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ORIGINAL RESEARCH

A new numerical scheme for the nonlinear Schrödinger equation with wave operator

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Abstract In this paper, a new compact finite difference scheme is proposed for a periodic initial value problem of the nonlinear Schrödinger equation with wave operator. This is an explicit scheme of four levels with a discrete conservation law. The unconditional stability and convergence in maximum norm with order $O(h^4 + \tau^2)$ are verified by the energy method. Those theoretical results are proved by a numerical experiment and it is also verified that this scheme is better than the previous scheme via comparison.

Keywords Nonlinear Schrödinger equation · Wave operator · Four-level explicit scheme · Conservation · Convergence · Stability

1 Introduction

The NLS equation with wave operator was presented in [1], when considering the nonlinear interaction of monochromatic waves. The same equation can also be deduced in discussing the problem of soliton in plasma physics. The aim of this work is to discuss the following periodic initial value problem of the NLS equation with wave operator:

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$$u_{tt} - u_{xx} + i\alpha u_t + \beta |u|^2 u = 0, \quad x \in R, \quad 0 < t < T, \tag{1}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in R, \tag{2}$$

$$u(x + L, t) = u(x, t), \quad x \in R, \quad 0 \le t \le T, \tag{3}$$

where $u(x, t)$ is a complex function, L is the period, α, β are two real constants, and $i^2 = -1$.

In order to solve the problem (1)–(3), we restrict it on $(-\frac{L}{2}, \frac{L}{2})$. Computing the inner product of (1) with u_t and taking the real part, the conservation law is obtained as

$$\|u_t\|_{L_2}^2 + \|u_x\|_{L_2}^2 + \frac{\beta}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} |u|^4 dx = const. \tag{4}$$

To the authors' knowledge, a number of methods are proposed to solve the NLS equation and related equations. For instance, the Adomian decomposition method (ADM, [2,3]), the variational iteration method (VIM, [4–6]), the homotopy perturbation method (HPM, [7–9]), the differential transform method (DTM, [10,11]). Furthermore, Fatoorehchi and Abolghasemi used these methods and the improvements in different areas of different equations (see [12–20]). All of the above are semi-analytical-techniques that transform the equation into a recurrence equation for solving. The finite difference method (FDM) is a discrete-numerical-technique. It is able to control the error of solution in a small area by variable discretization. In [21], a finite difference scheme is proposed, however, it is nonconservative and its accuracy is only $O(h^2 + \tau)$. It is desirable and natural to form numerical schemes keeping special properties of original problems, such as the conservation law. Zhang et al. presented a conservative difference scheme for the NLS equation in [22] when they found that the nonconservative schemes may easily show nonlinear blow-up. The conclusion is proved in the generalized NLS equations, Regularized long wave equations, Sine-Gordon equation, Klein–Gordon equation and Zakharov equations in [23–28], respectively. Since then, many conservative schemes for the NLS equation with wave operator are presented in [29–32]. However, the convergence order of all the schemes is $O(h^2 + \tau^2)$, referring to Wang in [33], Li et al. presented a nonlinear three-level iterative difference scheme and improved the accuracy order to $O(h^4 + \tau^2)$ in [34]. This paper aims to construct an explicit compact difference scheme of four levels for the NLS equation with wave operator, demonstrating its accuracy by both theory and numerical experiment. Through the comparison of computation time and infinite modulo error, the new scheme is verified to be better than the scheme in [34].

The layout of the paper is as follows. In Sect. 2, a new explicit conservative scheme of four levels is given. The discrete conservation law of the scheme is discussed in Sect. 3. The prior estimations for numerical solutions are made in Sect. 4. In Sect. 5, the convergence in maximum norm and unconditional stability for the new scheme are confirmed. Finally, numerical tests will be discussed in the last section.

2 Description of the finite difference scheme

In this section, we propose a new difference scheme for problem (1)–(3). As usual, the following notations are used:

$$x_j = -\frac{L}{2} + jh,\ t_n = n\tau,\ \Omega_h = \{x_j | 0 \le j \le J\},\ \Omega_\tau = \{t_n | 0 \le n \le N\},\ \Omega_h^\tau = \Omega_h \times \Omega_\tau,$$

where $h = \frac{L}{J}$ and $\tau = \frac{T}{N}$ denote the spatial and temporal step sizes respectively, and J, N are two positive integers, $u_j^n \equiv u(x_j, t_n)$, $U_j^n \cong u(x_j, t_n)$.

$$\delta_t V_j^n = \frac{V_j^{n+1} - V_j^n}{\tau}, \delta_{\bar{t}} V_j^n = \frac{V_j^n - V_j^{n-1}}{\tau}, \quad \delta_x V_j^n = \frac{V_{j+1}^n - V_j^n}{h},$$
$$\delta_{\bar{x}} V_j^n = \frac{V_j^n - V_{j-1}^n}{h},$$
$$\delta_x^2 V_j^n = \delta_x \delta_{\bar{x}} V_j^n = \frac{1}{h^2}(V_{j-1}^n - 2V_j^n + V_{j+1}^n),$$
$$\delta_t^2 V_j^n = \delta_t \delta_{\bar{t}} V_j^n = \frac{1}{\tau^2}(V_j^{n+1} - 2V_j^n + V_j^{n-1}),$$
$$A_h V_j^n = (1 + \frac{h^2}{12}\delta_x^2) V_j^n = V_j^n + \frac{h^2}{12}\delta_x^2 V_j^n = \frac{1}{12}(V_{j+1}^n + 10V_j^n + V_{j-1}^n).$$

We define the inner product and norms as

$$(\mathbf{U}^n, \mathbf{V}^n) = h \sum_{j=0}^{J-1} U_j^n \bar{V}_j^n, \quad (\mathbf{U}^n, \mathbf{V}^n \in \Omega_h^n = \{\mathbf{w}^n | \mathbf{w}^n = (\omega_0^n, \omega_1^n, \ldots, \omega_{J-1}^n)^T\}),$$
$$\|\mathbf{V}^n\|_p = \sqrt{[p]h \sum_{j=0}^{J-1} |V_j^n|^p}, \quad \|\delta_x \mathbf{V}^n\| = \sqrt{h \sum_{j=0}^{J-1} |\delta_x V_j^n|^2}, \|\mathbf{V}^n\|_\infty = \max_{0 \le j \le J-1} |V_j^n|,$$

and in the paper, C denotes a general positive constant which may have different values in different places.

