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Nonrelativistic limit in the energy space for nonlinear Klein-Gordon equations

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Abstract. We study the nonrelativistic limit of the Cauchy problem for the nonlinear Klein-Gordon equation and prove that any finite energy solution converges to the corresponding solution of the nonlinear Schrödinger equation in the energy space, after the infinite oscillation in time is removed. We also derive the optimal rate of convergence in L^2.

1. Introduction

In this paper we study the nonrelativistic limit of the Cauchy problem for the nonlinear Klein-Gordon equation:

$$\frac{\hbar}{2mc^2}\ddot{u} - \frac{\hbar}{2m}\Delta u + \frac{mc^2}{2}u + f(u) = 0, \tag{1.1}$$

where $u = u(t, x) : \mathbb{R}^{1+n} \to \mathbb{C}$, $f(u) = \lambda|u|^p u$ with $p > 0$ and $\lambda \in \mathbb{R}$, c is the speed of light, $\hbar$ is the Planck constant, and $m > 0$ is the mass of particle. We consider the pure power nonlinearity just for simplicity. Our arguments below are obviously applicable to more general functions $f(\cdot)$.

Rescaling t, x, u, λ and c, we can normalize the other constants as $\hbar = m = 2$. Before taking the nonrelativistic limit $c \to \infty$, we consider the modulated function $v := e^{-ic^2t}u$, which obeys the following modulated equation:

$$\ddot{v}/c^2 + 2i\dot{v} - \Delta v + f(v) = 0. \tag{1.2}$$

Then we may think of the nonlinear Schrödinger equation:

$$2i\dot{v} - \Delta v + f(v) = 0 \tag{1.3}$$

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as the singular limit as $c \to \infty$ of (1.2). The most important quantities of these equations are the energy and the charge:

$$\begin{aligned} E_{NK}(u) &:= \int_{\mathbb{R}^n} |\dot{u}/c|^2 + |\nabla u|^2 + F(u)dx, \\ E_{NS}(u) &:= \int_{\mathbb{R}^n} |\nabla u|^2 + F(u)dx, \\ Q_K(u) &:= \int_{\mathbb{R}^n} |u|^2 + \Im \dot{u}\bar{u}/c^2 dx, \\ Q_S(u) &:= \int_{\mathbb{R}^n} |u|^2 dx, \end{aligned} \tag{1.4}$$

where $F(u) := 2\lambda|u|^{p+2}/(p+2)$. E_{NK} and Q_K are conserved for any solution u of (1.2), and so are E_{NS} and Q_S for (1.3).

So the most natural question about the nonrelativistic limit of (1.2) is whether any solution with finite energy and charge converges to a solution of (1.3) in the topology induced by those quantities (H^1). However, to the best of our knowledge, there is no rigorous result on this problem in the literature. Indeed, there are a few papers on some weaker results. In [11], L^2 convergence was proved assuming H^2 convergence of the initial data in the case where $n \le 2$ and $p \le 2$. In [9,8], L^q convergence was shown for $2 \le q < 2n/(n-2)$ assuming H^1 boundedness and L^2 convergence of the initial data under the assumption $n \le 3$ and some restrictive assumptions on p.

Here we present an almost complete answer. Namely, we will prove H^1 convergence assuming H^1 convergence of the initail data, for any n and any $p < 4/(n-2)$. Perhaps the upper bound $p = 4/(n-2)$ might be allowed, but it would be a very delicate and difficult problem. Actually, we do not know even the uniqueness of finite energy solutions for (1.3) with $p = 4/(n-2)$ in general case (see [2] for the wellposedness in the radial case).

As we will show later, the H^1 convergence can be rather easily shown, at least when $\lambda \ge 0$, by a compactness argument. However, when we want to investigate the nonrelativistic limit in more details, such an argument can yield very little information. So we give another method of analizing the problem via the Strichartz estimate applied to the associated integral equation.

Here the main idea is to adjust the space-time norms to the nonrelativistic limit in order to get a uniform estimate. More precisely, we split the solution in the Fourier space into the lower frequency part $|\xi| < c$ and the higher frequency part $|\xi| > c$. The lower part is shown to behave as a solution of the Schrödinger, and the higher part vanishes at the rate of a certain power of c. Then one might be anxious about the compatibility of such a decomposition with the nonlinearity, but it will be efficiently dealt with the nonlinear estimate in sum spaces of Lebesgue-Besov type. At this step, we lose the information about the frequency separation, but we can recover it by exploiting the ∇/c-derivative gain.

By virtue of the analysis via the Strichartz estimate, we can show the convergence also in the case where $\lambda < 0$, as long as the solution to (1.3) exists. As another application, we will derive the optimal rate of the convergence in L^2, that is $1/\sqrt{c}$.

Our main results are the following, which cover those in [11,9,8].

Theorem 1.1 *Let $n \in \mathbb{N}$ and $0 < p < 4/(n-2)$ (if $n \geq 3$). Let $(\varphi_c, \psi_c) \in H^1 \oplus L^2$, $\varphi \in H^1$, u_c be the solution of* (1.2) *with $(u_c(0), \dot{u}_c(0)) = (\varphi_c, \psi_c)$, and v be the solution of* (1.3) *with $v(0) = \varphi$. Denote by T_c^* and T^* the maximal existence time of u_c and v, respectively. Assume that*

$$(\varphi_c, \psi_c/c) \to (\varphi, 0) \quad \text{in } H^1 \oplus L^2, \tag{1.5}$$

as $c \to \infty$. Then we have

$$\liminf_{c\to\infty} T_c^* \geq T^*, \tag{1.6}$$

and u_c converges to v in $C([0, T^*); H^1)$ (locally uniform convergence in time).

It is obvious that we have the same result for the negative time direction. The local wellposedness is well-known for (1.2) and (1.3) under the above conditions (see [4,3]). By a priori bounds from the conservation law, we easily observe the following: If $\lambda \geq 0$, then we have $T_c^* = T^* = \infty$. If $p < 4/n$, then we have $T^* = \infty$ and $T_c^* = \infty$ for sufficiently large c (depending on the size of the initial energy and charge).

Theorem 1.2 *Let $p, \varphi_c, \psi_c, \varphi, u_c, v, T^*$ and T_c^* be as in the above theorem. Instead of* (1.5), *assume that $(\varphi_c, \psi_c/c)$ is bounded in $H^1 \oplus L^2$ and $\varphi_c \to \varphi$ in L^2 as $c \to \infty$. Then, for any $T < T^*$ we have*

$$\sup_{t\in[0,T]} \|u_c(t) - v(t)\|_{L^2} \leq O(\|\varphi_c - \varphi\|_{L^2}) + q(c), \tag{1.7}$$

where $q = o(1/\sqrt{c})$, or more precisely, $\|\sqrt{c}q(c)\|_{\ell^2 L^\infty} < \infty$, where

$$\|g(c)\|_{\ell^2 L^\infty}^2 := \sum_{j\in\mathbb{N}} \sup_{2^j \leq c \leq 2^{j+1}} |g(c)|^2. \tag{1.8}$$

Moreover, for any $q(c)$ satisfying $\|\sqrt{c}q\|_{\ell^2 L^\infty} < \infty$, we can find $\varphi \in H^1$ such that we have

$$\lim_{c\to\infty} \|u_c(t) - v(t)\|_{L^2}/q(c) = \infty, \tag{1.9}$$

no matter how we choose $t > 0$ close to 0 and a bounded sequence $(\varphi_c, \psi_c/c)$ in $H^1 \oplus L^2$ satisfying $\|\varphi_c - \varphi\|_{L^2} \leq q(c)$.

