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The Nonrelativistic Limit of the Nonlinear Klein–Gordon Equation

By

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1. Introduction

We consider the nonlinear Klein–Gordon equation in space-time $\boldsymbol{R}^{n+1}$

$$\frac{\hbar^2}{2mc^2}u'' - \frac{\hbar^2}{2m}\Delta u + \frac{mc^2}{2}u + \lambda|u|^{\gamma-1}u = 0, \qquad x \in \boldsymbol{R}^n, t \in \boldsymbol{R}, \tag{1.1}$$

where $1 \leq n \leq 3$, $\lambda > 0$, $\hbar$ is the Planck constant, m is the mass of particle, c is the speed of light, and u'' is the second time derivative. When $n = 3$ and $\gamma = 3$, the equation (1.1) was introduced by Schiff [10] as the equation of classical neutral scalar mesons.

Substituting

$$u = ve^{-(imc^2t)/\hbar},$$

we obtain from (1.1) the following nonlinear Klein–Gordon equation for v:

$$\frac{\hbar}{2mc^2}v'' - i\hbar v' - \frac{\hbar^2}{2m}\Delta v + \lambda|v|^{\gamma-1}v = 0.$$

We now consider the nonlinear Schrödinger equation

$$-i\hbar v' - \frac{\hbar^2}{2m}\Delta v + \lambda|v|^{\gamma-1}v = 0.$$

Comparing the last two equations, we find that the nonlinear Klein–Gordon equation turns into the nonlinear Schrödinger equation if the first term of the nonlinear Klein–Gordon equation vanishes. So we expect that solutions of the nonlinear Klein–Gordon equation converge as $c \to \infty$ toward the corresponding solutions of the nonlinear Schrödinger equation. We regard the procedure $c \to \infty$ as "nonrelativistic limit."

We may think of the Klein–Gordon equation as a relativistic generalization for the Schrödinger equation. From this relation, we have a particular interest in the convergence of solutions of two equations. In this paper we study this problem in detail. Without loss of generality, we may set $\hbar = 1$, $m = 1/2$, $\varepsilon = 1/c^2$ and $f(v) = \lambda|v|^{\gamma-1}v$.

With given initial data, we rewrite the equations in question as

$$\varepsilon v'' - iv' - \Delta v + f(v) = 0, \qquad v(0) = v_{0\varepsilon}, v'(0) = v_{1\varepsilon}, \tag{1.2}$$

$$-iv' - \Delta v + f(v) = 0, \qquad v(0) = v_{00}. \tag{1.3}$$

We denote by v_ε and v_0 solutions of (1.2) and (1.3), respectively.

The purpose of this paper is to study how v_ε converges to v_0 as $\varepsilon \to 0$. There are a few results on the problem. In [11], Tsutsumi proved the convergence in $L^\infty(0, T; L^2)$ for $n = 2$ and $2 \le \gamma \le 3$. In [8], Najman prove the convergence in $L^\infty(0, T; L^q)$ with $2 < q < 2n/(n-2)$ for $n \le 3$ and $2 \le \gamma \le 3$. As regards the initial data, Tsutsumi required that $v_{0\varepsilon}, v_{00} \in H^2$, $v_{1\varepsilon} \in H^1$, and that $v_{0\varepsilon}$ converges to v_{00} in H^2, while Najman required that $v_{0\varepsilon}, v_{00} \in H^1$, $v_{1\varepsilon} \in L^2$, and that $v_{0\varepsilon}$ converges to v_{00} in L^2. But Najman's result does not cover Tsutsumi's one, since the mode of convergence of solutions is different. In respect to methods of proof, Tsutsumi used the conservation of energy and the Brezis–Gallouet inequality (see [1]). Najman's proof is based on the representation of the solution of the second order (in time) equation which was used by Fattorini [2] in treating the linear nonrelativistic limit, and on the L^p–L^q estimates for the Klein–Gordon equation (see [7]).

In this paper, we would improve Najman's results. Under the same assumptions on the data as Najman's, we give that the solutions converge in $L^\infty(0, T; L^2)$. From energy estimate and the Sobolev embedding theorem, we have convergence in $L^\infty(0, T; L^q)$ for any q with $2 \le q < 2n/(n-2)$ at once. Here we emphasize that the case $q = 2$ has been open and that L^2 convergence is of both mathematical and physical importance. We give an extension of admissible values of γ as well. We would follow almost the same line as in Najman's proof. We adopt Strichartz's type uniform estimate under procedure of nonrelativistic limit.

This paper is constructed as follows. In section 2, we state the main theorem. In section 3, we give Strichartz's estimate for the Klein–Gordon equation. Using this, we prove the main theorem in section 4.