Now, we present the following four-level compact finite difference scheme for problem (1)–(3):

$$\frac{1}{2} A_h \delta_t^2 (U_j^{n+1} + U_j^n) - \frac{1}{2}\delta_x^2 (U_j^{n+1} + U_j^n) + i\alpha A_h \delta_t U_j^n$$
$$+ \frac{\beta}{4} A_h (|U_j^{n+1}|^2 + |U_j^n|^2)(U_j^{n+1} + U_j^n) = 0,$$
$$j = 0, 1, \ldots, J-1; \quad n = 1, 2, \ldots, N-2, \tag{5}$$
$$U_j^0 = u_0(x_j), \quad \delta_t U_j^0 = u_1(x_j), \quad j = 0, 1, \ldots, J-1, \tag{6}$$
$$U_{j+J} = U_j. \tag{7}$$

Suppose

$$\mathbf{U}^n = (U_0^n, U_1^n, \ldots, U_{J-1}^n)^T, \quad |\mathbf{U}^{n+1}|^2 + |\mathbf{U}^n|^2$$
$$= \mathbf{diag}(|U_0^{n+1}|^2 + |U_0^n|^2, \ldots, |U_{J-1}^{n+1}|^2 + |U_{J-1}^n|^2).$$

(5) can be written as

$$\frac{1}{2}\mathbf{M}\delta_t^2(\mathbf{U}^{n+1} + \mathbf{U}^n) - \frac{1}{2}\delta_x^2(\mathbf{U}^{n+1} + \mathbf{U}^n) + i\alpha\mathbf{M}\delta_t\mathbf{U}^n$$
$$+\frac{\beta}{4}\mathbf{M}(|\mathbf{U}^{n+1}|^2 + |\mathbf{U}^n|^2)(\mathbf{U}^{n+1} + \mathbf{U}^n) = 0, \quad n = 1, 2, \ldots, N-2,$$

where $\mathbf{M}$ is a symmetric positive definite matrix and

$$\mathbf{M} = \frac{1}{12}\begin{pmatrix} 10 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 10 & 1 & \cdots & 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & 1 & 10 & 1 \\ 1 & 0 & \cdots & 0 & 1 & 10 \end{pmatrix}_{J\times J}.$$

Considering the vector form, the scheme (5)–(7) is equivalent to the following one:

$$\frac{1}{2}\delta_t^2(\mathbf{U}^{n+1} + \mathbf{U}^n) - \frac{1}{2}\mathbf{H}\delta_x^2(\mathbf{U}^{n+1} + \mathbf{U}^n) + i\alpha\delta_t\mathbf{U}^n$$
$$+\frac{\beta}{4}(|\mathbf{U}^{n+1}|^2 + |\mathbf{U}^n|^2)(\mathbf{U}^{n+1} + \mathbf{U}^n) = 0,$$
$$n = 1, 2, \ldots, N-2, \tag{8}$$
$$U_j^0 = u_0(x_j), \quad \delta_t U_j^0 = u_1(x_j), \quad j = 0, 1, \ldots, J-1, \tag{9}$$
$$U_{j+J} = U_j, \tag{10}$$

where $\mathbf{H} = \mathbf{M}^{-1}$ and $\mathbf{H}$ is also a symmetric positive definite matrix.

3 Discrete conservation law of the new scheme

To obtain the discrete conservation law, some lemmas are required in the subsequent analysis:

Lemma 3.1 ([30]) *For any mesh functions $\boldsymbol{U}^n$, the following equalities hold:*

(1) $2\boldsymbol{Re}(\delta_t\boldsymbol{U}^{n+1}, \delta_t\boldsymbol{U}^n) = \|\delta_t\boldsymbol{U}^{n+1}\|^2 + \|\delta_t\boldsymbol{U}^n\|^2 - \tau^2\|\delta_t^2\boldsymbol{U}^{n+1}\|^2.$

(2) $\boldsymbol{Re}(\delta_t^2(\boldsymbol{U}^{n+1} + \boldsymbol{U}^n), \delta_t\boldsymbol{U}^n) = \frac{1}{2\tau}(\|\delta_t\boldsymbol{U}^{n+1}\|^2 - \|\delta_t\boldsymbol{U}^{n-1}\|^2) - \frac{\tau}{2}(\|\delta_t^2\boldsymbol{U}^{n+1}\|^2 - \|\delta_t^2\boldsymbol{U}^n\|^2).$

(3) $\boldsymbol{Re}(\delta_x^2(\boldsymbol{U}^{n+1} + \boldsymbol{U}^n), \delta_t\boldsymbol{U}^n) = -\frac{1}{\tau}(\|\delta_x\boldsymbol{U}^{n+1}\|^2 - \|\delta_x\boldsymbol{U}^n\|^2).$

Lemma 3.2 ([33]) *For any real symmetric positive definite matrices* $\boldsymbol{H}$, *we can get*

$$\boldsymbol{Re}(\boldsymbol{H}\delta_x^2(\boldsymbol{u}^{n+1}+\boldsymbol{u}^{n-1}),\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n-1}) = -(\|\boldsymbol{R}\delta_x\boldsymbol{u}^{n+1}\|^2 - \|\boldsymbol{R}\delta_x\boldsymbol{u}^{n-1}\|^2),$$

where $\boldsymbol{R}$ *is obtained by Cholesky decomposition for* $\boldsymbol{H}$, *denoted as* $\boldsymbol{R} = Chol(\boldsymbol{H})$.