The contents of this paper are as follows. In Sect. 2, we give a generalized Strichartz estimates for (1.2). It can be seen as a full mixture of the well-known Strichartz estimates for the Schrödinger, the wave and the Klein-Gordon equations, but it does not seem a trivial combination. In Sect. 3, we derive a nonlinear estimate in sum spaces. In Sect. 4, we will derive a uniform boundedness of Strichartz type norms (which themselves depend on c). In Sect. 5, we prove the H^1 convergence. In Sect. 6, we prove the $1/\sqrt{c}$ rate of L^2 convergence and its optimality.

We abbreviate '$\le C$' to '$\lesssim$', where C is a positive constant dependent only on n, p and any other fixed parameter (except c, of course). For any Banach space X consisting of space-time functions and any time interval I, we denote

$$\|u\|_{X(I)} := \|\chi_I u\|_X, \tag{1.10}$$

where χ_I denotes the characteristic function of I. For any function φ, we denote its Fourier transform by $\tilde{\varphi} = \mathcal{F}\varphi$, and denote Fourier multipliers as $\varphi(\nabla) := \mathcal{F}^{-1}\varphi(i\xi)\mathcal{F}$. We denote $\langle a\rangle_b := \sqrt{|a|^2 + b^2}$, where we omit b when $b = 1$. Denote $D := \langle\nabla\rangle_1$. $B^\sigma_{q,r}$ and $\dot{B}^\sigma_{q,r}$ denote the inhomogeneous Besov space and the homogeneous one, respectively (see [1]). ℓ^σ_q denotes the function space of sequences with the norm $\|a\|_{\ell^\sigma_q} := \|2^{\sigma j} a_j\|_{\ell^q}$. $[\cdot,\cdot]_\theta$ and $(\cdot,\cdot)_{\theta,q}$ denote the complex and the real interpolation functors. We denote the L^2 inner product by $\langle\cdot,\cdot\rangle$. For any space X, we denote the dual space by X'. For any complex function g, we denote by $g'(z)$ its $\mathbb{R}$-linearization at z.

2. Uniform Strichartz estimates for nonrelativistic limit

In this section, we derive Strichartz-type estimates for the linear equation:

$$\ddot{u}/c^2 + 2i\dot{u} - \Delta u = f, \tag{2.1}$$

where function spaces depend on c but in those inequalities we can take positive constants independent of c. Those Strichartz estimates describe the transition of the space-time norms of Strichartz type along the nonrelativistic limit, from the Klein-Gordon to the Schrödinger.

The following separation of frequency is essential to know that transition. Let X and Y be Banach spaces which consist of space-time functions. Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\tilde{\chi}(\xi) = 1$ for $|\xi| < 1$ and $\tilde{\chi}(\xi) = 0$ for $|\xi| > 2$. Let $\tilde{\chi}^c(\xi) := \tilde{\chi}(\xi/c)$ and $\tilde{\chi}_c := 1 - \tilde{\chi}^c$. We introduce the Banach space $X \,|\, Y$ with norm defined as

$$\|u\|_{X\,|\,Y} := \|\chi^c * u\|_X + \|\chi_c * u\|_Y, \tag{2.2}$$

so that it measures the frequency lower than the speed of light by the X-norm and the higher frequency by the Y-norm.

Equation (2.1) can be rewritten in the following integral equation.

$$u = e^{-ic^2t}\left\{\cos(c\langle\nabla\rangle_c t) + i\frac{c}{\langle\nabla\rangle_c}\sin(c\langle\nabla\rangle_c t)\right\}u(0)$$
$$+ e^{-ic^2t}\sin(c\langle\nabla\rangle_c t)\frac{1}{c\langle\nabla\rangle_c}\dot{u}(0) + \int_0^t e^{-ic^2(t-s)}\sin(c\langle\nabla\rangle_c(t-s))\frac{c}{\langle\nabla\rangle_c}f(s)ds. \tag{2.3}$$

So it suffices to investigate the operators $K(t) := e^{\pm ic\langle\nabla\rangle_c t}$. Then we get the following Strichartz-type estimates.

Lemma 2.1 *For any $c > 0$, we have*

$$\begin{aligned} &\|K(t)\varphi\|_{S_0\,|\,(W_0\cap K_0)} \lesssim \|\varphi\|_{L^2}, \\ &\left\|\int_0^t K(t-s)f(s)ds\right\|_{S_0\,|\,(W_0\cap K_0)} \lesssim \|f\|_{S_1'\,|\,(W_1'+K_1')} \end{aligned} \tag{2.4}$$

for any space S_i, W_i, and K_i of the form $c^{-\mu}L^p(\mathbb{R};\dot{B}^\sigma_{q,2})$ satisfying the following conditions. Let $b := 1/p$ and $\alpha := 1/2 - 1/q$. All the spaces S_i, W_i and K_i must obey

$$-2b + n\alpha + \sigma + \mu = 0, \quad 0 \le 2b < 1, \quad 0 \le 2\alpha \le 1, \tag{2.5}$$

and each space should satisfy

$$S_i: \quad \mu = 0, \qquad 2b \le n\alpha, \tag{2.6}$$
$$W_i: \quad \mu = b, \qquad 2b \le (n-1)\alpha, \tag{2.7}$$
$$K_i: \quad \mu = (1+2/n)b, \quad 2b \le n\alpha, \tag{2.8}$$

respectively.

We will use the corresponding estimates for H^1 solutions, which are immediate from this lemma.

Proof. Denote the linear operators in (2.4) by T_c^1 and T_c^2, respectively. Then we have the following scaling property:

$$T_c^1\varphi = T_1^1[\varphi(x/c)](c^2t, cx), \quad T_c^2 f = T_1^2[f(t/c^2, x/c)](c^2t, cx)c^{-2}. \tag{2.9}$$

From this and (2.5), it is obvious that the estimate in general case follows from that in the special case $c = 1$. Then, the estimates in $S\,|\,W$ and $S\,|\,K$ reduce to the estimate in [5, Lemma 2.1] by the well-known argument (see, e.g., [6,7]). Thus we have only to derive the estimates from K' to W and W' to K.

Below we will show the estimate from W' to K with the following exponents:

$$W: \quad (b, \alpha, \sigma) = ((n-1)\beta, 2\beta, -(n+1)\beta), \tag{2.10}$$
$$K: \quad (b, \alpha, \sigma) = (n\gamma, 2\gamma, -(n+2)\gamma), \tag{2.11}$$

where $0 \le 2\beta \le 1/2$, $2\beta < 1/(n-1)$, $2\gamma < 1/n$ and $0 \le \gamma \le \beta$. This is the borderline case in (2.7) and (2.8) with a constraint $\gamma \le \beta$. Nevertheless, we get the estimate in the remaining cases from that in this special case, via Sobolev embedding, interpolations with the L^2 estimate and the duality argument. We denote $W' =: L^{p_0}(B'_W)$ and $K' =: L^{q_0}(B'_K)$. Since we are now considering only the frequency $|\xi| > 1$, we do not have to distinguish the homogenous Besov spaces from their inhomogeneous counterparts.

