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A compact finite difference scheme for the nonlinear Schrödinger equation with wave operator ☆

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ABSTRACT

In this paper, a compact finite difference scheme is presented for an periodic initial value problem of the nonlinear Schrödinger (NLS) equation with wave operator. This is a scheme of three levels with a discrete conservation law. The unconditional stability and convergence in maximum norm with order $O(h^4+\tau^2)$ are proved by the energy method. A numerical experiment is presented to support our theoretical results.

1. Introduction

The NLS equation with wave operator was presented in [1], when considering the nonlinear interaction of monochromatic waves. The same equation can also be deduced in discussing the problem of soliton in plasma physics. In this paper, the following periodic initial value problem of NLS equation with wave operator is considered:

$$u_{tt}-u_{xx}+i\alpha u_t+\beta|u|^2u=0,\quad x\in R,\quad 0<t<T, \tag{1}$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in R, \tag{2}$$

$$u(x+L,t)=u(x,t),\quad x\in R,\quad 0\leqslant t\leqslant T, \tag{3}$$

where $u(x,t)$ is a complex function, α, β are two real constants, L is the period, and $i^2=-1$.

To solve the periodic initial value problem (1)–(3), we restrict it on a bounded domain $(-\frac{L}{2},\frac{L}{2})$. Computing the inner product of (1) with u_t and then taking the real part, the conservation law is obtained as

$$\|u_t\|_{L_2}^2+\|u_x\|_{L_2}^2+\frac{\beta}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}}|u|^4dx=const.$$

The finite difference scheme in [2] is an implicit nonconservative one which needs lots of algebraic operators. An explicit conservative finite difference scheme was constructed in [3], but which is conditionally stable. It is known that the conservative schemes are better than the nonconservative ones. Zhang et al. found that the nonconservative schemes may easily show nonlinear blow-up when they study for NLS equation, so they presented a conservative difference scheme in [4]. Then, in [5–12] the conservative finite difference schemes were used for a system of the generalized NLS equations, Regularized long wave equations, Sine–Gordon equation, Klein–Gordon equation and Zakharov equations, respectively. Numerical results of all the schemes are very good. However, in [3,13] the convergence order of all the schemes about the NLS equation

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with wave operator is $O(h^2+\tau^2)$. Recently, Wang presented compact finite difference scheme for the NLS equation in [14], it provides us a new thinking on the theoretical proving about compact difference scheme. The purpose of this paper is to construct a compact conservative difference scheme for the NLS equation with wave operator.

This paper is organized as follows. A new conservative scheme is proposed in Section 2. The discrete conservation law of the difference scheme is discussed in Section 3. In Section 4, the prior estimations for numerical solutions are made. In Section 5, the convergence and stability for the new scheme are proved. In the last section, an iterative algorithm and numerical results will be discussed.

2. Finite difference scheme

In this section, we describe a new difference scheme for problem (1)–(3). For convenience, the following notations are used:

$$\delta_t w_j^n = \frac{w_j^{n+1}-w_j^n}{\tau},\quad \delta_{\hat{t}} w_j^n = \frac{w_j^{n+1}-w_j^{n-1}}{2\tau},\quad \delta_x w_j^n = \frac{w_{j+1}^n-w_j^n}{h},\quad \delta_{\bar{x}} w_j^n = \frac{w_j^n-w_{j-1}^n}{h},$$

$$\delta_x^2 w_j^n = \delta_x\delta_{\bar{x}} w_j^n = \frac{1}{h^2}(w_{j-1}^n-2w_j^n+w_{j+1}^n),\quad A_h w_j^n = w_j^n+\frac{h^2}{12}\delta_x^2 w_j^n = \frac{1}{12}(w_{j-1}^n+10w_j^n+w_{j+1}^n).$$

where $h=\frac{L}{J}$ and $\tau=\frac{T}{N}$ are step sizes of space and time respectively, and J, N are two positive integers.

For any $\mathbf{a},\mathbf{b}\in V_h=\{\mathbf{v}|\mathbf{v}=(v_0,v_1,\ldots,v_{J-1})^T\}$, we define the inner product as

$$(\mathbf{a},\mathbf{b}) = h\sum_{j=0}^{J-1} a_j\bar{b}_j.$$

Also, we define norms as

$$\|\mathbf{v}\|_p = \sqrt[p]{h\sum_{j=0}^{J-1}|v_j|^p},\quad \|\delta_x\mathbf{v}\| = \sqrt{h\sum_{j=0}^{J-1}|\delta_x v_j|^2},\quad \|\mathbf{v}\|_\infty = \max_{0\leqslant j\leqslant J-1}|v_j|.$$

In the paper, we define $\{U_j^n\}$ as the exact solution and $\{u_j^n\}$ as the numerical one. C denotes a general positive constant which may have different values in different places. For the exact solution of problem (1)–(3), there exists the following inequality:

$$max\{\|U^n\|,\quad \|\delta_x U^n\|,\quad \|U^n\|_\infty\}\leqslant C.$$

Now, we present the following compact finite difference scheme for problem (1)–(3):

$$A_h\delta_t^2 u_j^n - \delta_x^2\frac{u_j^{n+1}+u_j^{n-1}}{2} + i\alpha A_h\delta_{\hat{t}} u_j^n + \frac{\beta}{2}A_h\left[(|u_j^{n+1}|^2+|u_j^{n-1}|^2)\frac{u_j^{n+1}+u_j^{n-1}}{2}\right] = 0,\quad j=0,1,\ldots,J-1;$$
$$n=1,2,\ldots,N-1, \tag{4}$$

$$u_j^0 = u_0(x_j),\quad \delta_t u_j^0 = u_1(x_j),\quad j=0,1,\ldots,J-1, \tag{5}$$

$$u_{j+J} = u_j. \tag{6}$$

Suppose

$$\mathbf{u}^n = (u_0^n,u_1^n,\ldots,u_{J-1}^n)^T,$$

$$|\mathbf{u}^{n+1}|^2+|\mathbf{u}^{n-1}|^2 = \mathbf{diag}(|u_0^{n+1}|^2+|u_0^{n-1}|^2,\ldots,|u_{J-1}^{n+1}|^2+|u_{J-1}^{n-1}|^2).$$