Lemma 3.3 ([34]) *For any two mesh functions* $\boldsymbol{U}^n$, $\boldsymbol{V}^n \in \Omega_h^n$ *and satisfied (7), there is the identity*

$$h\sum_{j=0}^{J-1}(\delta_x^2 U_j^n)\bar{V}_j^n = -h\sum_{j=0}^{J-1}(\delta_x U_j^n)(\delta_x \bar{V}_j^n).$$

Theorem 3.1 *The difference scheme (5)–(7) admits the following invariant:*

$$E^n = \frac{1}{2}\left(\|\delta_t\boldsymbol{U}^{n+1}\|^2 + \|\delta_t\boldsymbol{U}^n\|^2 - \tau^2\|\delta_t^2\boldsymbol{U}^{n+1}\|^2\right) + \|\boldsymbol{R}\delta_x\boldsymbol{U}^{n+1}\|^2 + \frac{\beta}{2}\|\boldsymbol{U}^{n+1}\|_4^4 = const.$$

Proof Computing the inner product of (8) with $\mathbf{U}^{n+1}-\mathbf{U}^n$, and taking the real part, we obtain

$$I_1 - I_2 + I_3 + I_4 = 0,$$

where

$$\begin{aligned}
I_1 &= \frac{1}{2}\mathbf{Re}(\delta_t^2(\mathbf{U}^{n+1}+\mathbf{U}^n),\mathbf{U}^{n+1}-\mathbf{U}^n)\\
&= \frac{1}{4}(\|\delta_t\mathbf{U}^{n+1}\|^2 - \|\delta_t\mathbf{U}^{n-1}\|^2 - \tau^2\|\delta_t^2\mathbf{U}^{n+1}\|^2 + \tau^2\|\delta_t^2\mathbf{U}^n\|^2),\\
I_2 &= \frac{1}{2}\mathbf{Re}(\mathbf{H}\delta_x^2(\mathbf{U}^{n+1}+\mathbf{U}^n),\mathbf{U}^{n+1}-\mathbf{U}^n) = -\frac{1}{2}(\|\mathbf{R}\delta_x\mathbf{U}^{n+1}\|^2 - \|\mathbf{R}\delta_x\mathbf{U}^n\|^2),\\
I_3 &= \mathbf{Re}(i\alpha\delta_t\mathbf{U}^n,\mathbf{U}^{n+1}-\mathbf{U}^n) = \frac{\alpha}{\tau}\mathbf{Im}(\mathbf{U}^{n+1}-\mathbf{U}^n,\mathbf{U}^{n+1}-\mathbf{U}^n) = 0,\\
I_4 &= \frac{\beta}{4}\mathbf{Re}((|\mathbf{U}^{n+1}|^2+|\mathbf{U}^n|^2)(\mathbf{U}^{n+1}+\mathbf{U}^n),\mathbf{U}^{n+1}-\mathbf{U}^n) = \frac{\beta}{4}(\mathbf{U}^{n+1}\|_4^4 - \|\mathbf{U}^n\|_4^4).
\end{aligned}$$

We can get

$$\begin{aligned}
&\frac{1}{2}\left(\|\delta_t\mathbf{U}^{n+1}\|^2 - \|\delta_t\mathbf{U}^{n-1}\|^2 - \tau^2\|\delta_t^2\mathbf{U}^{n+1}\|^2 + \tau^2\|\delta_t^2\mathbf{U}^n\|^2\right)\\
&\quad + \left(\|\mathbf{R}\delta_x\mathbf{U}^{n+1}\|^2 - \|\mathbf{R}\delta_x\mathbf{U}^n\|^2\right) + \frac{\beta}{2}\left(\|\mathbf{U}^{n+1}\|_4^4 - \|\mathbf{U}^n\|_4^4\right) = 0.
\end{aligned}$$

Let

$$E^n = \frac{1}{2}(\|\delta_t\mathbf{U}^{n+1}\|^2 + \|\delta_t\mathbf{U}^n\|^2 - \tau^2\|\delta_t^2\mathbf{U}^{n+1}\|^2) + \|\mathbf{R}\delta_x\mathbf{U}^{n+1}\|^2 + \frac{\beta}{2}\|\mathbf{U}^{n+1}\|_4^4. \tag{11}$$

Then $E^n = E^{n-1} = \ldots = E^0 = const.$

This completes the proof of Theorem 3.1. □

4 The prior estimations for the numerical solution

In this section, the difference solution will be estimated. First, we introduce the following lemmas:

Lemma 4.1 ([30]) *For any mesh functions $\boldsymbol{U}^n$, there is*

$$\|\boldsymbol{U}^{n+1}\|^2 - \|\boldsymbol{U}^n\|^2 \leq \tau \left[\|\delta_t \boldsymbol{U}^n\|^2 + \frac{1}{2}(\|\boldsymbol{U}^n\|^2 + \|\boldsymbol{U}^{n+1}\|^2) \right].$$

Summing up for n, we know

$$\left(1 - \frac{\tau}{2}\right) \|\boldsymbol{U}^{n+1}\|^2 \leq \|\boldsymbol{U}^0\|^2 + \tau \sum_{k=1}^{n} (\|\delta_t \boldsymbol{U}^k\|^2 + \boldsymbol{U}^k\|^2). \tag{12}$$

Lemma 4.2 ([33]) *For any real symmetric positive definite matrices $\boldsymbol{H}$, there exist two positive numbers C_0, C_1, s.t.*

$$C_0 \|\boldsymbol{u}^n\|^2 \leq (\boldsymbol{H}\boldsymbol{u}^n, \boldsymbol{u}^n) \leq C_1 \|\boldsymbol{u}^n\|^2.$$

Lemma 4.3 (Gronwall's inequality [35]) *Suppose that the nonnegative mesh function $\{w(n), \rho(n), n = 1, 2, \ldots, N, N_\tau = T\}$ satisfy the inequality*

$$w(n) \leq \rho(n) + \tau \sum_{l=1}^{n} B_l w(l),$$

where $B_l (l = 1, 2, \ldots, N)$ are nonnegative constant. Then for any $0 \leq n \leq N$, there is

$$w(n) \leq \rho(n) exp\left(n\tau \sum_{l=1}^{N} B_l \right).$$

Lemma 4.4 (Discrete Sobolev's inequality [36]) *Suppose that $\{u_j\}$ is mesh functions. Given $\varepsilon > 0$, there exists a constant C dependent on ε such that*

$$\|u\|_\infty \leq \varepsilon \|u_x\| + C\|u\|.$$

Lemma 4.5 *Suppose that $u_0(x) \in H_0^1,\ u_1(x) \in L_2,\ \beta > 0$, then the following estimations hold:*

$$\|\boldsymbol{U}^n\| \leq C, \quad \|\delta_t \boldsymbol{U}^n\| \leq C, \quad \|\delta_x \boldsymbol{U}^n\| \leq C.$$

Proof From (11), we get

$$E^n = \frac{1}{2}(\|\delta_t \mathbf{U}^{n+1}\|^2 + \|\delta_t \mathbf{U}^n\|^2 - \tau^2\|\delta_t^2 \mathbf{U}^{n+1}\|^2) + \|\mathbf{R}\delta_x \mathbf{U}^{n+1}\|^2 + \frac{\beta}{2}\|\mathbf{U}^{n+1}\|_4^4 = C,$$