By the duality argument, it suffices to estimate $\iint_{s<t} \langle K(t-s)f(s), g(t)\rangle ds dt$. The double integration in $\{s < t\}$ can be decomposed as follows.

$$\iint_{s<t} ds dt F(s,t) = C \int_0^\infty \frac{dr}{r} \int_{\mathbb{R}} \frac{da}{r} \int_{a-3r}^{a-r} ds \int_{a+r}^{a+3r} dt F(s,t), \tag{2.12}$$

where C is a certain positive constant. Denote

$$I := \int_{a-3r}^{a-r} \int_{a+r}^{a+3r} \langle K(-s)f(s), K(-t)g(t)\rangle ds dt. \tag{2.13}$$

If we apply the Schwarz inequality to the x-integral and use the L^2 estimate, then we can not dominate the integral for r, but we get only a bound of the integrand. We will recover its integrability by the real interpolation for bilinear operators; this idea was inspired by [7]. We denote

$$J := \int_{a-3r}^{a-r} K(t-s)f(s)ds. \tag{2.14}$$

Applying the decay estimate in [5, Lemma 2.1] directly, we obtain

$$\|J\|_{B^{-(n+2)\beta}_{2/(1-4\beta),2}} \lesssim r^{-(2n-1)\beta} \|f\|_{L^1(a-3r,a-r;B'_W)}. \tag{2.15}$$

On the other hand, by the L^2 estimate we have

$$\|J\|_{L^2} \lesssim \|f\|_{W'(a-3r,a-r)}, \tag{2.16}$$

Interpolating between (2.16) and (2.15) by $[\cdot,\cdot]_{\gamma/\beta}$, we obtain

$$\|J\|_{B_K} \lesssim r^{-(2n-1)\gamma} \|f\|_{L^{p_1}(a-3r,a-r;B'_W)}, \tag{2.17}$$

where $1/p_1 := 1-(n-1)(\beta-\gamma)$. Then, by the Hölder inequality we get

$$|I| \lesssim r^\nu \|f\|_{L^p(a-3r,a-r;B'_W)} \|g\|_{L^q(a+r,a+3r;B'_K)}, \tag{2.18}$$

for $p \geq p_1$, $q \geq 1$ and $\nu = 2 - (n-1)\beta - n\gamma - 1/p - 1/q$. Using the Hölder and the Minkowski inequalities, we obtain

$$\begin{aligned}\int \frac{da}{r}|I| \lesssim r^{\nu-1} &\|\{|t-a+2r|<r\}f(t)\|_{L_t^p L_a^{p'}(B'_W)} \\ &\times \|\{|t-a-2r|<r\}g(t)\|_{L_t^q L_a^{q'}(B'_K)} \\ \lesssim r^{\nu} &\|f\|_{L^p(B'_W)}\|g\|_{L^q(B'_K)},\end{aligned} \tag{2.19}$$

where $p \leq p'$, $q \leq q'$ and $1/p' + 1/q' = 1$, so that we need an additional restriction $1/p + 1/q \geq 1$. Denoting the left hand side by H, we discretize the integral for r as

$$\int_0^\infty H\frac{dr}{r} = \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} H\frac{dr}{r}, \tag{2.20}$$

and denote the summand by $\tilde{H}_j$. We have obtained

$$\|\tilde{H}\|_{\ell_\infty^{-\nu}} \lesssim \|f\|_{L^p(B'_W)}\|g\|_{L^q(B'_K)}, \tag{2.21}$$

when $p \geq p_1$, $q \geq 1$ and $1/p + 1/q \geq 1$.

Now, to get the desired estimate for $\|\tilde{H}\|_{\ell_1^0}$, we use the following real interpolation theorem for bilinear operators. This theorem is essentially due to O'Neil [10] and formulated as below in [1, 3.13.5(b)].

Theorem 2.2 *Let T be a bilinear operator, (X_0, X_1), (Y_0, Y_1), (Z_0, Z_1) be interpolation couples of Banach spaces. Assume*

$$\|T(f,g)\|_{Z_{i+j}} \lesssim \|f\|_{X_i}\|g\|_{Y_j}, \tag{2.22}$$

for $0 \leq i, j, i+j \leq 1$. Then, for any $0 < \theta_0, \theta_1 < 1$ and $1 \leq p_0, p_1 \leq \infty$ satisfying $\theta_0 + \theta_1 < 1$ and $1 \leq p := 1/(1/p_0 + 1/p_1)$, we have

$$\|T(f,g)\|_{(Z_0,Z_1)_{\theta_0+\theta_1,p}} \lesssim \|f\|_{(X_0,X_1)_{\theta_0,p_0}}\|g\|_{(Y_0,Y_1)_{\theta_1,p_1}}. \tag{2.23}$$

Since the point $(1/p, 1/q) = P := (1/p_0, 1/q_0)$ is included inside the triangle $\triangle_0 := \triangle(0,1)(1/p_1, 1)(1/p_1, 1-1/p_1)$, we can find another triangle $\triangle_1 = \triangle(b_2, d_2)(b_2, d_3)(b_3, d_2)$ which is contained inside $\triangle_0$ and surrounds P. Applying the above theorem to (2.21) on $\triangle_1$ with appropriate θ_0 and θ_1, we obtain

$$\|\tilde{H}\|_{\ell_1^0} \lesssim \|f\|_{L^{p_0,2}(B'_W)}\|g\|_{L^{q_0,2}(B'_K)}, \tag{2.24}$$

where $L^{p,q}$ denotes the Lorentz space. Then, by the embedding $L^p \subset L^{p,2}$ ($p \leq 2$), we obtain the desired result. □

3. Nonlinear estimate in sum spaces

In this section, we derive an estimate for power nonlinearities in sum spaces of Lebesgue-Besov type. The argument is quite standard; its complexity comes only from the summation of the spaces. We will derive the estimate in the homogeneous Besov spaces, which together with trivial estimates from Hölder's inequality gives the corresponding estimate in the inhomogeneous spaces. Now we introduce the following notation to state and prove the estimate both simply and systematically.

Definition 3.1 We define

$$\begin{aligned} \mathcal{L}_0 &:= \{\alpha L^s(\mathbb{R}; L^q(\mathbb{R}^n)) \mid \alpha > 0, 0 < s, q \le \infty\}, \\ \mathcal{L}_1 &:= \{\alpha L^s(\mathbb{R}; \ell^\sigma_r(\mathbb{Z}; L^q(\mathbb{R}^n))) \mid \alpha, \sigma > 0, 1 \le s, q, r \le \infty\}, \\ \mathcal{B} &:= \{\alpha L^s(\mathbb{R}; \dot{B}^\sigma_{q,r}(\mathbb{R}^n)) \mid \alpha, \sigma > 0, 1 \le s, q, r \le \infty\}. \end{aligned} \tag{3.1}$$

You can see that the set of spaces αL^s for t variable is pretty superfluous in the following argument. It can be easily generalized to the set of order preserving function spaces, but here we restrict it to the case that we need. With any $B = \alpha L^s \dot{B}^\sigma_{q,r} \in \mathcal{B}$, we associate $\pi_b(B) := (\log \alpha, 1/s, 1/q, 1/r, \sigma)$ and $\sigma(B) := \sigma$. Similarly, we define $\pi_1(X)$ for any $X \in \mathcal{L}_1$ and define $\pi_0(X)$ for any $X \in \mathcal{L}_0$, for which we regard as $(1/r, \sigma) = 0$. For any $X \in \mathcal{L}_i$ and $\beta > 0$, we define $X^\beta := \pi_i^{-1}(\beta \pi_i(X))$. For any X in $\mathcal{L}_0$, any $Y \in \mathcal{L}_1$ and any $B \in \mathcal{B}$, we define $XY := \pi_1^{-1}(\pi_0(X) + \pi_1(Y))$ and $XB := \pi_b^{-1}(\pi_0(X) + \pi_b(B))$, and we denote $\bar{X} := \pi_b^{-1}\pi_1(X)$ and $\underline{B} := \pi_1^{-1}\pi_b(B)$.

