac{1}{2}(||e^{n+1}||_1^2 + ||e^n||_1^2) + (||e^{n+1}||^2 + ||e^n||^2) \leq C|\tau^2 + h^2|^2. \tag{27}$$

We get

$$||e^n||^2 \leq C|\tau^2 + h^2|^2.$$

Similarly, by the same means of Theorem 2.1 and the nonlinear term (23), we have

$$||\Delta_h e^n||^2 \leq C|\tau^2 + h^2|^2.$$

Finally, we have

$$||e^n||_\infty^2 \leq C|\tau^2 + h^2|^2.$$

■

Similarly, we can prove the stability of the difference solution. i.e.

Theorem 3.3. *Under the conditions of theorem 3.2, the solution of the difference scheme (5)–(7) is unconditionally stable in the L_∞ norm for initial value.*

IV. NUMERICAL EXPERIMENT

In this section, we consider the following example:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + i\frac{\partial u}{\partial t} + |u|^2 u = (\sqrt{2}\pi + sin^2(\pi x)sin^2(\pi y))u,$$
$$(x, y) \in [-4, 4] \times [-4, 4], \quad t \in [0, 1], \tag{28}$$
$$u(x, y, t) = 0, \quad (x, y) \in \Gamma, \quad t \in [0, 1], \tag{29}$$
$$u(x, y, 0) = sin(\pi x)sin(\pi y), \quad \frac{\partial u}{\partial t}(x, y, 0) = -\sqrt{2}\pi i sin(\pi x)sin(\pi y),$$
$$(x, y) \in [-4, 4] \times [-4, 4]. \tag{30}$$

The exact solution of the equation is

$$u(x, y, t) = sin(\pi x)sin(\pi y)e^{-\sqrt{2}\pi i t}. \tag{31}$$

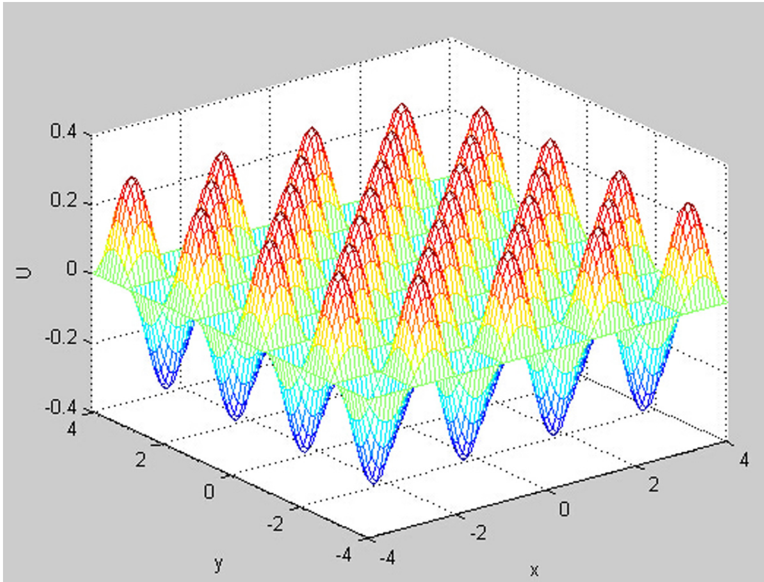

FIG. 1. Real part of U $t=1$, $h=0.1$, $\tau=0.01$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

For problems (28) $-$ (30), we have the following difference scheme

$$\delta_t^2 U_{km}^n + i\delta_{\hat{t}} U_{km}^n - \frac{1}{2}[\Delta_h U_{km}^{n+1} + \Delta_h U_{km}^{n-1}] + \frac{1}{4}(|U_{km}^{n+1}|^2 + |U_{km}^{n-1}|^2)(U_{km}^{n+1} + U_{km}^{n-1})$$
$$= f_{km}^n \frac{U_{km}^{n+1} + U_{km}^{n-1}}{2}, \quad (k,m = 1,2,\cdots,M-1,\ n = 1,2,\cdots,N-1), \tag{32}$$

where $f_{km}^n = \sqrt{2}\pi + sin^2(\pi kh)sin^2(\pi mh)$.