So we have

$$\frac{1}{2}(\|\delta_t \mathbf{U}^{n+1}\|^2 + \|\delta_t \mathbf{U}^n\|^2) + \|\mathbf{R}\delta_x \mathbf{U}^{n+1}\|^2 \le C + \frac{\tau^2}{2}\|\delta_t^2 \mathbf{U}^{n+1}\|^2.$$

From Lemma 4.2, we get

$$\|\mathbf{R}\delta_x \mathbf{U}^{n+1}\|^2 = (\mathbf{H}\delta_x \mathbf{U}^{n+1}, \delta_x \mathbf{U}^{n+1}) \ge C_0\|\delta_x \mathbf{U}^{n+1}\|^2.$$

So we can obtain

$$\frac{1}{2}(\|\delta_t \mathbf{U}^{n+1}\|^2 + \|\delta_t \mathbf{U}^n\|^2) + C_0(\|\delta_x \mathbf{U}^{n+1}\|^2 \le C + \frac{\tau^2}{2}\|\delta_t^2 \mathbf{U}^{n+1}\|^2. \tag{13}$$

From (8), we have

$$\begin{aligned}
\delta_t^2 \mathbf{U}^{n+1} &= -\delta_t^2 \mathbf{U}^n + \mathbf{H}\delta_x^2(\mathbf{U}^{n+1} + \mathbf{U}^n) - 2i\alpha\delta_t \mathbf{U}^n - \frac{\beta}{2}(|\mathbf{U}^{n+1}|^2 + |\mathbf{U}^n|^2)(\mathbf{U}^{n+1} + \mathbf{U}^n) \\
&= \delta_t^2 \mathbf{U}^{n-1} - \mathbf{H}\delta_x^2(\mathbf{U}^n + \mathbf{U}^{n-1}) + 2i\alpha\delta_t \mathbf{U}^{n-1} + \frac{\beta}{2}(|\mathbf{U}^n|^2 + |\mathbf{U}^{n-1}|^2)(\mathbf{U}^n + \mathbf{U}^{n-1}) \\
&\quad + \mathbf{H}\delta_x^2(\mathbf{U}^{n+1} + \mathbf{U}^n) - 2i\alpha\delta_t \mathbf{U}^n - \frac{\beta}{2}(|\mathbf{U}^{n+1}|^2 + |\mathbf{U}^n|^2)(\mathbf{U}^{n+1} + \mathbf{U}^n) \\
&= \ldots\ldots \\
&= (-1)^n\delta_t^2 \mathbf{U}^1 + \mathbf{H}\delta_x^2 \mathbf{U}^{n+1} - (-1)^n\mathbf{H}\delta_x^2 \mathbf{U}^1 + 2i\alpha\sum_{k=1}^{n}(-1)^{n-k+1}\delta_t \mathbf{U}^k \\
&\quad + \frac{\beta}{2}\sum_{k=1}^{n}(-1)^{n-k+1}(|\mathbf{U}^{k+1}|^2 + |\mathbf{U}^k|^2)(\mathbf{U}^{k+1} + \mathbf{U}^k).
\end{aligned}$$

So

$$\begin{aligned}
\|\delta_t^2 \mathbf{U}^{n+1}\|^2 &\le 8\|\delta_t^2 \mathbf{U}^1\|^2 + C_1\|\delta_x^2 \mathbf{U}^{n+1}\|^2 + C_2\|\delta_x^2 \mathbf{U}^1\|^2 + 32\alpha^2 n\sum_{k=1}^{n}\|\delta_t \mathbf{U}^k\|^2 \\
&\quad + 2n\beta^2\sum_{k=1}^{n}\|(|\mathbf{U}^{k+1}|^2 + |\mathbf{U}^k|^2)(\mathbf{U}^{k+1} + \mathbf{U}^k)\|^2 \\
&\le 8\|\delta_t^2 \mathbf{U}^1\|^2 + C_1\|\delta_x^2 \mathbf{U}^{n+1}\|^2 + C_2\|\delta_x^2 \mathbf{U}^1\|^2 + 32\alpha^2 n\sum_{k=1}^{n}\|\delta_t \mathbf{U}^k\|^2 \\
&\quad + 32\beta^2 n M^2\sum_{k=1}^{n+1}\|\mathbf{U}^k\|^2.
\end{aligned}$$

Adding it to (13), we get

$$
\begin{aligned}
&\frac{1}{2}(\|\delta_t \mathbf{U}^{n+1}\|^2 + \|\delta_t \mathbf{U}^n\|^2) + C_0\|\delta_x \mathbf{U}^{n+1}\|^2 \\
&\quad \le C + 4\tau^2\|\delta_t^2 \mathbf{U}^1\|^2 + \frac{\tau^2}{2}C_1\|\delta_x^2 \mathbf{U}^{n+1}\|^2 + \frac{\tau^2}{2}C_2\|\delta_x^2 \mathbf{U}^1\|^2 \\
&\quad +16T\tau\left(\alpha^2 \sum_{k=1}^{n}\left(\|\delta_t \mathbf{U}^k\|^2 + \beta^2 M^2 \sum_{k=1}^{n+1}\|\mathbf{U}^k\|^2\right)\right).
\end{aligned} \tag{14}
$$

Since

$$
\|\delta_x^2 \mathbf{U}^{n+1}\|^2 \le \frac{4}{h^2}\|\delta_x \mathbf{U}^{n+1}\|^2,
$$

So

$$
\begin{aligned}
&\frac{1}{2}(\|\delta_t \mathbf{U}^{n+1}\|^2 + \|\delta_t \mathbf{U}^n\|^2) + C_0\|\delta_x \mathbf{U}^{n+1}\|^2 \\
&\quad \le C + 8(\|\delta_t \mathbf{U}^1\|^2 + \|\delta_t \mathbf{U}^0\|^2) + 2C_1 r^2\|\delta_x \mathbf{U}^{n+1}\|^2 + 2C_2 r^2\|\delta_x \mathbf{U}^1\|^2 \\
&\quad +16T\tau\left(\alpha^2 \sum_{k=1}^{n}\left(\|\delta_t \mathbf{U}^k\|^2 + \beta^2 M^2 \sum_{k=1}^{n+1}\|\mathbf{U}^k\|^2\right)\right).
\end{aligned}
$$