We close this section by giving several notation. We abbreviate $L^q(\boldsymbol{R}^n)$ to L^q and $L^r(I; L^q(\boldsymbol{R}^n))$ to $L_t^r L_x^q$, where I is a time interval. We denote by $H^{s,q}$ the usual Sobolev space of order s. We abbreviate $H^{s,2}$ to H^s. For any p with $1 < p < \infty$, p' stands for its Hölder conjugate, i.e. $p' = p/(p-1)$.

2. Main theorem

We state our main theorem.

Theorem 1. *We assume that*

$$v_{0\varepsilon} \in H^1, \qquad v_{1\varepsilon} \in L^2, \tag{2.1}$$

$$v_{00} \in H^1, \tag{2.2}$$

$$\sup_{\varepsilon>0}(\|v_{0\varepsilon}\|_{H^1}+\varepsilon^{1/2}\|v_{1\varepsilon}\|_{L^2})<\infty, \tag{2.3}$$

$$\lim_{\varepsilon\to 0}\|v_{0\varepsilon}-v_{00}\|_{L^2}=0. \tag{2.4}$$

Then for every $T>0$, we have

$$\lim_{\varepsilon\to 0}\|v_\varepsilon-v_0\|_{L^\infty(0,T;L^2)}=0, \tag{2.5}$$

here

$$1<\gamma\leq\frac{17}{5}\qquad\text{for } n=3, \tag{2.6}$$

$$1<\gamma<\infty\qquad\text{for } n=1,2. \tag{2.7}$$

Remark 1. From this Theorem, we can obtain that the solutions converge in also $L^\infty(0,T;L^q)$ for $2<q<2n/(n-2)$. We have it by Sobolev's embedding theorem and interpolation theory with the fact that the solutions of NLKG are bounded in $L^\infty(0,T;H^1)$ uniformly with ε, which will appear in section 4.

3. Strichartz's type estimate for the Klein–Gordon equation

In this section we study the space–time integrability properties of solutions of the free Klein–Gordon equation. In order to use in proof of Theorem 1, we construct the estimate including the parameter ε for equation (1.2). From the Duhamel principle, the solution v_ε of (1.2) satisfies the integral equation,

$$v_\varepsilon(t)=I_\varepsilon(t)v_{0\varepsilon}+J_\varepsilon(t)v_{1\varepsilon}-\frac{1}{\varepsilon}\int_0^t J_\varepsilon(t-s)f(v_\varepsilon(s))ds, \tag{3.1}$$

where

$$I_\varepsilon(t)=e^{it/2\varepsilon}\left(\cos tA_\varepsilon-\frac{i}{2\varepsilon}A_\varepsilon^{-1}\sin tA_\varepsilon\right),$$

$$J_\varepsilon(t)=e^{it/2\varepsilon}A_\varepsilon^{-1}\sin tA_\varepsilon,$$

$$A_\varepsilon=\frac{1}{2\varepsilon}(1-4\varepsilon\Delta)^{1/2}.$$

We investigate the operator $J_\varepsilon(t)$.

Proposition 2. *For any interval $I\subset R$ with $0\in\bar{I}$, $u\in C_0(I\times R^n)$ and pair (q_i,r_i), $i=1,2$, such that*

$$\frac{2}{r_i}=\frac{n}{2}-\frac{n}{q_i},\qquad \frac{1}{2}-\frac{1}{n+2}\leq\frac{1}{q_i}\leq\frac{1}{2}, \tag{3.2}$$

the following estimate holds:

$$\left\| \int_0^t \frac{1}{\varepsilon} J_\varepsilon(t-s)u(s)ds \right\|_{L^{r_1}(I;L^{q_1})} \leq c\|u\|_{L^{r_2'}(I;L^{q_2'})}, \tag{3.3}$$

where c is independent of u, I, and ε.

Proof of Proposition 2.

We introduce the results on decay of solution of Klein–Gordon equation. (see [9]) For any $1 < q' \leq 2 \leq q < \infty$, the following inequality holds:

$$\|(I-\Delta)^{-1/2}\sin((I-\Delta)^{1/2}t)u\|_{L^q(R^n)} \leq ct^{-n(1/2-1/q)}\|u\|_{H^{n/2-(n+2)/q,q'}(R^n)}. \tag{3.4}$$

We define

$$\mathfrak{J}_\alpha(t) = (\alpha I - \Delta)^{-1/2}\sin t(\alpha I - \Delta)^{1/2}.$$

We consider the mapping property of $J_\varepsilon(t)$ on the basis of the identity

$$J_\varepsilon(t) = e^{it/2\varepsilon}\varepsilon^{1/2}\mathfrak{J}_{1/4\varepsilon}(\varepsilon^{-1/2}t).$$

For $\beta > 0$, we define $(U_\beta f)(x) = f(\beta x)$ and we use the facts that $U_\beta^{-1} = U_{1/\beta}$, that $\beta^{n/p}U_\beta$ is an isometry on L^p and that

$$\mathfrak{J}_\alpha(t) = \alpha^{-1/2}U_{\alpha^{1/2}}\mathfrak{J}_1(\alpha^{1/2}t)U_{\alpha^{1/2}}^{-1}.$$