(4)–(6) can be written as

$$\mathbf{M}\delta_t^2\mathbf{u}^n - \delta_x^2\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2} + i\alpha\mathbf{M}\delta_{\hat{t}}\mathbf{u}^n + \frac{\beta}{2}\mathbf{M}(|\mathbf{u}^{n+1}|^2+|\mathbf{u}^{n-1}|^2)\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2} = 0,\quad n=1,2,\ldots,N-1,$$

where

$$M = \frac{1}{12}\begin{pmatrix} 10 & 1 & 0 & \cdots & 0 & 1\\ 1 & 10 & 1 & \cdots & 0 & 0\\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & 1 & 10 & 1\\ 1 & 0 & \cdots & 0 & 1 & 10 \end{pmatrix}_{J\times J}.$$

Obviously, $\mathbf{M}$ is a symmetric positive definite matrix and there is a symmetric positive definite matrix $\mathbf{H}$ such that $\mathbf{H} = \mathbf{M}^{-1}$. So the scheme (4)–(6) is equivalent to the following vector form:

$$\delta_t^2 \mathbf{u}^n - \mathbf{H}\delta_x^2 \frac{\mathbf{u}^{n+1} + \mathbf{u}^{n-1}}{2} + i\alpha\delta_{\hat{t}}\mathbf{u}^n + \frac{\beta}{2}(|\mathbf{u}^{n+1}|^2 + |\mathbf{u}^{n-1}|^2)\frac{\mathbf{u}^{n+1} + \mathbf{u}^{n-1}}{2} = 0, \quad n = 1, 2, \ldots, N-1, \tag{7}$$

$$u_j^0 = u_0(x_j), \quad \delta_{\hat{t}} u_j^0 = u_1(x_j), \quad j = 0, 1, \ldots, J-1, \tag{8}$$

$$u_{j+J} = u_j. \tag{9}$$

3. Discrete conservation law of the new scheme

To obtain the discrete conservation law, we introduce the following lemmas:

Lemma 3.1. *For any two mesh functions* $\mathbf{u}, \mathbf{v} \in V_h$ *and satisfied (6), there is the identity*

$$h\sum_{j=0}^{J-1}(\delta_x^2 u_j)\bar{v}_j = -h\sum_{j=0}^{J-1}(\delta_x u_j)(\delta_x \bar{v}_j).$$

Proof.

$$\begin{aligned} h\sum_{j=0}^{J-1}(\delta_x^2 u_j)\bar{v}_j &= \frac{1}{h}\sum_{j=0}^{J-1}(u_{j+1} - 2u_j + u_{j-1})\bar{v}_j = \frac{1}{h}\sum_{j=0}^{J-1}u_{j+1}\bar{v}_j - \frac{1}{h}\sum_{j=-1}^{J-2}u_{j+1}\bar{v}_{j+1} - \frac{1}{h}\sum_{j=0}^{J-1}u_j\bar{v}_j + \frac{1}{h}\sum_{j=-1}^{J-2}u_j\bar{v}_{j+1} \\ &= \frac{1}{h}\sum_{j=0}^{J-1}u_{j+1}\bar{v}_j - \frac{1}{h}\sum_{j=0}^{J-1}u_{j+1}\bar{v}_{j+1} - u_0\bar{v}_0 + u_J\bar{v}_J - \frac{1}{h}\sum_{j=0}^{J-1}u_j\bar{v}_j + \frac{1}{h}\sum_{j=0}^{J-1}u_j\bar{v}_{j+1} + u_{-1}\bar{v}_0 - u_{J-1}\bar{v}_J \\ &= \frac{1}{h}\sum_{j=0}^{J-1}u_{j+1}(\bar{v}_j - \bar{v}_{j+1}) + \frac{1}{h}\sum_{j=0}^{J-1}u_j(\bar{v}_{j+1} - \bar{v}_j) = -h\sum_{j=0}^{J-1}(\delta_x u_j)(\delta_x\bar{v}_j). \end{aligned}$$

This completes the proof of Lemma 3.1. □

Lemma 3.2. *For all mesh functions* $\{\mathbf{u}^n\}$ *satisfied (6), the following equalities hold:*

$$\mathbf{Re}(\delta_t^2\mathbf{u}^n, \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) = \|\delta_t\mathbf{u}^n\|^2 - \|\delta_t\mathbf{u}^{n-1}\|^2,$$

$$\mathbf{Re}(\delta_x^2(\mathbf{u}^{n+1} + \mathbf{u}^{n-1}), \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) = -(\|\delta_x\mathbf{u}^{n+1}\|^2 - \|\delta_x\mathbf{u}^{n-1}\|^2).$$

Proof.

$$\begin{aligned} \mathbf{Re}(\delta_t^2\mathbf{u}^n, \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) &= \frac{1}{\tau^2}\mathbf{Re}(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}, \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) \\ &= \frac{1}{\tau^2}\mathbf{Re}((\mathbf{u}^{n+1} - \mathbf{u}^n) - (\mathbf{u}^n - \mathbf{u}^{n-1}), (\mathbf{u}^{n+1} - \mathbf{u}^n) + (\mathbf{u}^n - \mathbf{u}^{n-1})) = \frac{1}{\tau^2}(\|\mathbf{u}^{n+1} - \mathbf{u}^n\|^2 - \|\mathbf{u}^n - \mathbf{u}^{n-1}\|^2) \\ &= \|\delta_t\mathbf{u}^n\|^2 - \|\delta_t\mathbf{u}^{n-1}\|^2. \end{aligned}$$

$$\mathbf{Re}(\delta_x^2(\mathbf{u}^{n+1} + \mathbf{u}^{n-1}), \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) = -\mathbf{Re}(\delta_x(\mathbf{u}^{n+1} + \mathbf{u}^{n-1}), \delta_x(\mathbf{u}^{n+1} - \mathbf{u}^{n-1})) = -(\|\delta_x\mathbf{u}^{n+1}\|^2 - \|\delta_x\mathbf{u}^{n-1}\|^2).$$

This completes the proof of Lemma 3.2. □

Lemma 3.3 [14]. *For any real symmetric positive definite matrices* $\mathbf{H}$, *we can get*

$$\mathbf{Re}(\mathbf{H}\delta_x^2(\mathbf{u}^{n+1} + \mathbf{u}^{n-1}), \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) = -(\|\mathbf{R}\delta_x\mathbf{u}^{n+1}\|^2 - \|\mathbf{R}\delta_x\mathbf{u}^{n-1}\|^2),$$

where $\mathbf{R}$ *is obtained by Cholesky decomposition for* $\mathbf{H}$, *denoted as* $\mathbf{R} = Chol(\mathbf{H})$.