Adding (12) to (14), we have

$$
\begin{aligned}
&\frac{1}{2}(\|\delta_t \mathbf{U}^{n+1}\|^2 + \|\delta_t \mathbf{U}^n\|^2) + C_0\|\delta_x \mathbf{U}^{n+1}\|^2 + (1-\frac{\tau}{2})\|\mathbf{U}^{n+1}\|^2 \\
&\quad \le C + 8(\|\delta_t \mathbf{U}^1\|^2 + \|\delta_t \mathbf{U}^0\|^2) + 2C_1 r^2\|\delta_x \mathbf{U}^{n+1}\|^2 \\
&\quad +2C_2 r^2\|\delta_x \mathbf{U}^1\|^2 + 16T\tau\left(\alpha^2 \sum_{k=1}^{n}\left(\|\delta_t \mathbf{U}^k\|^2 + \beta^2 M^2 \sum_{k=1}^{n+1}\|\mathbf{U}^k\|^2\right)\right) \\
&\quad +\tau\sum_{k=1}^{n}(\|\delta_t \mathbf{U}^k\|^2 + \|\mathbf{U}^k\|^2).
\end{aligned}
$$

Clearing it up,

$$
\begin{aligned}
&\frac{1}{2}(\|\delta_t \mathbf{U}^{n+1}\|^2 + \|\delta_t \mathbf{U}^n\|^2) + (C_0 - 2C_1 r^2)\|\delta_x \mathbf{U}^{n+1}\|^2\| \\
&\quad +(1-\frac{\tau}{2} - 16T\tau\beta^2 M^2)\|\mathbf{U}^{n+1}\|^2 \le C + \tau C\sum_{k=0}^{n}(\|\delta_t \mathbf{U}^k\|^2 + \|\mathbf{U}^k\|^2).
\end{aligned}
$$

According to the Lemma 4.3, the lemma holds when τ is small enough.

Corollary *If the conditions of Lemma 4.5 are satisfied, then,*

$$
\|\boldsymbol{U}^n\|_\infty \le C.
$$

Proof According to the Lemmas 4.4 and 4.5, the corollary follows. □

5 Convergence and stability of the difference scheme

In this section, we assume that the truncation error

$$\mathbf{R}^n = (R_0^n, R_1^n, \ldots, R_{J-1}^n)^T \in \Omega_h^n,$$

and we have

$$\mathbf{R}^n = \frac{1}{2}\delta_t^2(\mathbf{u}^{n+1} + \mathbf{u}^n) - \frac{1}{2}\mathbf{H}\delta_x^2(\mathbf{u}^{n+1} + \mathbf{u}^n) + i\alpha\delta_t\mathbf{u}^n$$
$$+\frac{\beta}{4}(|\mathbf{u}^{n+1}|^2 + |\mathbf{u}^n|^2)(\mathbf{u}^{n+1} + \mathbf{u}^n). \tag{15}$$

According to Taylor's expansion, it can be easily obtained that

Lemma 5.1 *Assume that $u(x,t) \in C^{6,3}$, the truncation error of the scheme (5)–(7) is of order $O(h^4 + \tau^2)$.*

Theorem 5.1 *Suppose that the conditions of the Lemma 4.5 and the Lemma 5.1 are fulfilled, then the numerical solution of the scheme (5)–(7) converges to the solution of the problem (1)–(3) with order $O(h^4 + \tau^2)$ by the $\|\cdot\|_\infty$ norm.*

Proof Let

$$\mathbf{e}^n = \mathbf{u}^n - \mathbf{U}^n.$$

Subtracting (8) from (15), we get

$$\mathbf{R}^n = \frac{1}{2}\delta_t^2(\mathbf{e}^{n+1} + \mathbf{e}^n) - \frac{1}{2}\mathbf{H}\delta_x^2(\mathbf{e}^{n+1} + \mathbf{e}^n)$$
$$+i\alpha\delta_t\mathbf{e}^n + \frac{\beta}{4}\big[(|\mathbf{u}^{n+1}|^2 + |\mathbf{u}^n|^2)(\mathbf{u}^{n+1} + \mathbf{u}^n)$$
$$-(|\mathbf{U}^{n+1}|^2 + |\mathbf{U}^n|^2)(\mathbf{U}^{n+1} + \mathbf{U}^n)\big]. \tag{16}$$

Computing the inner product of (16) with $\delta_t\mathbf{e}^n$, and taking the real part, we obtain

$$\mathbf{Re}(\mathbf{R}^n, \delta_t\mathbf{e}^n) = II_1 - II_2 + II_3 + II_4,$$

where

$$II_1 = \frac{1}{2}\mathbf{Re}(\delta_t^2(\mathbf{e}^{n+1} + \mathbf{e}^n), \delta_t\mathbf{e}^n)$$
$$= \frac{1}{4\tau}(\|\delta_t\mathbf{e}^{n+1}\|^2 - \|\delta_t\mathbf{e}^{n-1}\|^2 - \tau^2\|\delta_t^2\mathbf{e}^{n+1}\|^2 + \tau^2\|\delta_t^2\mathbf{e}^n\|^2),$$