We estimate the Besov norms via difference operators. We denote the k-th unit vector in $\mathbb{R}^n$ by e_k. For any function u we define

$$\begin{aligned} [u]_{k,j} &:= |u(x + 2^{-j}e_k)| + |u(x)| + |u(x - 2^{-j}e_k)|, \\ \delta_{k,j} u &:= |u(x + 2^{-j}e_k) - u(x)| + |u(x) - u(x - 2^{-j}e_k)|, \\ \delta^2_{k,j} u &:= |u(x + 2^{-j}e_k) - 2u(x) + u(x - 2^{-j}e_k)|. \end{aligned} \tag{3.2}$$

It is well-known that the Besov norms can be represented by the differences. Actually, we have the following retraction from $\mathcal{L}_1$ to $\mathcal{B}$, which is obvious from the usual proof of the equivalence of the norms (see, e.g., [1]).

Lemma 3.2 *Define operators S^m for $m = 1, 2$ by*

$$(S^m f)_{k,j} := \delta^m_{k,j} f. \tag{3.3}$$

Then, S^m is bounded from $B \in \mathcal{B}$ to $(\underline{B})^n$ if $\sigma(B) < m$. Moreover, there exists a sequence $\{R^m_{k,j}\} \subset \mathcal{S}(\mathbb{R}^n)$ satisfying $R^m_{k,j} = 2^{nj} R^m_{k,0}(2^j x)$ and the following properties. Define operators R^m by

$$R^m f := \sum_{k,j} R^m_{k,j} * f_{k,j}. \tag{3.4}$$

Then we have $R^m S^m f = f$ and R^m is bounded from $(\underline{B})^n$ to B.

By this retraction, we have in particular the equivalence also for sum spaces and interpolation spaces.

The following basic lemma is necessary to consider estimates in sum spaces.

Lemma 3.3 *Let $N \in \mathbb{N}$ and B_i, $i = 1, \dots, N$, be a compatible tuple of Banach function spaces. Suppose that we have $\|u\|_{B_i} \le \|v\|_{B_i}$ for any i, u and v satisfying $|u| \le |v|$. Assume $|u| \lesssim \sum_i |v_i|$. Then we have*

$$\|u\|_{\sum_i B_i} \lesssim \sum_i \|v_i\|_{B_i}. \tag{3.5}$$

Proof. Define u_i as follows. If $|v_i(x)| > |v_j(x)|$ for any $j < i$ and $|v_i(x)| \ge |v_j(x)|$ for any $j \ge i$, then let $u_i(x) = u(x)$. Otherwise, let $u_i(x) = 0$. Then, we have

$$u(x) = \sum_i u_i(x), \quad |u_i(x)| \lesssim |v_i(x)|. \tag{3.6}$$

Thus we obtain $\|u\|_{\sum_i B_i} \le \sum_i \|u_i\|_{B_i} \lesssim \sum_i \|v_i\|_{B_i}$. □

We introduce the following assumption about the nonlinear function g with a parameter $p > 0$. A typical example is $g(u) = |u|^p u$.

$$\begin{aligned} &g : \mathbb{C} \to \mathbb{C}, \quad g(0) = 0, \\ &|g(a) - g(b)| \lesssim |a - b|(|a| + |b|)^p, \\ &|g'(a) - g'(b)| \lesssim \begin{cases} |a - b|(|a| + |b|)^{p-1}, & (p > 1), \\ |a - b|^p, & (p \le 1), \end{cases} \end{aligned} \tag{3.7}$$

Now we can prove the desired nonlinear estimate.

Lemma 3.4 *Let $p > 0$ and assume* (3.7). *Let $X_i \in \mathcal{L}_0$ and $Z_i \in \mathcal{B}$ for $i = 0, \dots, 3$. Suppose that $\sigma(Z_i) < \min(2, p + 1)$ and $X_i^p Z_i \in \mathcal{B}$ for any i. Then we have*

$$\|g(u)\|_{\sum_i X_i^p Z_i} \lesssim \inf_{u=a+b} \left(\|a\|_{X_0 \cap X_1} + \|b\|_{X_2 \cap X_3}\right)^p \left(\|a\|_{Z_0 \cap Z_2} + \|b\|_{Z_1 \cap Z_3}\right). \tag{3.8}$$

Proof. By elementary calculations, we have

$$|\delta_{k,j}^2 g(u)| \lesssim [u]_{k,j}^p |\delta_{k,j}^2 u| + [u]_{k,j}^{p-1} |\delta_{k,j} u|^2, \tag{3.9}$$

if $p > 1$. Substituting $u = a + b$, we obtain eight terms. We estimate only two typical terms. We omit the subindices for a while. By Hölder's inequality, we have

$$\|[a]^p |\delta^2 b|\|_{X_1^p \underline{Z_1}} \lesssim \||a|^p\|_{X_1^p} \|\delta^2 b\|_{\underline{Z_1}} \lesssim \|a\|_{X_1}^p \|b\|_{Z_1}, \tag{3.10}$$

and

$$\begin{aligned}\|[a]^{p-1}|\delta b|^2\|_{[X_1^p\underline{Z_1},X_3^p\underline{Z_3}]_{1/p}} &\lesssim \||a|^{p-1}\|_{X_1^{p-1}}\||\delta b|^2\|_{[X_3\underline{Z_1},X_3\underline{Z_3}]_{1/p}} \\ &\lesssim \|a\|_{X_1}^{p-1}\|\delta b\|^2_{[(X_3\underline{Z_1})^{1/2},(X_3\underline{Z_3})^{1/2}]_{1/p}},\end{aligned} \tag{3.11}$$

where the last factor can be estimated as follows.

$$\begin{aligned}\|\delta b\|_{[(X_3\underline{Z_1})^{1/2},(X_3\underline{Z_3})^{1/2}]_{1/p}} &\lesssim \|b\|_{[\overline{(X_3\underline{Z_1})^{1/2}},\overline{(X_3\underline{Z_3})^{1/2}}]_{1/p}} \\ &\lesssim \|b\|^{1-1/p}_{\overline{(X_3\underline{Z_1})^{1/2}}}\|b\|^{1/p}_{\overline{(X_3\underline{Z_3})^{1/2}}} \\ &\lesssim \|b\|^{1-1/p}_{[X_3,Z_1]_{1/2}}\|b\|^{1/p}_{[X_3,Z_3]_{1/2}} \\ &\lesssim \|b\|^{1/2}_{X_3}\|b\|^{1/2-1/(2p)}_{Z_1}\|b\|^{1/(2p)}_{Z_3},\end{aligned} \tag{3.12}$$

where, in the third inequality, we have used the embedding

$$L^q(\mathbb{R}^n) \subset \dot{B}^0_{q,\infty}(\mathbb{R}^n) \tag{3.13}$$

for X_3. Thus we obtain a bound in $[X_1^pZ_1, X_3^pZ_3]_{1/p} \subset X_1^pZ_1 + X_3^pZ_3$. The remaining six terms are estimated in the same way.