Theorem 3.1. *The difference scheme (4)–(6) admits the following invariant:*

$$E^n = \|\delta_t \mathbf{u}^n\|^2 + \frac{1}{2}(\|\mathbf{R}\delta_x \mathbf{u}^{n+1}\|^2 + \|\mathbf{R}\delta_x \mathbf{u}^n\|^2) + \frac{\beta}{4}(\|\mathbf{u}^{n+1}\|_4^4 + \|\mathbf{u}^n\|_4^4) = const.$$

Proof. Computing the inner product of (7) with $\mathbf{u}^{n+1} - \mathbf{u}^{n-1}$, and then taking the real part,we get

$$I_1 - I_2 + I_3 + I_4 = 0,$$

where

$$I_1 = \mathbf{Re}(\delta_t^2 \mathbf{u}^n, \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) = \|\delta_t \mathbf{u}^n\|^2 - \|\delta_t \mathbf{u}^{n-1}\|^2,$$

$$I_2 = \frac{1}{2}\mathbf{Re}(\mathbf{H}\delta_x^2(\mathbf{u}^{n+1} + \mathbf{u}^{n-1}), \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) = -\frac{1}{2}(\|\mathbf{R}\delta_x \mathbf{u}^{n+1}\|^2 - \|\mathbf{R}\delta_x \mathbf{u}^{n-1}\|^2),$$

$$I_3 = \mathbf{Re}(i\alpha\delta_{\hat{t}} \mathbf{u}^n, \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) = \frac{\alpha}{2\tau}\mathbf{Im}(\mathbf{u}^{n+1} - \mathbf{u}^{n-1}, \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) = 0,$$

$$I_4 = \frac{\beta}{4}\mathbf{Re}((|\mathbf{u}^{n+1}|^2 + |\mathbf{u}^{n-1}|^2)(\mathbf{u}^{n+1} + \mathbf{u}^{n-1}), \mathbf{u}^{n+1} - \mathbf{u}^{n-1}) = \frac{\beta}{4}(\|\mathbf{u}^{n+1}\|_4^4 - \|\mathbf{u}^{n-1}\|_4^4).$$

We can obtain

$$\|\delta_t \mathbf{u}^n\|^2 - \|\delta_t \mathbf{u}^{n-1}\|^2 + \frac{1}{2}(\|\mathbf{R}\delta_x \mathbf{u}^{n+1}\|^2 - \|\mathbf{R}\delta_x \mathbf{u}^{n-1}\|^2) + \frac{\beta}{4}(\|\mathbf{u}^{n+1}\|_4^4 - \|\mathbf{u}^{n-1}\|_4^4) = 0.$$

Let

$$E^n = \|\delta_t \mathbf{u}^n\|^2 + \frac{1}{2}(\|\mathbf{R}\delta_x \mathbf{u}^{n+1}\|^2 + \|\mathbf{R}\delta_x \mathbf{u}^n\|^2) + \frac{\beta}{4}(\|\mathbf{u}^{n+1}\|_4^4 + \|\mathbf{u}^n\|_4^4). \tag{10}$$

Then $E^n = E^{n-1} = \cdots = E^0 = const.$

This completes the proof of Theorem 3.1. □

4. The prior estimations for the numerical solution

In this section, we will estimate the difference solution. First, some lemmas are introduced.

Lemma 4.1 (Discrete Sobolev's inequality [15]). *Suppose that* $\{u_j\}$ *is mesh functions. Given* $\varepsilon > 0$, *there exists a constant C dependent on* ε *such that*

$$\|u\|_\infty \leqslant \varepsilon\|u_x\| + C\|u\|.$$

Lemma 4.2 (Gronwall's inequality [16]). *Suppose that the nonnegative mesh function* $\{w(n), \rho(n), n = 1, 2, \ldots, N,\ N_\tau = T\}$ *satisfy the inequality*

$$w(n) \leqslant \rho(n) + \tau\sum_{l=1}^{n} B_l w(l),$$

where $B_l (l = 1, 2, \ldots, N)$ *are nonnegative constant. Then for any* $0 \leqslant n \leqslant N$, *there is*

$$w(n) \leqslant \rho(n) exp(n\tau\sum_{l=1}^{N} B_l).$$

Lemma 4.3 [3]. *For any mesh functions* $\{\mathbf{u}^n\}$, *there is*

$$\|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 \leqslant \tau[\|\delta_t \mathbf{u}^n\|^2 + \frac{1}{2}(\|\mathbf{u}^n\|^2 + \|\mathbf{u}^{n+1}\|^2)].$$

Lemma 4.4 [14]. *For any real symmetric positive definite matrices* **H**, *there exists two positive numbers* C_0, C_1, *s.t.*

$$C_0\|\mathbf{u}^n\|^2 \leqslant (\mathbf{H}\mathbf{u}^n, \mathbf{u}^n) \leqslant C_1\|\mathbf{u}^n\|^2.$$

Lemma 4.5. *Suppose that* $u_0(x) \in H_0^1,\ u_1(x) \in L_2,\ \beta > 0$, *then the following estimates hold:*

$$\|\mathbf{u}^n\| \leqslant C, \quad \|\delta_t \mathbf{u}^n\| \leqslant C, \quad \|\delta_x \mathbf{u}^n\| \leqslant C.$$

Proof. From (10), we get

$$E^n = \|\delta_t \mathbf{u}^n\|^2 + \frac{1}{2}(\|\mathbf{R}\delta_x \mathbf{u}^{n+1}\|^2 + \|\mathbf{R}\delta_x \mathbf{u}^n\|^2) + \frac{\beta}{4}(\|\mathbf{u}^{n+1}\|_4^4 + \|\mathbf{u}^n\|_4^4) = C,$$

so we have

$$\|\delta_t \mathbf{u}^n\|^2 + \frac{1}{2}(\|\mathbf{R}\delta_x \mathbf{u}^{n+1}\|^2 + \|\mathbf{R}\delta_x \mathbf{u}^n\|^2) \leqslant C.$$