$$
\begin{aligned}
II_2 &= \frac{1}{2}\mathbf{Re}(\mathbf{H}\delta_x^2(\mathbf{e}^{n+1}+\mathbf{e}^n),\delta_t\mathbf{e}^n) = -\frac{1}{2\tau}(\|\mathbf{R}\delta_x\mathbf{e}^{n+1}\|^2-\|\mathbf{R}\delta_x\mathbf{e}^n\|^2),\\
II_3 &= \mathbf{Re}(i\alpha\delta_t\mathbf{e}^n,\delta_t\mathbf{e}^n) = \alpha\mathbf{Im}(\delta_t\mathbf{e}^n,\delta_t\mathbf{e}^n)=0,\\
II_4 &= \frac{\beta}{4}\mathbf{Re}((|\mathbf{u}^{n+1}|^2+|\mathbf{u}^n|^2)(\mathbf{u}^{n+1}+\mathbf{u}^n)-(|\mathbf{U}^{n+1}|^2+|\mathbf{U}^n|^2)(\mathbf{U}^{n+1}+\mathbf{U}^n),\delta_t\mathbf{e}^n)\\
&= \frac{\beta}{4}\mathbf{Re}((|\mathbf{u}^{n+1}|^2+|\mathbf{u}^n|^2)(\mathbf{e}^{n+1}+\mathbf{e}^n)+[(|\mathbf{u}^{n+1}|^2+|\mathbf{u}^n|^2)\\
&\quad -(|\mathbf{U}^{n+1}|^2+|\mathbf{U}^n|^2)](\mathbf{U}^{n+1}+\mathbf{U}^n),\delta_t\mathbf{e}^n)\\
&= \frac{\beta}{4}\mathbf{Re}((|\mathbf{u}^{n+1}|^2+|\mathbf{u}^n|^2)(\mathbf{e}^{n+1}+\mathbf{e}^n),\delta_t\mathbf{e}^n)\\
&\quad +\frac{\beta}{4}\mathbf{Re}([(|\mathbf{u}^{n+1}|^2+|\mathbf{u}^n|^2)-(|\mathbf{U}^{n+1}|^2+|\mathbf{U}^n|^2)](\mathbf{U}^{n+1}+\mathbf{U}^n),\delta_t\mathbf{e}^n).
\end{aligned}
$$

We can obtain

$$
\begin{aligned}
\mathbf{Re}(\mathbf{R}^n,\delta_t\mathbf{e}^n) = {}& \frac{1}{4\tau}(\|\delta_t\mathbf{e}^{n+1}\|^2-\|\delta_t\mathbf{e}^{n-1}\|^2-\tau^2\|\delta_t^2\mathbf{e}^{n+1}\|^2+\tau^2\|\delta_t^2\mathbf{e}^n\|^2)\\
&+\frac{\beta}{4}\mathbf{Re}((|\mathbf{u}^{n+1}|^2+|\mathbf{u}^n|^2)(\mathbf{e}^{n+1}+\mathbf{e}^n),\delta_t\mathbf{e}^n)\\
&+\frac{1}{4\tau}(\|\mathbf{R}\delta_x\mathbf{e}^{n+1}\|^2-\|\mathbf{R}\delta_x\mathbf{e}^n\|^2)+\frac{\beta}{4}\mathbf{Re}([(|\mathbf{u}^{n+1}|^2+|\mathbf{u}^n|^2)\\
&-(|\mathbf{U}^{n+1}|^2+|\mathbf{U}^n|^2)](\mathbf{U}^{n+1}+\mathbf{U}^n),\delta_t\mathbf{e}^n),
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
&\mathbf{Re}(\mathbf{R}^n,\delta_t\mathbf{e}^n)\le C(\|\mathbf{R}^n\|^2+\|\delta_t\mathbf{e}^n\|^2),\\
&\frac{\beta}{4}\mathbf{Re}((|\mathbf{u}^{n+1}|^2+|\mathbf{u}^n|^2)(\mathbf{e}^{n+1}+\mathbf{e}^n),\delta_t\mathbf{e}^n)\\
&= \frac{\beta}{4}\mathbf{Re}((|\mathbf{u}^{n+1}|^2+|\mathbf{u}^n|^2)(\mathbf{e}^{n+1}+\mathbf{e}^n),\delta_t\mathbf{e}^n)\\
&\le C(\|\mathbf{e}^{n+1}\|^2+\|\mathbf{e}^n\|^2+\|\delta_t\mathbf{e}^n\|^2),\\
&\frac{\beta}{4}\mathbf{Re}([(|\mathbf{u}^{n+1}|^2+|\mathbf{u}^n|^2)-(\mathbf{U}^{n+1}|^2+|\mathbf{U}^n|^2)](\mathbf{U}^{n+1}+\mathbf{U}^n),\delta_t\mathbf{e}^n)\\
&= \frac{\beta}{4}\mathbf{Re}([\mathbf{u}^{n+1}\bar{\mathbf{e}}^{n+1}+\mathbf{e}^{n+1}\bar{\mathbf{U}}^{n+1}+\mathbf{u}^n\bar{\mathbf{e}}^n+\mathbf{e}^n\bar{\mathbf{U}}^n](\mathbf{U}^{n+1}+\mathbf{U}^n),\delta_t\mathbf{e}^n)\\
&\le C(\|\mathbf{e}^{n+1}\|^2+\|\mathbf{e}^n\|^2+\|\delta_t\mathbf{e}^n\|^2).
\end{aligned}
$$

From Lemma 4.1, we have

$$
\frac{1}{\tau}(\|\mathbf{e}^n\|^2-\|\mathbf{e}^{n-1}\|^2)\le C(\|\delta_t\mathbf{e}^{n-1}\|^2+\|\mathbf{e}^n\|^2+\|\mathbf{e}^{n-1}\|^2).
$$

So we can get

$$\frac{1}{4\tau}(\|\delta_t \mathbf{e}^{n+1}\|^2 - \|\delta_t \mathbf{e}^{n-1}\|^2 - \tau^2\|\delta_t^2 \mathbf{e}^{n+1}\|^2 + \tau^2\|\delta_t^2 \mathbf{e}^n\|^2)$$
$$+\frac{1}{2\tau}(\|\mathbf{R}\delta_x \mathbf{e}^{n+1}\|^2 - \|\mathbf{R}\delta_x \mathbf{e}^n\|^2) + \frac{1}{\tau}(\|\mathbf{e}^n\|^2 - \|\mathbf{e}^{n-1}\|^2)$$
$$\leq C(\|\mathbf{R}^n\|^2 + \|\delta_t \mathbf{e}^n\|^2 + \|\delta_t \mathbf{e}^{n-1}\|^2 + \|\mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n-1}\|^2). \tag{18}$$

Summing (18) up for n, we obtain

$$\frac{1}{4}(\|\delta_t \mathbf{e}^{n+1}\|^2 + \|\delta_t \mathbf{e}^n\|^2) + C_1\|\mathbf{R}\delta_x \mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^n\|^2 \leq [O(h^4+\tau^2)]^2.$$

From Lemma 4.2,

$$\|\mathbf{R}\delta_x \mathbf{e}^{n+1}\|^2 = (\mathbf{H}\delta_x \mathbf{e}^{n+1}, \delta_x \mathbf{e}^{n+1}) \geq C_0\|\delta_x \mathbf{e}^{n+1}\|^2.$$

So we can obtain

$$\frac{1}{4}(\|\delta_t \mathbf{e}^{n+1}\|^2 + \|\delta_t \mathbf{e}^n\|^2) + C\|\delta_x \mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^n\|^2 \leq [O(h^4+\tau^2)]^2.$$

Then

$$\|\mathbf{e}^n\| \leq O(h^4+\tau^2), \quad \|\delta_t \mathbf{e}^n\| \leq O(h^4+\tau^2), \quad \|\delta_x \mathbf{e}^n\| \leq O(h^4+\tau^2). \tag{19}$$

Lastly, according to the Lemma 4.4, it is follows that

$$\|\mathbf{e}^n\|_\infty \leq O(h^4+\tau^2). \tag{20}$$

Then the convergence is proved. □

Similarly, we can prove the stability of the difference solution. i.e.