Obviously, the scheme (32) is an implicit and nonlinear one. In order to obtain the numerical solution U_{km}^n, we need define the following iterative algorithm

$$AU_{k,m-1}^{n+1(s+1)} + AU_{k-1,m}^{n+1(s+1)} + C_{km}^{n+1(s)} U_{km}^{n+1(s+1)} + AU_{k+1,m}^{n+1(s+1)} + AU_{k,m+1}^{n+1(s+1)} = D_{km}^{n+1(s)}, \tag{33}$$

$$U_{k0}^n = U_{kM}^n = 0, \quad U_{0m}^n = U_{Mm}^n = 0, \tag{34}$$

$$U_{km}^0 = \varphi(x_k, y_m), \quad U_{km}^{-1} = U_{km}^1 - 2\tau\phi(x_k, y_m), \tag{35}$$

where

$$A = -\frac{r^2}{2}, \quad C_{km}^{n+1(s)} = 1 + 2r^2 + i\frac{\tau}{2} + \frac{\tau^2}{4}(|U_{km}^{n+1(s)}|^2 + |U_{km}^{n-1}|^2) - \frac{\tau^2}{2} f_{km}^n,$$

$$D_{km}^{n+1(s)} = -\frac{r^2}{2}(U_{k,m-1}^{n-1} + U_{k-1,m}^{n-1} + U_{k,m+1}^{n-1} + U_{k+1,m}^{n-1})$$
$$+ \left(-1 - 2r^2 + i\frac{\tau}{2} - \frac{\tau^2}{4}(|U_{km}^{n+1(s)}|^2 + |U_{km}^{n-1}|^2) + \frac{\tau^2}{2} f_{km}^n\right) U_{km}^{n-1} + 2U_{km}^n,$$

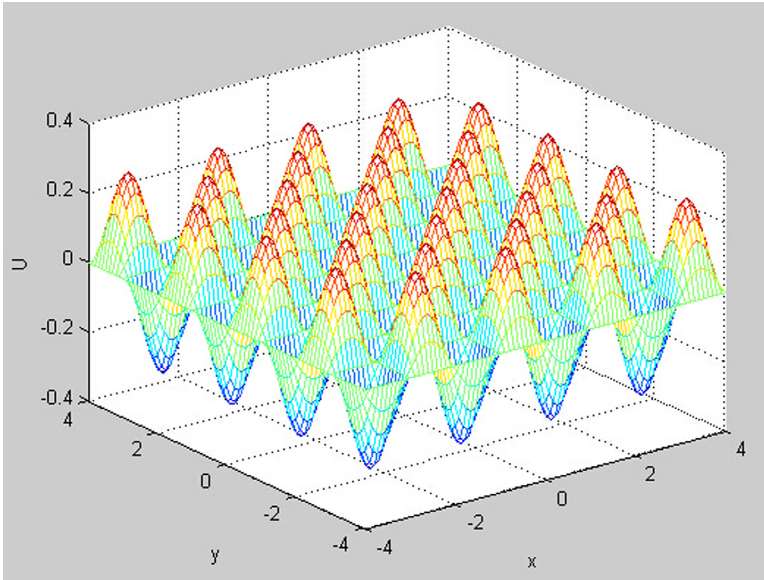

FIG. 2. Real part of u $t=1$, $h = 0.1, \tau = 0.01$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

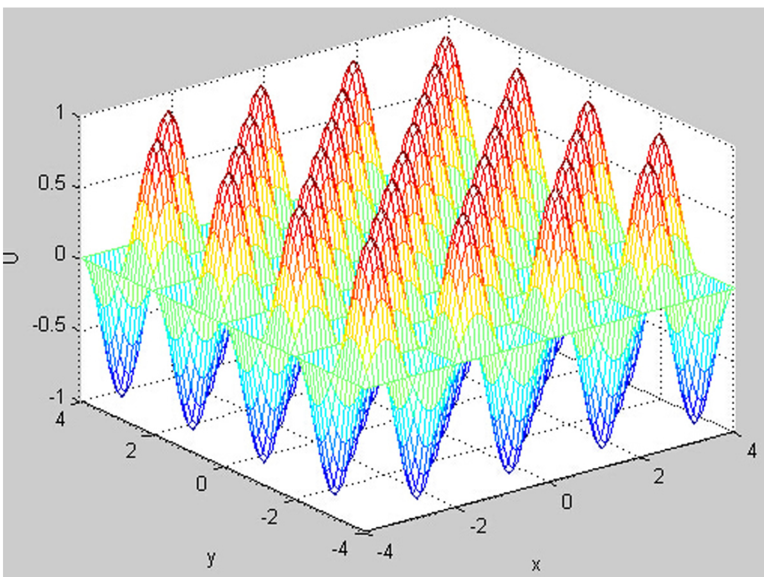

FIG. 3. Imaginary part of U $t=1$, $h = 0.1, \tau = 0.01$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

where s said the number of the iterative algorithm. The initial value of the iterative algorithm $U_{km}^{n+1(0)} = U_{km}^{n}$. All iteration processes are terminated, when $||U^{n+1(s+1)} - U^{n+1(s)}||_{\infty} < 10^{-6}$.