From Lemma 4.4, we get

$$\|\mathbf{R}\delta_x \mathbf{u}^{n+1}\|^2 = (\mathbf{H}\delta_x \mathbf{u}^{n+1}, \delta_x \mathbf{u}^{n+1}) \geqslant C_0 \|\delta_x \mathbf{u}^{n+1}\|^2.$$

So we can obtain

$$\|\delta_t \mathbf{u}^n\|^2 + \frac{C_0}{2}(\|\delta_x \mathbf{u}^{n+1}\|^2 + \|\delta_x \mathbf{u}^n\|^2) \leqslant C. \tag{11}$$

From Lemma 4.3, we have

$$\|\mathbf{u}^{n+1}\|^2 - \|\mathbf{u}^n\|^2 \leqslant \tau\left[\|\delta_t \mathbf{u}^n\|^2 + \frac{1}{2}(\|\mathbf{u}^n\|^2 + \|\mathbf{u}^{n+1}\|^2)\right].$$

Summing up for n, we know

$$\left(1 - \frac{\tau}{2}\right)\|\mathbf{u}^{n+1}\|^2 \leqslant \|\mathbf{u}^0\|^2 + \tau\sum_{k=1}^{n}(\|\delta_t \mathbf{u}^k\|^2 + \|\mathbf{u}^k\|^2). \tag{12}$$

Adding (12) to (11), we get

$$\|\delta_t \mathbf{u}^n\|^2 + \frac{C_0}{2}(\|\delta_x \mathbf{u}^{n+1}\|^2 + \|\delta_x \mathbf{u}^n\|^2) + \left(1 - \frac{\tau}{2}\right)\|\mathbf{u}^{n+1}\|^2 \leqslant C + \tau C\sum_{k=0}^{n}(\|\delta_t \mathbf{u}^k\|^2 + \|\mathbf{u}^k\|^2).$$

According to the Lemma 4.2, the lemma holds when τ is small enough. □

Corollary. *If the conditions of Lemma 4.5 are satisfied, then,*

$$\|\mathbf{u}^n\|_\infty \leqslant C.$$

Proof. According to the Lemma 4.1 and Lemma 4.5, the corollary follows. □

5. Convergence and stability of the difference scheme

Assume that the truncation error

$$\mathbf{R}^n = (R_0^n, R_1^n, \ldots, R_{J-1}^n)^T \in \mathrm{V}_h,$$

then we have

$$\mathbf{R}^n = \delta_t^2 \mathbf{U}^n - \mathbf{H}\delta_x^2 \frac{\mathbf{U}^{n+1} + \mathbf{U}^{n-1}}{2} + i\alpha\delta_{\hat{t}} \mathbf{U}^n + \frac{\beta}{2}(|\mathbf{U}^{n+1}|^2 + |\mathbf{U}^{n-1}|^2)\frac{\mathbf{U}^{n+1} + \mathbf{U}^{n-1}}{2}. \tag{13}$$

According to Taylor's expansion, it can be easily obtained that

Lemma 5.1. *Assume that $u(x,t) \in C^{6,3}$, then the truncation error of the scheme (4)–(6) is of order $O(h^4 + \tau^2)$.*

Theorem 5.1. *Suppose that the conditions of the Lemma 4.5 and the Lemma 5.1 are fulfilled, then the numerical solution of the scheme (4)–(6) converge to the solution of the problem (1)–(3) with order $O(h^4 + \tau^2)$ by the $\|\cdot\|_\infty$ norm.*

Proof. Let

$$\mathbf{e}^n = \mathbf{U}^n - \mathbf{u}^n.$$

Subtracting (7) from (13), we obtain

$$\mathbf{R}^n = \delta_t^2\mathbf{e}^n - \mathbf{H}\delta_x^2\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2} + i\alpha\delta_{\hat{t}}\mathbf{e}^n + \frac{\beta}{2}\left[(|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2)\frac{\mathbf{U}^{n+1}+\mathbf{U}^{n-1}}{2} - (|\mathbf{u}^{n+1}|^2+|\mathbf{u}^{n-1}|^2)\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}\right]. \quad (14)$$

Computing the inner product of (14) with $\delta_{\hat{t}}\mathbf{e}^n$, and taking the real part, we get

$$\mathbf{Re}(\mathbf{R}^n, \delta_{\hat{t}}\mathbf{e}^n) = I_5 - I_6 + I_7 + I_8,$$

where

$$I_5 = \mathbf{Re}(\delta_t^2\mathbf{e}^n, \delta_{\hat{t}}\mathbf{e}^n) = \frac{1}{2\tau}\mathbf{Re}(\delta_t^2\mathbf{e}^n, \mathbf{e}^{n+1}-\mathbf{e}^{n-1}) = \frac{1}{2\tau}(\|\delta_t\mathbf{e}^n\|^2 - \|\delta_t\mathbf{e}^{n-1}\|^2),$$

$$I_6 = \frac{1}{4\tau}\mathbf{Re}(\mathbf{H}\delta_x^2(\mathbf{e}^{n+1}+\mathbf{e}^{n-1}), \mathbf{e}^{n+1}+\mathbf{e}^{n-1}) = -\frac{1}{4\tau}(\|\mathbf{R}\delta_x\mathbf{e}^{n+1}\|^2 - \|\mathbf{R}\delta_x\mathbf{e}^{n-1}\|^2),$$

$$I_7 = \mathbf{Re}(i\alpha\delta_{\hat{t}}\mathbf{e}^n, \delta_{\hat{t}}\mathbf{e}^n) = \alpha\mathbf{Im}(\delta_{\hat{t}}\mathbf{e}^n, \delta_{\hat{t}}\mathbf{e}^n) = 0,$$

$$\begin{aligned}
I_8 &= \frac{\beta}{2}\mathbf{Re}\left((|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2)\frac{\mathbf{U}^{n+1}+\mathbf{U}^{n-1}}{2} - (|\mathbf{u}^{n+1}|^2+|\mathbf{u}^{n-1}|^2)\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}, \delta_{\hat{t}}\mathbf{e}^n\right)\\
&= \frac{\beta}{2}\mathbf{Re}\left((|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2)\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2} + [(|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2) - (|\mathbf{u}^{n+1}|^2+|\mathbf{u}^{n-1}|^2)]\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}, \delta_{\hat{t}}\mathbf{e}^n\right)\\
&= \frac{\beta}{2}\mathbf{Re}((|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2)\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \delta_{\hat{t}}\mathbf{e}^n) + \frac{\beta}{2}\mathbf{Re}\left([(|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2) - (|\mathbf{u}^{n+1}|^2+|\mathbf{u}^{n-1}|^2)]\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}, \delta_{\hat{t}}\mathbf{e}^n\right).
\end{aligned}$$