If $p \le 1$, we have

$$|\delta^2_{k,j}g(u)| \lesssim [u]^p_{k,j}|\delta^2_{k,j}u| + |\delta_{k,j}u|^{p+1}. \tag{3.14}$$

Substituting $u = a + b$, we have two new terms. We estimate by Hölder's inequality as

$$\||\delta a|^{p+1}\|_{X_0^p\underline{Z_0}} \lesssim \|\delta a\|^{p+1}_{X_0^{p/(p+1)}\underline{Z_0}^{1/(p+1)}}. \tag{3.15}$$

By (3.13) and the complex interpolation we have

$$\overline{X_0^{p/(p+1)}\underline{Z_0}^{1/(p+1)}} \supset [X_0, Z_0]_{1/(p+1)}, \tag{3.16}$$

and then the interpolation inequality gives the desired estimate. The other term is estimated in the same way. □

4. Uniform boundedness of strichartz norms

In this section, we prove uniform boundedness of solutions for (1.2) in the Strichatz type norms appearing in Sect. 2. The procedure of our proof goes as follows. We derive an iterative estimate for the associated integral equation (2.3) with $f = f(u)$. At first we have an estimate for the homogeneous term by the result in Sect. 2. It is known that such a norm is finite for any solution u_c of (1.2) with finite energy and charge (see, e.g., [3]). So we can estimate the nonlinear term $f(u_c)$ in sum spaces by the result in Sect. 3. Since we have $c/\langle\nabla\rangle_c$ in the inhomogeneous term, we can gain 1 derivative for the higher frequency part at the cost of c weight. For the lower frequency, we can gain as many derivatives as we like, at the cost of the same power of c. Thereby we can come back to the frequency separated spaces, and then we get the desired closed estimate, where the norms depend on c but the constants are independent of c.

The precise result is the following:

Lemma 4.1 *Let n, p and λ be as in Theorem 1.1. Let S, K and W be as in Lemma 2.4. Then, there exist positive continuous functions $T(\cdot)$ and $M(\cdot)$ satisfying the following. Let $c \geq 1$ and u be a solution of* (1.2) *with*

$$\|u(0)\|_{H^1} + \|\dot{u}(0)/c\|_{L^2} \leq E < \infty. \tag{4.1}$$

Then u exists at least on $[0, T(E)]$ and we have

$$\|Du\|_{S|W\cap K(0,T(E))} \leq M(E). \tag{4.2}$$

Proof. First we prove the above lemma when $n \geq 3$. The other case is much easier. We estimate the solution u in the following spaces:

$$\begin{aligned} Z &:= S_2 \,|\, W_2 \cap K_1, \\ X &:= S_1 \,|\, W_1, \end{aligned} \tag{4.3}$$

where S_2, W_2 and K_1 are of the form $c^{-\mu} L^{1/b}(\mathbb{R}; B^{\sigma}_{1/b,2}(\mathbb{R}^n))$, and S_1 and W_1 are of the form $c^{-\mu} L^{1/b}(\mathbb{R}^{1+n})$, with the exponents b, σ, μ listed in Table 1.

Table 1. Exponents for $n \geq 3$ ($p^* = 4/(n-2)$)

u	$1/2-b$	σ	μ	$f(u)$	$b-1/2$	σ	μ
S_1	$2/(n+2)$	0	0	$S_1^{p^*} S_2$	$1/(n+2)$	1	0
S_2	$1/(n+2)$	1	0	$S_1^{p^*} K_1$	$1/(n+2)$	$1/2$	$1/2$
W_1	$1.5/(n+1)$	0	$=b$	$W_1^{p^*} W_2$	$1/(n+1)$	$1/2$	$=b$
W_2	$1/(n+1)$	$1/2$	$=b$	$W_1^{p^*} S_2$	$\frac{2}{n+1} - \frac{1}{n+2}$	1	$2/(n+1)$
K_1	$1/(n+2)$	$1/2$	$1/2$				

$c\langle\nabla\rangle_c^{-1} f(u)$	$b-1/2$	σ_0	σ_1	μ
R_1	$1/(n+2)$	1	$3/2$	$-1/2$
R_2	$1/(n+1)$	$1+1/(n+1)$	$3/2$	$-1/2+1/(n+1)$
R_3	$\frac{2}{n+1} - \frac{1}{n+2}$	$1+\frac{2}{n+1}$	$\frac{3}{2}+\frac{1}{n+2}$	$-\frac{1}{2}+\frac{2}{n+1}-\frac{1}{n+2}$

By the Strichartz estimate (Lemma 2.4), we have

$$\|v\|_{L^\infty H^1 \cap Z \cap X} \lesssim \|u(0)\|_{H^1} + \|\dot{u}(0)/c\|_{L^2}, \tag{4.4}$$

where v solves the free equation ((1.2) with $f \equiv 0$) with the same initial data as u. For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ depending only on f and ε such that we can decompose the nonlinearity as $f(u) = f_0(u) + f_1(u)$ where

$$|f_0(u)| \le C_\varepsilon |u|, \quad |f_0'(u)| \le C_\varepsilon, \tag{4.5}$$

and $g = f_1/\varepsilon$ satisfies (3.7) with $p = p^* := 4/(n-2)$. By the nonlinear estimate (Lemma 3.4), we have

$$\|f_1(u)\|_{S_1^{p^*} S_2 + S_1^{p^*} K_1 + W_1^{p^*} W_2 + W_1^{p^*} S_2} \lesssim \varepsilon \|u\|_Z \|u\|_X^{p^*}, \tag{4.6}$$

where the exponents of the spaces on the left hand side are given in Table 1, and

$$\|f_0(u)\|_{L^1(0,T;H^1)} \lesssim C_\varepsilon T \|u\|_{L^\infty(0,T;H^1)}, \tag{4.7}$$

for any $T < T_c^*$. By the regularization effect of $\langle\nabla\rangle_c^{-1}$, we have

$$\|c\langle\nabla\rangle_c^{-1} f_1(u)\|_{R_1+R_2+R_3} \lesssim \|f_1(u)\|_{S_1^{p^*} S_2 + S_1^{p^*} K_1 + W_1^{p^*} W_2 + W_1^{p^*} S_2}, \tag{4.8}$$

where the spaces R_i are of the form

$$L^{1/b} B^{\sigma_0}_{1/b,2} \,|\, c^{-\mu} L^{1/b} B^{\sigma_1}_{1/b,2} \tag{4.9}$$

with the exponents $(b, \sigma_0, \sigma_1, \mu)$ given in Table 1. Combining these estimates with the Strichartz estimate, we finally obtain

$$\begin{aligned} &\left\| \int_0^t e^{-ic^2(t-s)} \sin(c\langle\nabla\rangle_c (t-s)) \frac{c}{\langle\nabla\rangle_c} f(u(s)) ds \right\|_{L^\infty H^1 \cap Z \cap X(0,T)} \\ &\qquad \lesssim \varepsilon \|u\|^{p^*}_{X(0,T)} \|u\|_{Z(0,T)} + C_\varepsilon T \|u\|_{L^\infty(0,T;H^1)}. \end{aligned} \tag{4.10}$$

Denote $\Omega := L^\infty(H^1) \cap Z \cap X$. The above estimate implies

$$\|u\|_{\Omega(0,T)} \le \|v\|_{\Omega(0,T)} + C\varepsilon \|u\|^{p^*+1}_{\Omega(0,T)} + CTC_\varepsilon \|u\|_{\Omega(0,T)}, \tag{4.11}$$

as long as u exists until $t = T$. If we take ε sufficiently small, which depends only on E, and take $T(E)$ sufficiently small, then we obtain from (4.11),

$$\|u\|_{\Omega(0,T)} \le 2\|v\|_{\Omega(0,T)} \tag{4.12}$$

for $T < T(E)$, which also implies that u can be extended until $t = T(E)$. Repeating the estimate (4.11) with the left hand side replaced with any space allowed by the Strichartz estimate, we obtain the desired result.