The pictures shown in Figs. 1–4 display good agreement between the exact and numerical solutions. Figure 5 shows error E is effective in $t = 1, h = 0.1, \tau = 0.01$. When we choose two

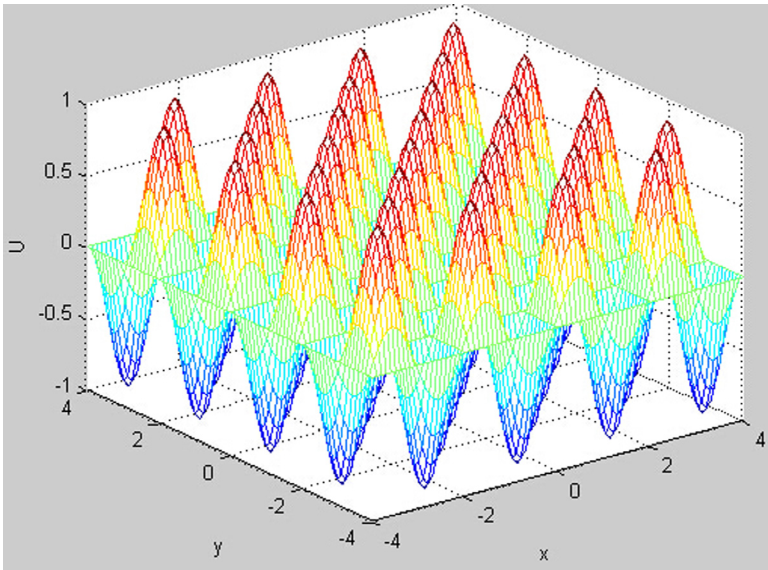

FIG. 4. Imaginary part of u $t = 1$, $h = 0.1$, $\tau = 0.01$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

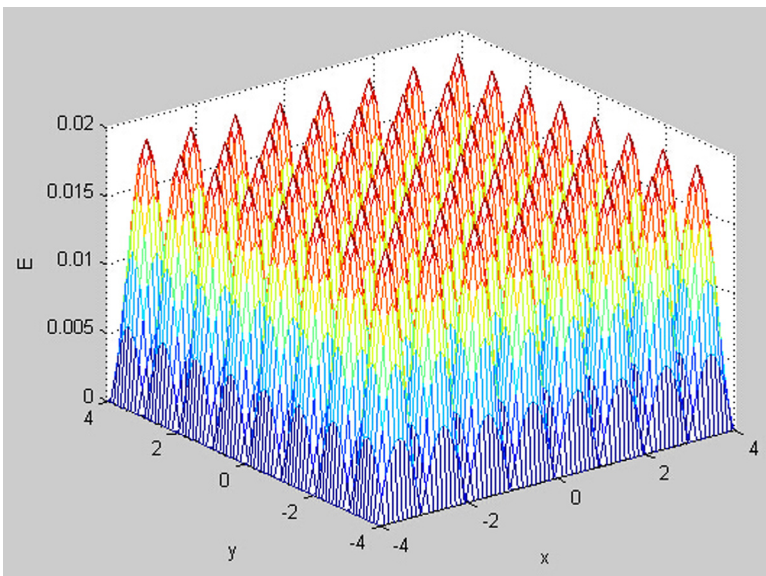

FIG. 5. Error E $t = 1$, $h = 0.1$, $\tau = 0.01$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

different step sizes with $h = 0.1, \tau = 0.01$, $h = 0.1, \tau = 0.001$, Figs. 6 and 7 show change of error e at different times, respectively.

We compute the discrete conservation law. Here, we choose $h = 0.1, \tau = 0.01$, and $h = 0.1, \tau = 0.005$. Table I shows the value of the scheme E^n at different times. It indicates that the conservation of the scheme (5) is very good.