We can obtain

$$\begin{aligned}
\mathbf{Re}(\mathbf{R}^n, \delta_{\hat{t}}\mathbf{e}^n) &= \frac{1}{2\tau}(\|\delta_t\mathbf{e}^n\|^2 - \|\delta_t\mathbf{e}^{n-1}\|^2) + \frac{1}{4\tau}(\|\mathbf{R}\delta_x\mathbf{e}^{n+1}\|^2 - \|\mathbf{R}\delta_x\mathbf{e}^{n-1}\|^2)\\
&+ \frac{\beta}{2}\mathbf{Re}\left((|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2)\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \delta_{\hat{t}}\mathbf{e}^n\right)\\
&+ \frac{\beta}{2}\mathbf{Re}\left([(|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2) - (|\mathbf{u}^{n+1}|^2+|\mathbf{u}^{n-1}|^2)]\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}, \delta_{\hat{t}}\mathbf{e}^n\right),
\end{aligned} \quad (15)$$

where

$$\mathbf{Re}(\mathbf{R}^n, \delta_{\hat{t}}\mathbf{e}^n) = \frac{1}{2}\mathbf{Re}\left(\mathbf{R}^n, \frac{\mathbf{e}^{n+1}-\mathbf{e}^{n-1}}{\tau}\right) = \frac{1}{2}\mathbf{Re}(\mathbf{R}^n, \delta_t\mathbf{e}^n + \delta_t\mathbf{e}^{n-1}) \leqslant C(\|\mathbf{R}^n\|^2 + \|\delta_t\mathbf{e}^n\|^2 + \|\delta_t\mathbf{e}^{n-1}\|^2),$$

$$\begin{aligned}
\frac{\beta}{2}\mathbf{Re}\left((|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2)\frac{\mathbf{e}^{n+1}+\mathbf{e}^{n-1}}{2}, \delta_{\hat{t}}\mathbf{e}^n\right) &= \frac{\beta}{8}\mathbf{Re}((|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2)(\mathbf{e}^{n+1}+\mathbf{e}^{n-1}), \delta_t\mathbf{e}^n + \delta_t\mathbf{e}^{n-1})\\
&\leqslant C(\|\mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^{n-1}\|^2 + \|\delta_t\mathbf{e}^n\|^2 + \|\delta_t\mathbf{e}^{n-1}\|^2),
\end{aligned}$$

$$\begin{aligned}
&\frac{\beta}{2}\mathbf{Re}\left([(|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2) - (|\mathbf{u}^{n+1}|^2+|\mathbf{u}^{n-1}|^2)]\frac{\mathbf{u}^{n+1}+\mathbf{u}^{n-1}}{2}, \delta_{\hat{t}}\mathbf{e}^n\right)\\
&= \frac{\beta}{8}\mathbf{Re}([(|\mathbf{U}^{n+1}|^2+|\mathbf{U}^{n-1}|^2) - (|\mathbf{u}^{n+1}|^2+|\mathbf{u}^{n-1}|^2)](\mathbf{u}^{n+1}+\mathbf{u}^{n-1}), \delta_t\mathbf{e}^n + \delta_t\mathbf{e}^{n-1})\\
&= \frac{\beta}{8}\mathbf{Re}([\mathbf{U}^{n+1}\bar{\mathbf{e}}^{n+1} + \mathbf{e}^{n+1}\bar{\mathbf{u}}^{n+1} + \mathbf{U}^{n-1}\bar{\mathbf{e}}^{n-1} + \mathbf{e}^{n-1}\bar{\mathbf{u}}^{n-1}](\mathbf{u}^{n+1}+\mathbf{u}^{n-1}), \delta_t\mathbf{e}^n + \delta_t\mathbf{e}^{n-1})\\
&\leqslant C(\|\mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^{n-1}\|^2 + \|\delta_t\mathbf{e}^n\|^2 + \|\delta_t\mathbf{e}^{n-1}\|^2).
\end{aligned}$$

From Lemma 4.3, we have

$$\frac{1}{\tau}(\|\mathbf{e}^n\|^2 - \|\mathbf{e}^{n-1}\|^2) \leqslant C(\|\delta_t\mathbf{e}^{n-1}\|^2 + \|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n-1}\|^2).$$

So we can obtain

$$\begin{aligned}
&\frac{1}{2\tau}(\|\delta_t\mathbf{e}^n\|^2 - \|\delta_t\mathbf{e}^{n-1}\|^2) + \frac{1}{4\tau}(\|\mathbf{R}\delta_x\mathbf{e}^{n+1}\|^2 - \|\mathbf{R}\delta_x\mathbf{e}^{n-1}\|^2) + \frac{1}{\tau}(\|\mathbf{e}^n\|^2 - \|\mathbf{e}^{n-1}\|^2)\\
&\leqslant C(\|\mathbf{R}^n\|^2 + \|\delta_t\mathbf{e}^n\|^2 + \|\delta_t\mathbf{e}^{n-1}\|^2 + \|\mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n-1}\|^2).
\end{aligned} \quad (16)$$

Summing (16) up for n, we get

$$\frac{1}{2}\|\delta_t \mathbf{e}^n\|^2 + \frac{1}{4}(\|\mathbf{R}\delta_x \mathbf{e}^{n+1}\|^2 + \|\mathbf{R}\delta_x \mathbf{e}^n\|^2) + \|\mathbf{e}^n\|^2 \leqslant [O(h^4+\tau^2)]^2 + \tau C\sum_{i=1}^{n}\left[\frac{1}{2}\|\delta_t \mathbf{e}^i\|^2 + \frac{1}{4}(\|\mathbf{R}\delta_x \mathbf{e}^{i+1}\|^2 + \|\mathbf{R}\delta_x \mathbf{e}^i\|^2) + \|\mathbf{e}^i\|^2\right].$$