If $n \le 2$, then we define $\Omega := L^\infty H^1 \cap L^q L^\infty$, where $q > \max(p, 2)$. By the Strichartz estimate we have $\|v\|_\Omega \lesssim E$, and

$$\|u - v\|_{\Omega(0,T)} \lesssim \|f(u)\|_{L^1 H^1(0,T)} \lesssim T^{1-p/q} \|u\|^{p+1}_{\Omega(0,T)}, \tag{4.13}$$

from which the desired result follows. □

5. H^1 convergence

Lemma 4.1 means in particular that there exists some $T > 0$ independent of c such that $(u_c, \dot{u}_c/c)$ is uniformly bounded in $H^1 \oplus L^2$ on the time interval $t \in [0, T]$. So, in order to prove the H^1 convergence (Theorem 1.1), it suffices to derive the H^1 convergence on $[0, T]$ under the additional assumption of the boundedness in $H^1 \oplus cL^2$. Using this repeatedly on consecutive intervals, we obtain Theorem 1.1. In particular, if $\lambda \geq 0$, we can prove the theorem directly without the estimate in the previous sections, since we have a uniform global a priori bound for the $H^1 \oplus cL^2$ norm by the energy and charge conservation.

Hence we suppose, in addition to the assumptions of Theorem 1.1, that

$$\|u_c(t)\|_{H^1} + \|\dot{u}_c(t)/c\|_{L^2} < M \tag{5.1}$$

on $t \in [0, T]$ for some $M < \infty$ independent of c and t, and prove the convergence $(u_c, \dot{u}_c/c) \to (v, 0)$ in $C([0, T]; H^1 \oplus L^2)$ via a compactness argument.

Let $A \subset C_0^\infty(\mathbb{R}^n)$ be an enumerable set which is dense in H^{-1}. Then, for any $\rho \in A$, $\{\langle \rho, u_c(t)\rangle\}_{c>1}$ is a bounded set in $C([0, T])$. The equicontinuity (for $c \to \infty$) can be seen as follows: From the equation, we have

$$\begin{aligned}\langle \rho, u_c(t_0) - u_c(t_1)\rangle &= \langle \rho, \int_{t_0}^{t_1} \dot{u}_c dt\rangle \\ &= \langle \rho, i\int_{t_0}^{t_1} \ddot{u}_c/c^2 - \Delta u_c + f(u_c)dt\rangle/2 \\ &= \langle \rho, i(\dot{u}_c(t_1) - \dot{u}_c(t_0))/c^2 + i\int_{t_0}^{t_1} -\Delta u_c + f(u_c)dt\rangle/2,\end{aligned} \tag{5.2}$$

so that we can estimate

$$|\langle \rho, u_c(t_0) - u_c(t_1)\rangle| \lesssim c^{-1}\|\rho\|_{L^2} + |t_0 - t_1|\|\rho\|_{H^1}. \tag{5.3}$$

Now the Ascoli-Arzelà theorem implies that if we extract an appropriate subsequence, then $\langle \rho, u_c(t)\rangle$ converges in $C([0, T])$ for any $\rho \in A$, and so $u_c(t)$ converges in $C([0, T]; w\text{-}H^1)$, where $w\text{-}H^1$ denotes the weakly topologized H^1. Then it is easy to see that the limit function $u_\infty(t)$ satisfies (1.3) and the initial condition $u_\infty(0) = \varphi$. But the uniqueness of such solutions is well-known (see [4]), so that we have $u_\infty = v$. It is indeed not an easy task to get the uniqueness only from the finiteness of H^1, but in our case, we have the uniform boundedness of the space-time norms from Lemma 4.1:

$$\|Du_c\|_{S\,|\,W\cap K(0,T)} \leq N. \tag{5.4}$$

for some finite N. Since the higher frequency vanishes in $\mathcal{S}'(\mathbb{R}^{1+n})$ as $c \to \infty$, we obtain by the weak convergence,

$$\|Du_\infty\|_{S(0,T)} \le N, \tag{5.5}$$

for any S satisfying the conditions in Lemma 2.4 (N may depend on S). Then it is quite easy to show $u_\infty = v$.

By the conservation of charge and boundedness of $\dot{u}/c$, we have

$$Q_S(v) = \lim_{c\to\infty} Q_K(u_c) = \lim_{c\to\infty} Q_S(u_c; t), \tag{5.6}$$

uniformly on $[0, T]$. Thus we obtain $u_c \to v$ in $C([0, T]; L^2)$. By the interpolation, we also have the convergence in L^{p+2}. Then, we have

$$\int |\nabla v(t)|^2 dx = \lim_{c\to\infty} \int |\dot{u}/c(t)|^2 + |\nabla u_c(t)|^2 dx, \tag{5.7}$$

uniformly on $[0, T]$. On the other hand, the uniform weak convergence implies that, for any $\varepsilon > 0$, there exists c_0 such that

$$\int |\nabla v(t)|^2 dx \le \inf_{c>c_0} \int |\nabla u_c(t)|^2 dx + \varepsilon \tag{5.8}$$

for any $t \in [0, T]$. These two facts make us conclude that

$$\|\dot{u}_c(t)/c\|_{L^2} \to 0, \quad \|\nabla u_c(t)\|_{L^2} \to \|\nabla v(t)\|_{L^2} \tag{5.9}$$

in $C([0, T])$, which in turn enhance the weak convergence into the desired strong one; $u_c \to v$ in $C([0, T]; H^1)$.

Since the limit is unique, we do not need to extract any subsequence. Thus we obtain the desired convergence result. □

Interpolating the convergence in the energy space and the uniform boundedness in Lemma 4.1, we obtain also the convergence in the space-time norms.

Corollary 5.1 *Under the same assumptions as in Theorem 1.1, we have*

$$\|D(u_c - v)\|_{S|W\cap K(0,T)} \to 0, \tag{5.10}$$

for any $T < T^*$ and any S, W and K satisfying the conditions in Lemma 2.4.

Remark 5.2. If $p \ge 1$, then we can prove the convergence also by such direct estimates as in the previous section. But if $p < 1$, then we can not avoid some compactness argument more or less, because of the singularity of $f(\cdot)$ at the origin.