Theorem 5.2 *Under the conditions of Theorem 5.1, the solution of the difference scheme (5)–(7) is unconditionally stable for initial data by the* $\|\cdot\|_\infty$ *norm.*

6 Numerical experiment and comparison

In this section, some numerical tests are carried out to verify the performance of the new scheme. We use the numerical example in [34] and compare the new difference scheme with the one in [34]. Taking $\alpha = \beta = 1, L = 80$, and considering the following problem:

$$u_{tt} - u_{xx} + iu_t + |u|^2u = 0, \quad -40 < x < 40, \quad 0 < t < T, \tag{21}$$
$$u(x,0) = (1+i)x exp(-10(1-x)^2), \quad u_t(x,0) = 0, \quad -40 < x < 40. \tag{22}$$

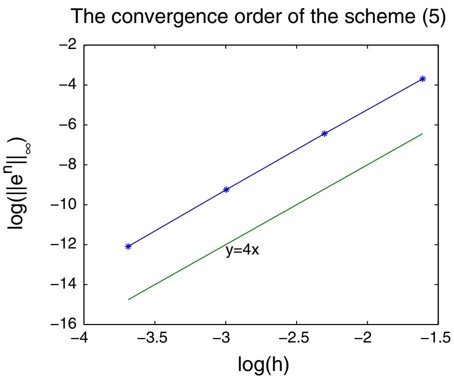

Fig. 1 The convergence order of the Scheme (5)

For problems (21)–(22), we have the following difference scheme

$$\frac{1}{2\tau^2}A_h(U_j^{n+2}-U_j^{n+1}-U_j^n+U_j^{n-1})-\frac{1}{2}\delta_x^2(U_j^{n+1}+U_j^n)+\frac{i}{\tau}A_h(U_j^{n+1}-U_j^n)$$
$$+\frac{1}{2}A_h(|U_j^{n+1}|^2+|U_j^n|^2)(U_j^{n+1}+U_j^n)=0,\quad j=0,1,\ldots,J-1, n=1,2,\ldots,N-1, \tag{23}$$
$$U_j^0=(1+i)x_j exp(-10(1-x_j)^2),\quad \delta_t U_j^0=0,\quad j=0,1,\ldots,J-1, \tag{24}$$

Obviously, the scheme (23) is an explicit four-level scheme. We can obtain U_j^0 and U_j^1 from (24) easily. For U_j^2, we use the iterative algorithm in [34]. Thus, the new scheme (23) can be used to solve the problems (21)–(22).

6.1 Numerical experiment

6.1.1 Convergence order

Firstly, we demonstrate the convergence order of scheme (5) which is stated in Theorem 5.1. We take $t=1$, and choose the numerical solution ($h=0.0125$, $\tau=h^2$) as the approximate exact solution used in [34]. The following convergence order figure is obtained with $h=0.2, 0.1, 0.05, 0.025$, $\tau=h^2$. From Fig. 1, it is obvious that the scheme (5) is convergent in maximum norm, and the convergence order is $O(h^4+\tau^2)$.

6.1.2 Discrete conservation law

Secondly, the discrete conservation law is calculated with $h=0.05$, $\tau=h^2$, $t=10$. Table 1 shows the value of E at different time. It indicates that the conservation of the scheme (5) is very good and it is suitable for long-time computation.

Table 1 The value of E at different time with $h = 0.05$, $\tau = h^2$

t	E	t	E
1	9.123106418942955	6	9.123106418942761
2	9.123106418942822	7	9.123106418942800
3	9.123106418942237	8	9.123106418942665
4	9.123106418942228	9	9.123106418942488
5	9.123106418942465	10	9.123106418942395

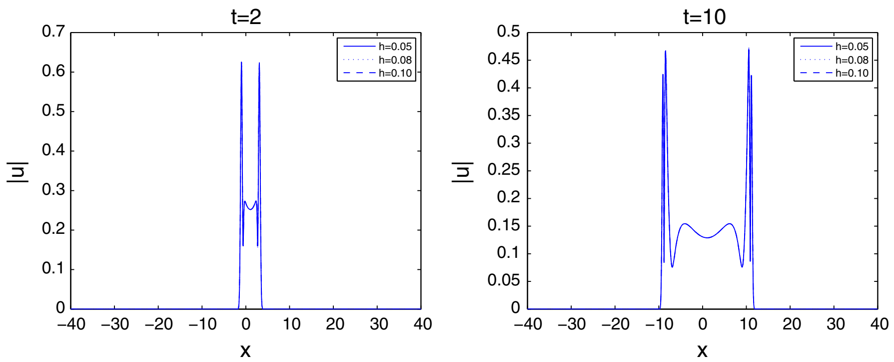

Fig. 2 Movement of $|U|$ with different step sizes: (*left*) $t = 2$ and (*right*) $t = 10$

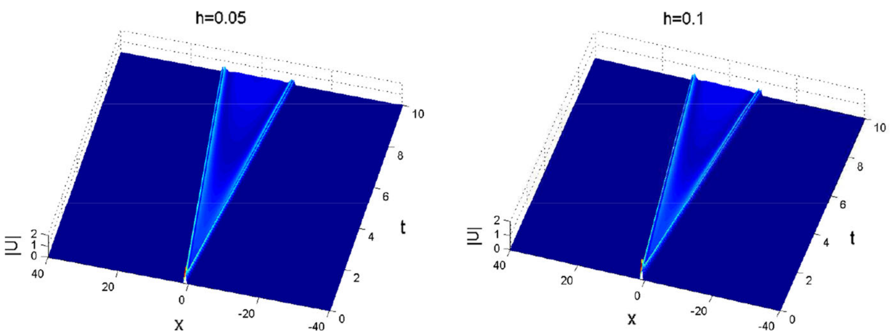

Fig. 3 Wave propagation with different step sizes from $t = 0$ to $t = 10$: (*left*) $h = 0.05$ and (*right*) $h = 0.1$

6.1.3 Long-time computation

Thirdly, we give the pictures about the movement of $|U|$ with three different step sizes at $t = 2, 10$, respectively. As shown in Fig. 2, we can conclude that the new scheme (5) is not affected by grid ratios. Then, the pictures of wave propagation from $t = 0$ to $t = 10$ are revealed in Fig. 3. Here we choose two different sizes with $h = 0.05, 0.1$, $\tau = h^2$, respectively. Compared with the figures in [34], Fig. 3 proves that the new scheme (5) is suitable for long-time computation as the one in [34].