According to the Lemma 4.2, when τ is small enough, it is follows that

$$\frac{1}{2}\|\delta_t \mathbf{e}^n\|^2 + \frac{1}{4}(\|\mathbf{R}\delta_x \mathbf{e}^{n+1}\|^2 + \|\mathbf{R}\delta_x \mathbf{e}^n\|^2) + \|\mathbf{e}^n\|^2 \leqslant [O(h^4+\tau^2)]^2.$$

From Lemma 4.4, we obtain

$$\|\mathbf{R}\delta_x \mathbf{e}^{n+1}\|^2 = (\mathbf{H}\delta_x \mathbf{e}^{n+1}, \delta_x \mathbf{e}^{n+1}) \geqslant C_0\|\delta_x \mathbf{e}^{n+1}\|^2.$$

So we can get

$$\frac{1}{2}\|\delta_t \mathbf{e}^n\|^2 + C(\|\delta_x \mathbf{e}^{n+1}\|^2 + \|\delta_x \mathbf{e}^n\|^2) + \|\mathbf{e}^n\|^2 \leqslant [O(h^4+\tau^2)]^2.$$

Then

$$\|\mathbf{e}^n\| \leqslant O(h^4+\tau^2), \quad \|\delta_t \mathbf{e}^n\| \leqslant O(h^4+\tau^2), \quad \|\delta_x \mathbf{e}^n\| \leqslant O(h^4+\tau^2). \tag{17}$$

Lastly, according to the Lemma 4.1, it is follows that

$$\|\mathbf{e}^n\|_\infty \leqslant O(h^4+\tau^2). \tag{18}$$

Then the convergence is proved. □

Similarly, we can prove the stability of the difference solution. i.e.

Theorem 5.2. *Under the conditions of Theorem 5.1, the solution of the difference scheme (4)–(6) is unconditionally stable for initial data by the* $\|\cdot\|_\infty$ *norm.*

6. Algorithm and numerical experiment

In this section, we take $\alpha = \beta = 1$, $L = 80$, and consider the following problem:

$$u_{tt} - u_{xx} + iu_t + |u|^2 u = 0, \quad -40 < x < 40, \ 0 < t < T, \tag{19}$$

$$u(x,0) = (1+i)xe^{-10(1-x)^2}, \quad u_t(x,0) = 0, \ -40 < x < 40. \tag{20}$$

6.1. Iterative algorithm

For problems (19),(20), we have the following difference scheme

$$\frac{2}{\tau^2}A_h(u_j^{n+1} - 2u_j^n + u_j^{n-1}) - \delta_x^2(u_j^{n+1} + u_j^{n-1}) + \frac{i}{\tau}A_h(u_j^{n+1} - u_j^{n-1}) + \frac{1}{2}A_h B_j^{n+1} = 0, \quad j = 0,1,\ldots,J-1, \ n = 1,2,\ldots,N-1, \tag{21}$$

where

$$B_j^{n+1} = (|u_j^{n+1}|^2 + |u_j^{n-1}|^2)(u_j^{n+1} + u_j^{n-1}), \quad j = 0,1,\ldots,J-1.$$

And the vector form of the scheme is

$$\frac{2}{\tau^2}(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) - \mathbf{H}\delta_x^2(\mathbf{u}^{n+1} + \mathbf{u}^{n-1}) + \frac{i}{\tau}(\mathbf{u}^{n+1} - \mathbf{u}^{n-1}) + \frac{1}{2}\mathbf{B}^{n+1} = 0, \quad n = 1,2,\ldots,N-1, \tag{22}$$

where

$$\mathbf{B}^{n+1} = (B_0^{n+1}, B_1^{n+1}, \ldots, B_{J-1}^{n+1})^T = (|\mathbf{u}^{n+1}|^2 + |\mathbf{u}^{n-1}|^2)(\mathbf{u}^{n+1} + \mathbf{u}^{n-1}).$$

Obviously, the scheme (21) is an implicit and nonlinear one. In order to obtain the numerical solution $\{u_j^{n+1}\}$, an iterative algorithm can be used.

Let

$$B_j^{n+1(s)} = (|u_j^{n+1(s)}|^2 + |u_j^{n-1}|^2)(u_j^{n+1(s)} + u_j^{n-1}), \quad j = 0,1,\ldots,J-1, \quad s = 0,1,2,\ldots,$$

we define the following iterative algorithm

$$\frac{2}{\tau^2}A_h(u_j^{n+1(s+1)} - 2u_j^n + u_j^{n-1}) - \delta_x^2(u_j^{n+1(s+1)} + u_j^{n-1}) + \frac{i}{\tau}A_h(u_j^{n+1(s+1)} - u_j^{n-1}) + \frac{1}{2}A_hB_j^{n+1(s)} = 0,$$
$$j = 0,1,\ldots,J-1, \; n = 1,2,\ldots,N-1, \; s = 1,2,\ldots, \tag{23}$$

where

$$u_j^{n+1(0)} = \begin{cases} u_j^n, & n = 0 \\ 2u_j^n - u_j^{n-1}, & n \geqslant 1 \end{cases}.$$

For $n = 0$, from (5) and (21), we obtain

$$\frac{2}{\tau^2}A_h(u_j^{1(s+1)} - u_j^0 - \tau u_1(x_j)) - \delta_x^2(u_j^{1(s+1)} - \tau u_1(x_j)) + iA_h(u_1(x_j)) + \frac{1}{4}A_hB_j^{1(s)} = 0, \quad j = 0,1,\ldots,J-1. \tag{24}$$

The vector form of the iterative algorithm (23) is

$$\frac{2}{\tau^2}(\mathbf{u}^{n+1(s+1)} - 2\mathbf{u}^n + \mathbf{u}^{n-1}) - \mathbf{H}\delta_x^2(\mathbf{u}^{n+1(s+1)} + \mathbf{u}^{n-1}) + \frac{i}{\tau}(\mathbf{u}^{n+1(s+1)} - \mathbf{u}^{n-1}) + \frac{1}{2}\mathbf{B}^{n+1(s)} = 0,$$
$$n = 1,2,\ldots,N-1, \; s = 1,2,\ldots, \tag{25}$$

for $n = 0$, we have

$$\frac{2}{\tau^2}(\mathbf{u}^{1(s+1)} - \mathbf{u}^0 - \tau u_1(\mathbf{x})) - \mathbf{H}\delta_x^2(\mathbf{u}^{1(s+1)} - \tau u_1(\mathbf{x})) + i(u_1(\mathbf{x})) + \frac{1}{4}\mathbf{B}^{1(s)} = 0, \quad \mathbf{x} = (x_0, x_1, \ldots, x_{J-1}). \tag{26}$$