6. L^2 convergence rate

In this section, we prove the optimal rate of convergence in L^2 (Theorem 1.2). The dominant term is the free part, and in order to prove the optimality, we employ an idea from the scattering theory to dominate the strongest term among the nonlinear ones. Denote $[a]_c := (\langle a\rangle_c + c)/2$ and

$$K_c^{\pm}(t) := e^{-ic(c\pm\langle\nabla\rangle_c)t}, \tag{6.1}$$

where $c(c - \langle\xi\rangle_c) = c|\xi|^2/[\xi]_c \to |\xi|^2/2$ and $c(c + \langle\xi\rangle_c) \to \infty$ as $c \to \infty$. Below we will often omit the subscript c.

First we show the upper bound $o(1/\sqrt{c})$. We estimate the convergence in $\Upsilon := Q \cap DS_2$, where $Q := L^\infty L^2$ and S_2 is as given in Sect. 4. Define $\alpha \in (0, 1)$ by $p = \alpha p^*$ for $n \geq 3$, where $p^* = 4/(n-2)$, and $\alpha := \min(p, 1)/3$ for $n \leq 2$. We define an auxiliary space $Y := [Q, DS_2]_\alpha \supset \Upsilon$. Hereafter, every norm for t is taken on the interval $(0, T)$, which we will not write explicitly.

Let $r(c) = c^{1/2+\varepsilon}$ with $\varepsilon > 0$ sufficiently small. By the boundedness in $L^\infty H^1 \cap S_2 \mid K_1$, we have

$$\|\chi_r * (u_c, v)\|_\Upsilon = O(1/r) + O((cr)^{-1/2}) = O(c^{-1/2-\varepsilon}). \tag{6.2}$$

Since $1 - c/\langle\xi\rangle_c = |\xi|^2/(2\langle\xi\rangle_c[\xi]_c)$, we have for any Besov or Lebesgue space B,

$$\left\|(1 - c/\langle\nabla\rangle_c)\chi^r *\right\|_{\mathcal{L}(B)} \lesssim r^2/c^2 = O(c^{-1+2\varepsilon}). \tag{6.3}$$

From these estimates, we obtain

$$\begin{aligned} u(t) - v(t) = \chi^r * R(t)\varphi + \frac{i}{2}\chi^r * \int_0^t R(t-s)f(v(s)) - K^+(t-s)f(u(s))ds \\ + \frac{i}{2}\chi^r * \int_0^t K^-(t-s)\{f(u(s)) - f(v(s))\}ds \\ + O(c^{-1/2-\varepsilon}) + O(\|\varphi_c - \varphi\|_{L^2}), \end{aligned} \tag{6.4}$$

in Υ, where $R_c(t) := K_c^-(t) - e^{-i\Delta t/2}$. Differentiating by c, we get

$$R_c(t)\varphi = -i\int_c^\infty \frac{|\nabla|^4}{4\langle\nabla\rangle_\gamma[\nabla]_\gamma^2} K_\gamma^-(t)t\varphi\, d\gamma. \tag{6.5}$$

Let $\varphi_j := \chi^{2^{j+1}} - \chi^{2^j}$ for $j \in \mathbb{N}$ and $\varphi_0 := \chi^1$. By the Strichartz estimate, we obtain

$$\|\varphi_j * R_c(t)\varphi\|_\Upsilon \lesssim \min(1, \int_c^\infty 2^{4j}\gamma^{-3}t\, d\gamma)\|\varphi_j * \varphi\|_{L^2} \tag{6.6}$$

Taking the square summation for j and using the Minkowski inequality, we get

$$\|\chi^r * R_c(t)\varphi\|_{\Upsilon} \lesssim \langle T\rangle \left\|\min(1, 2^{4j}c^{-2})\|\varphi_j * \varphi\|_{L^2}\right\|_{\ell^2}. \tag{6.7}$$

Denote the right hand side by $\rho(c)$. Then we have

$$\|\sqrt{c}\rho\|^2_{\ell^2 L^\infty} \lesssim \langle T\rangle^2 \sum_{k,j} \min(2^{k-2j}, 2^{-3(k-2j)})2^{2j}\|\varphi_j * \varphi\|^2_{L^2} \lesssim \langle T\rangle^2\|\varphi\|^2_{H^1}. \tag{6.8}$$

Applying the Strichartz and the nonlinear estimates to $f(v)$ as in Sect. 4, we obtain in a similar way,

$$\left\|\sqrt{c}\int_0^t \chi^r * R_c(t-s)f(v(s))ds\right\|_{\ell^2 L_c^\infty(\Upsilon)} < \infty. \tag{6.9}$$

Integrating by parts, we get

$$-\frac{i}{2}\chi^r * \int_0^t K^+(t-s)f(u(s))ds = \chi^r * \left[[\nabla]_c^{-1}K^+(t-s)f(u(s))/c\right]_0^t \\ - \chi^r * \int_0^t [\nabla]_c^{-1}K^+(t-s)f'(u(s))\dot{u}(s)/c\, ds. \tag{6.10}$$

Then the first term can be estimated as

$$\|\chi^r * [\nabla]_c^{-1}K^+(t-s)f(u(s))/c\|_{L^q} \lesssim c^{-2}r^{1+n(1/2-1/q)}\|f(u(s))\|_{H^{-1}}, \tag{6.11}$$

from which we obtain a bound of $o(c^{-1+2\varepsilon})$ in Υ. For the second term, we first consider the case where $n \le 3$. We have

$$u \in L_t^{p_0}L_x^\infty + c^{-1/p_0}L_t^{p_0}L_x^{p_1} =: B, \tag{6.12}$$

where $p_0 := \max(3, p)$, $1/p_1 := 1/2 - (1+1/p_0)/n$ for $n = 3$ and $p_1 = \infty$ for $n \le 2$. Since $|f'(u)| \lesssim 1 + |u|^{p_0}$, we have

$$\|\chi^r * f'(u)\dot{u}/c\|_{L^1L^2} \lesssim \|f'(u)\dot{u}/c\|_{L^1L^2 + c^{-1}L^1L^{p_2}} \lesssim (T + \|u\|_B^{p_0})\|\dot{u}/c\|_Q, \tag{6.13}$$

where $1/p_2 := 1/2 + p_0/p_1 < 1/2 + 1/n$. So the second term on the right hand side of (6.10) is bounded by $O(1/c)$ in Υ. If $n \ge 4$, we have

$$\|f'(u)\dot{u}/c\|_{L_t^{p_0}L_x^{p_1}} \lesssim T^{1/p_0}\|u\|^p_{L_t^\infty L_x^{2n/(n-2)}}\|\dot{u}/c\|_{L_t^\infty L_x^2}, \tag{6.14}$$

where $1/p_0 := 1 - \alpha/2$ and $1/p_1 := 1/2 + 2\alpha/n$. So we have

$$\|\chi^r * [\nabla]_c^{-1}f'(u)\dot{u}/c\|_{L^{p_0}B^\alpha_{p_1,2}} = O(c^{-1+\alpha(1/2+\varepsilon)}). \tag{6.15}$$

Then, the Strichartz estimate yields the same order for (6.10) in Υ.

Thus we have obtained

$$u(t) - v(t) = \frac{i}{2} \int_0^t \chi^r * K^-(t-s)\{f(u(s)) - f(v(s))\}ds + o, \tag{6.16}$$

in Υ, where $o = q(c) + O(\|\varphi_c - \varphi\|_{L^2})$ and $\sqrt{c}q(c) \in \ell^2 L^\infty$.