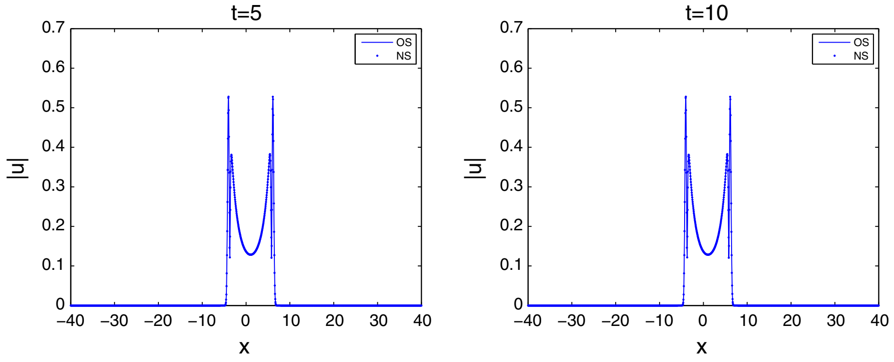

Fig. 4 Comparison of soliton $|U|$ ($h = 0.05$, $\tau = h^2$): (*left*) $t = 5$ and (*right*) $t = 10$

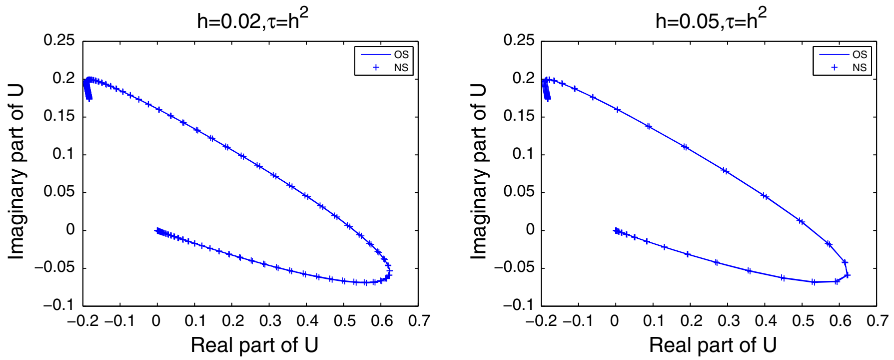

Fig. 5 Comparison of phasic picture of U ($t = 1$): (*left*) $h = 0.02$, $\tau = h^2$ and (*right*) $h = 0.05$, $\tau = h^2$

6.2 Comparison

In this subsection, we compare the new scheme (5) with the one in [34]. For convenience, the new scheme is NS and the one in [34] is OS for short. The comparison is proposed in three aspects: effectiveness, computation time and infinite modulo error.

6.2.1 Effectiveness

At first, the comparison of solitons $|U|$ which are formed by two schemes with the same step size ($h = 0.05$, $\tau = h^2$) at the different times ($t = 5, 10$) are given in Fig. 4. Figure 5 shows the comparison of phasic pictures of U which are formed by two schemes with different step sizes ($h = 0.02, 0.05$, $\tau = h^2$) at the same time ($t = 2$). Both two figures certify that the new scheme (5) is effective.

6.2.2 Computation time

Then, two schemes (OS and NS) in computation time are compared. Table 2 lists the computation time of OS and NS in different step sizes at $t = 1$. We can see the time

Table 2 Comparison of the two schemes on computation time with $t = 1, \tau = h^2$

h	0.2 (s)	0.1 (s)	0.05 (s)	0.025 (s)
OS	0.67	1.14	3.26	46.90
NS	0.28	0.35	2.22	40.39

Table 3 Comparison of the two schemes on computation time with $t = 5, \tau = h^2$

h	0.2 (s)	0.1 (s)	0.05 (s)	0.025 (s)
OS	1.28	2.80	31.33	1058.64
NS	0.36	1.77	29.44	1005.66

Table 4 Comparison of the two schemes on infinite modulo error with $t = 1, \tau = h^2$

h	0.2	0.1	0.05	0.025
OS	0.0599	0.0041	2.5113e−04	1.4726e−05
NS	0.0290	0.0016	9.6600e−05	5.6211e−06

Table 5 Comparison of the two schemes on infinite modulo error with $t = 5, \tau = h^2$

h	0.2	0.1	0.05	0.025
OS	0.2085	0.0179	0.0012	7.4984e−05
NS	0.0762	0.0080	5.2179e−04	3.2597e−05

of the latter is shorter than that of the former. For long-time computation, it is obvious that computation time of NS is shorter than that of OS from Table 3.

6.2.3 *Infinite modulo error*

At last, two schemes are compared in infinite modulo error. Also, we choose the numerical solution with $h = 0.0125, \tau = h^2$ as the approximate exact solution and calculate the infinite modulo error for $t = 1, h = 0.2,\ 0.1,\ 0.05,\ 0.025, \tau = h^2$. It is indicated in Tables 4 and 5 that the error of the second line is distinctly less than that of the first line, almost as half as the error of the first line.

7 Conclusion

In this paper, a four-level compact finite difference scheme is presented for the nonlinear Schrödinger equation with wave operator. The conservation, convergence, and stability of the new scheme are demonstrated. Numerical tests are carried out to confirm the theoretical proving. In addition, the new scheme is compared with the scheme in [34]. As for the aspect of either computation time or infinite modulo error, numerical results prove the new scheme is better than the scheme in [34].

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