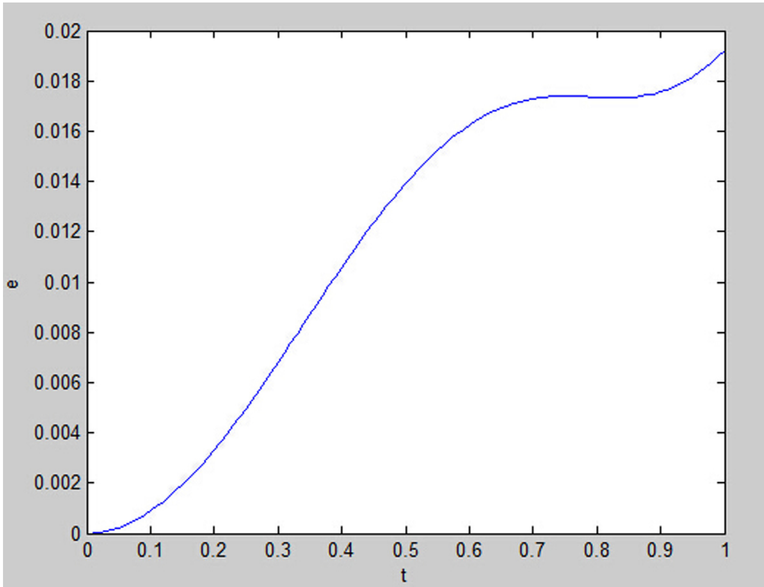

FIG. 6. Error e $t = [0,1]$, $h = 0.1, \tau = 0.01$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

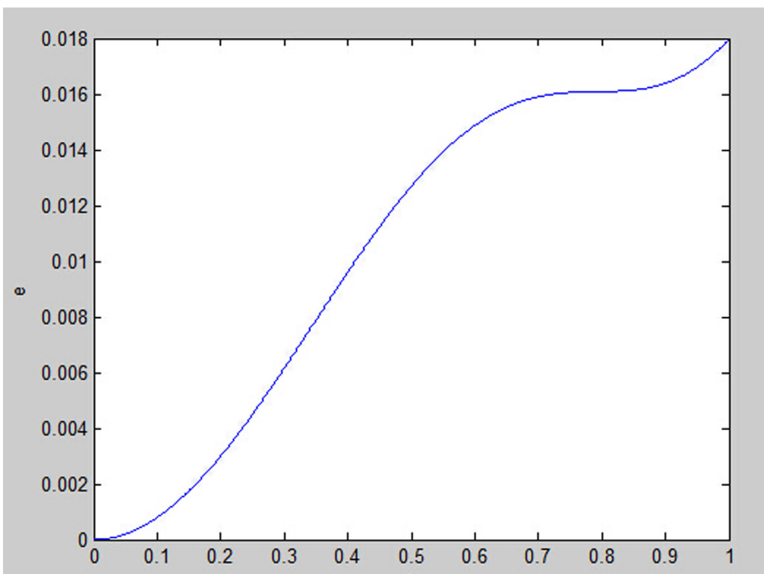

FIG. 7. Error e $t = [0,1]$, $h = 0.1, \tau = 0.001$. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

From Table II, it is obvious to see that the scheme (5) is convergent in maximum norm, and the convergent order is $O(\tau^2 + h^2)$. Table III show that the scheme (5) is very effective in term of CPU time.

TABLE I. The value of the scheme E^n.

t	$h = 0.1, \tau = 0.01$	$h = 0.1, \tau = 0.005$
0.2	0.0201	0.0201
0.4	0.0202	0.0202
0.6	0.0201	0.0201
0.8	0.0200	0.0200
1.0	0.0201	0.0201

TABLE II. Error and convergence order.

h	τ	$E_\infty(h,\tau)$	$E_\infty(2h,2\tau)/E_\infty(h,\tau)$
0.1	0.01	0.019232	
0.05	0.005	0.004815	3.994
0.025	0.0025	0.0012046	3.996

TABLE III. CPU time of the scheme.

h	τ	CPU time	h	τ	CPU time
0.2	0.01	5.656s	0.1	0.02	9.578s
0.1	0.01	14.141s	0.1	0.01	14.141s
0.05	0.01	56.172s	0.1	0.005	22.922s
0.025	0.01	248.672s	0.1	0.0025	41.734s

V. CONCLUSION

In this article, a conservative difference scheme is constructed for two dimensional NLS equation with wave operator. We prove the estimate of the numerical solution in maximum norm by the energy method and Lemma 2.3. The conservation, convergence, and stability are certified. In numerical experiment, an iterative algorithm is used to solve the implicit and nonlinear scheme, and numerical results are carried out to confirm the theoretical analysis.

It is easy to extend our estimates and algorithm for other derivatives of NLS equation. Moreover, it is not difficult to extend our method and algorithm for high dimensional other derivatives of NLS equations by the embedding theory. Future work is to construct and prove the higher accuracy of scheme and effective algorithm.

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