The remaining nonlinear term is estimated by Hölder's inequality and the Strichartz estimate as

$$\begin{aligned} \|u - v\|_\Upsilon &\lesssim T^{1-\alpha} \|\chi^r * (f(u) - f(v))\|_{[Q,DR_1]_\alpha + [Q,DR_3]_\alpha} + o \\ &\lesssim T^{1-\alpha} \|(u, v)\|_X^p \|u - v\|_Y + o, \end{aligned} \tag{6.17}$$

if $n \geq 3$. Since $\Upsilon \subset Y$ and $\|(u, v)\|_X$ is bounded, we obtain

$$\|u - v\|_\Upsilon \lesssim o, \tag{6.18}$$

if we take T sufficiently small. Repeating this argument, we can extend this result for any $T < T^*$.

If $n \leq 2$, then by Hölder's inequality and the Strichartz estimate we have

$$\|u - v\|_\Upsilon \lesssim \|f(u) - f(v)\|_{L^1 L^2} + o \lesssim T^{1-\alpha/2} \|(u, v)\|_{L^q L^q}^p \|u - v\|_Y + o, \tag{6.19}$$

where $q := (n+2)p/\alpha > 2(n+2)/n$. Since $\|(u, v)\|_{L^q L^q}$ is bounded, we obtain the desired upper bound as in the case $n \geq 3$.

Next we prove the optimality. Let $q_j := \sup_{2^{2j} < c < 2^{2j+1}} q(c)$. Since $2^j q_j \in \ell^2$, we can find q' satisfying $2^j q'_j \in \ell^2$ and $\lim_{j \to \infty} q'_j / q_j = \infty$. The above argument implies that it suffices to choose φ such that

$$u - v = \frac{i}{2} \int_0^t \chi^r * K^-(t-s)\{f(u(s)) - f(v(s))\}ds + \mathcal{R}, \tag{6.20}$$

with $\|\mathcal{R}\|_Y \lesssim q(c)$ and $\inf_{2^{2j} < c < 2^{2j+1}} \|\mathcal{R}(t)\|_{L^2} \gtrsim q'_j$. Now we have only to get the same order for the free part $I_0 := R(t)\varphi$ and to dominate the nonlinear term $I_1 := \int_0^t R(t-s) f(v(s))ds$ in Υ, for the above arguments show that the other terms decay faster.

First we find φ satisfying

$$\inf_{2^{2j} < c < 2^{2j+1}} \|I_0(t)\|_{L^2} \gtrsim q'_j. \tag{6.21}$$

Define $\varphi \in H^1$ by $\tilde{\varphi}(\xi) := q'_j$ for $2^j \leq |\xi| < 2^{j+1}$.Then we have $\|\varphi\|_{H^1} \lesssim \|2^j q'_j\|_{\ell^2}$, and (6.21) follows from

$$|\mathcal{F} I_0| = |e^{i|\xi|^4 t/(4[\xi]_c^2)} - 1||\tilde{\varphi}(\xi)| \gtrsim |\xi|^4 t/[\xi]_c^2 |\tilde{\varphi}(\xi)| \tag{6.22}$$

for $|\xi|^4 t/[\xi]_c^2 < 1$.

Next we reduce the space-time norms of v and $f(v)$ by the following lemma.

Lemma 6.1 *Let $S = L^q(B^\sigma_{r,2})$ satisfy $q < \infty$ and the conditions in Lemma 2.4. Then, for any $\varphi \in L^2$, any $s > 0$, any $T < \infty$ and any ε, we can find $\psi \in L^2$ satisfying*

$$\|D^s U(t)\psi\|_{S(0,T)} + \sup_{c>1} \|D^s \chi_c * K_c^-(t)\psi\|_{S(0,T)} < \varepsilon, \tag{6.23}$$

where $U(t) := e^{-i\Delta t/2}$, and

$$|\tilde{\psi}(\xi)| \equiv |\tilde{\varphi}(\xi)|. \tag{6.24}$$

Proof. Denote $U_c(t) := \chi_c * K_c^-(t)$. First we show that for any $\varphi \in L^2$ we have

$$\sup_{c>1} \|U_c(t)U(\tau)\varphi\|_{S(0,T)} \to 0, \tag{6.25}$$

as $\tau \to \infty$. Since the above supremum is bounded by $\|\varphi\|_{L^2}$, it suffices to show this convergence when $\tilde{\varphi} \in C_0^\infty$. Then, by the stationary phase method, we have

$$\|U(\tau)U_c(t)\varphi\|_{B^\sigma_{r,2}} \lesssim \tau^{-n(1/2-1/r)}, \tag{6.26}$$

so that we obtain (6.25).

Now define $\varphi_j \in L^2$ by $\varphi = \sum_{j\geq 0} \varphi_j$, $\operatorname{supp} \tilde{\varphi}_j \subset \{2^j \leq |\xi| \leq 2^{j+1}\}$ and $\operatorname{supp} \tilde{\varphi}_0 \subset \{|\xi| \leq 1\}$. Let $\psi_j := U(T_j)\varphi_j$ and $\psi := \sum_{j\geq 0} \psi_j$, where $T_j > 0$ should be chosen sufficiently large. Then (6.24) is obvious. Moreover, we have

$$\|D^s U_c(t)\psi_j\|_{S(0,T)} \lesssim 2^{sj} \|U(T_j)U_c(t)\varphi_j\|_{S(0,T)}, \tag{6.27}$$

which tends to 0 as $T_j \to \infty$, uniformly for $c > 1$. For example, we can choose T_j so large that (6.27) is smaller than $2^{-3-j}\varepsilon$. Hence we obtain the desired result by the triangle inequality. □

We choose $s \in (0,1)$ satisfying $s < p$. If $n \geq 3$, then let $Y_1 := (D^{-s-1}Y) \cap S_1$. By the above lemma, for any $\varepsilon > 0$, we can replace $\varphi \in H^1$ without violating (6.21) such that we have

$$\|U(t)\varphi\|_{Y_1} < \varepsilon. \tag{6.28}$$

Then the nonlinear and the Strichartz estimates yield

$$\|v\|_{Y_1} \lesssim \varepsilon + T^{1-\alpha}\|v\|_{Y_1}^{p+1}. \tag{6.29}$$

Since $p > 0$, if we choose ε sufficiently small depending on T and p, the above inequality implies a bound $\lesssim \varepsilon$ for the left hand side. Then, from (6.5) and the Strichartz estimate, we obtain

$$\begin{aligned} \|\chi^r * I_1\|_{\Upsilon} &\lesssim r^{3-s}c^{-2}T^{1-\alpha}\|f(v)\|_{[L^\infty H^{1+s}, D^{-s}S_1^{p^*}S_2]_\alpha} \\ &\lesssim r^{3-s}c^{-2}T^{1-\alpha}\|v\|_{Y_1}^{p+1} = O(c^{-2+(3-s)(1/2+\varepsilon)}) = o(c^{-1/2-\varepsilon}). \end{aligned} \tag{6.30}$$

Thus we have obtained the desired optimality.

If $n \leq 2$, we put $Y_1 := (D^{-s-1}Y) \cap L^q(\mathbb{R}^{1+n})$, where q is the same as in (6.19). Then, by the same argument, we obtain the desired optimality. □

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