Theorem 6.1. *Suppose that $u_0(x) \in H_0^1, u(x,t) \in C^{6,3}$, the solution of the iterative algorithm (23,24) is converged to the solution of the scheme(4)when h and τ are all small enough.*

Proof. The proof of the Theorem 6.1 refers to the Theorem 5.1 in [14]. □

6.2. Numerical experiment

6.2.1. Convergence order

Firstly, we verify the convergence order of scheme (4) which is stated in Theorem 5.1. We take $t = 1$, and choose the numerical solution with $h = 0.0125, \; \tau = h^2$ as the approximate exact solution. Then we obtain the following convergence order figure for $h = 0.2, 0.1, 0.05, 0.025, \; \tau = h^2$. From Fig. 1, it is obvious that the scheme (4) is convergent in maximum norm, and the convergence order is $O(h^4 + \tau^2)$.

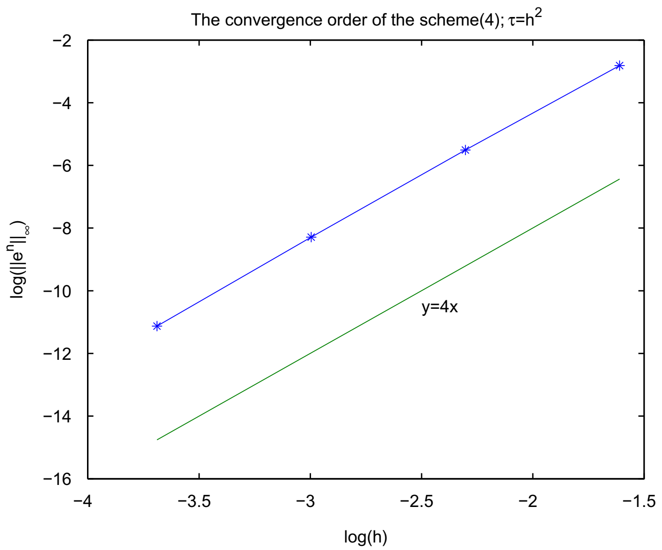

Fig. 1. The convergence order of the scheme (4); $\tau = h^2$.

6.2.2. Comparison

In this subsection, we give the pictures of the wave propagation from $t = 0$ to $t = 10$ in Fig. 2. Here we choose three different step sizes with $h = 0.05, 0.08, 0.10,\ \tau = h^2$, respectively. From the first three pictures, we can conclude that the wave propagation is independent of the step sizes. The last three pictures in Fig. 2 show the movement of $|U|$ with three different step sizes at $t = 2, 5, 10$, respectively. The wave curves of three step sizes completely coincide with one another. Consequently, Fig. 2 proves that the new scheme is not affected by grid ratios and it is suitable for long-term computation.

Then, we consider the periodic initial value problem in [2], and compare the scheme in [2] with the scheme (4). Here we note the former as S1, and the latter as S2. Fig. 3 shows the comparison of phasic picture of U which is formed by two schemes with different step sizes at the same time. Fig. 4 shows the comparison of $|U|$ which is formed by two schemes with the same step size at the different time. Both two figures demonstrate that the new scheme is effective.

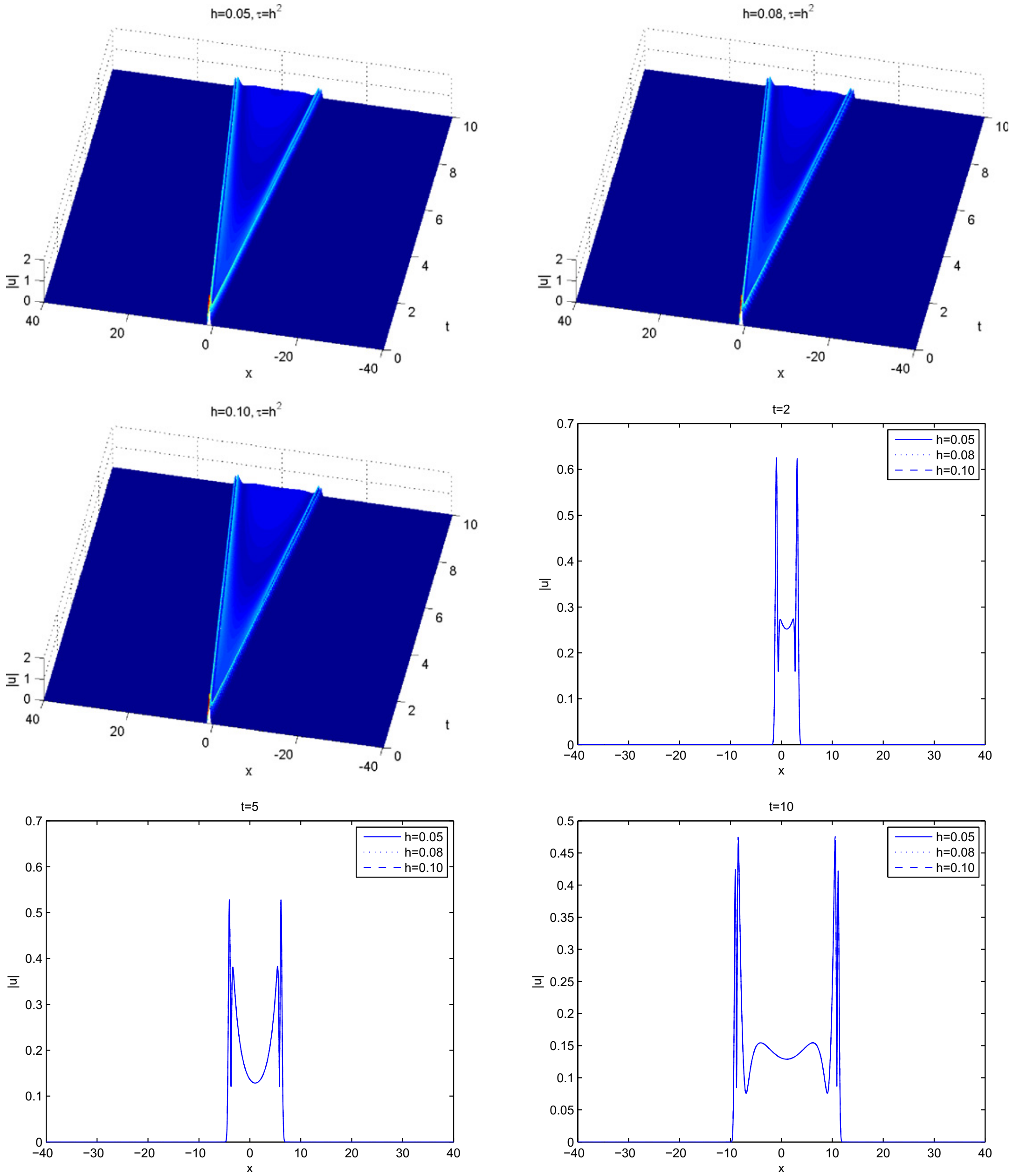

Fig. 2. The wave propagation with different step sizes from $t = 0$ to $t = 10$.

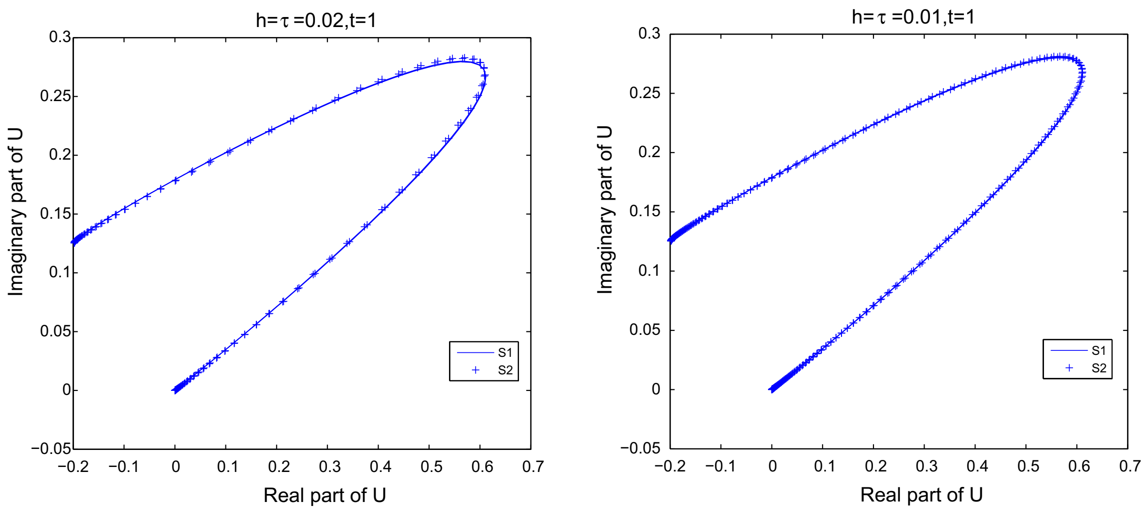

Fig. 3. Comparison of phasic picture of U: (left) $h = \tau = 0.02,\ t = 1$ and (right) $h = \tau = 0.01,\ t = 1$.

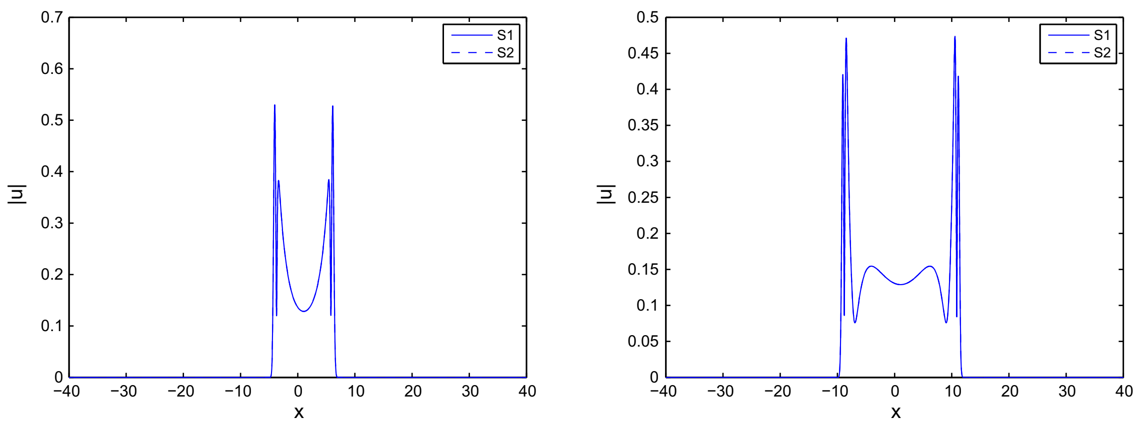

Fig. 4. Comparison of $|U|$: (left) $h = \tau = 0.01,\ t = 5$ and (right) $h = \tau = 0.01,\ t = 10$.

Table 1
The value of E at different time with $h = 0.05,\ \tau = h^2$.

t	E	t	E
1	9.123449881568437	6	9.123449881508652
2	9.123449881550876	7	9.123449881506540
3	9.123449881535114	8	9.123449881506801
4	9.123449881522152	9	9.123449881509744
5	9.123449881513384	10	9.123449881515821

6.2.3. Discrete conservation law

At last,we compute the discrete conservation law. Here we choose $t = 10,\ h = 0.05,\ \tau = h^2$. Table 1 shows the value of E at different time. It indicates that the conservation of the scheme (4) is very good and it is suitable for long-term computation.

7. Conclusion

In this paper, a compact finite difference scheme is constructed for the nonlinear Schrödinger equation with wave operator. The conservation, convergence, and stability are certified. In numerical experiment, an iterative algorithm is used to solve the implicit and nonlinear scheme and numerical results are carried out to confirm the theoretical proving.

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