Therefore we have

$$\frac{1}{\varepsilon}J_\varepsilon(t) = 2e^{it/2\varepsilon}U_{(1/4\varepsilon)^{1/2}}\mathfrak{J}_1\left(\frac{t}{2\varepsilon}\right)U_{(1/4\varepsilon)^{1/2}}^{-1}. \tag{3.5}$$

From this identity and (3.4), we obtain,

$$\begin{aligned}
\left\|\frac{1}{\varepsilon}J_\varepsilon(t)u\right\|_{L^q} &= c\left\|U_{(1/4\varepsilon)^{1/2}}\mathfrak{J}_1\left(\frac{t}{2\varepsilon}\right)U_{(1/4\varepsilon)^{1/2}}^{-1}u\right\|_{L^q} \\
&= c\varepsilon^{n/2q}\left\|\mathfrak{J}_1\left(\frac{t}{2\varepsilon}\right)U_{(1/4\varepsilon)^{1/2}}^{-1}u\right\|_{L^q} \\
&\leq c\varepsilon^{n/2q}\left(\frac{t}{2\varepsilon}\right)^{-n(1/2-1/q)}\|U_{(1/4\varepsilon)^{1/2}}^{-1}u\|_{H^{n/2-(n+2)/q,q'}} \\
&= ct^{-n(1/2-1/q)}\|(1-4\varepsilon\Delta)^{1/2(n/2-(n+2)/q)}u\|_{L^{q'}}.
\end{aligned} \tag{3.6}$$

Thus

$$\begin{aligned}
&\left\|\int_0^t \frac{1}{\varepsilon}J_\varepsilon(t-s)u(s)ds\right\|_{L^q} \\
&\quad\leq c\int_0^t |t-s|^{-n(1/2-1/q)}\|(1-4\varepsilon\Delta)^{1/2(n/2-(n+2)/q)}u\|_{L^{q'}}ds.
\end{aligned} \tag{3.7}$$

The Hardy-Littlewood-Sobolev inequality in time implies

$$\left\| \int_0^t \frac{1}{\varepsilon} J_\varepsilon(t-s)u(s)ds \right\|_{L_t^r L_x^q} \le c\|(1-4\varepsilon\Delta)^{1/2(n/2-(n+2)/q)}u\|_{L_t^{r'} L_x^{q'}}, \tag{3.8}$$

with

$$\frac{2}{r} = \frac{n}{2} - \frac{n}{q}.$$

There exists $c > 0$ which is independent of ε such that, for any $\theta \ge 0$, $1 < p < \infty$,

$$\|(1-4\varepsilon\Delta)^{-\theta}u\|_{L^p} \le c\|u\|_{L^p}. \tag{3.9}$$

So we have

$$\left\| \int_0^t \frac{1}{\varepsilon} J_\varepsilon(t-s)u(s)ds \right\|_{L_t^r L_x^q} \le c\|u\|_{L_t^{r'} L_x^{q'}}, \tag{3.10}$$

if

$$\frac{n}{2} - \frac{n+2}{q} \le 0.$$

Then we obtain desired estimate from duality argument and interpolation with the energy estimate.

4. Proof of the main theorem

At first, the assumptions (2.1), (2.2) ensure that there exists a unique solution v_ε of (1.2) such that $v_\varepsilon \in L^\infty(0,T;H^1)$, $v_\varepsilon' \in L^\infty(0,T;L^2)$ (see, e.g. [6]), and the equation (1.3) has a unique solution $v_0 \in L^\infty(0,T;H^1)$ (see, e.g. [5]).

From the energy conservation for (1.2) and the assumption (2.3), we obtain

$$\sup_{\varepsilon>0} \|v_\varepsilon\|_{L^\infty(0,T;H^1)} < \infty. \tag{4.1}$$

From the conservation laws of energy and charge for (1.3), we obtain

$$\|v_0\|_{L^\infty(0,T;H^1)} < \infty. \tag{4.2}$$

The Sobolev embedding theorem allows us that

$$\sup_{\varepsilon\ge 0} \|v_\varepsilon\|_{L^\infty(0,T;L^s)} < \infty, \tag{4.3}$$

for any s such that

$$\begin{aligned} &2 \le s \le 6 && \text{for } n = 3, \\ &2 \le s < \infty && \text{for } n = 2, \\ &2 \le s \le \infty && \text{for } n = 1. \end{aligned}$$

The solution v_0 of (1.3) satisfies

$$v_0(t) = I_0(t)v_{00} - i\int_0^t I_0(t-s)f(v_0(s))ds \tag{4.4}$$

with

$$I_0(t) = e^{i\Delta t}.$$

Proof of Theorem 1.
To study $v_\varepsilon(t) - v_0(t)$, we divide it into 5 parts:

$$v_\varepsilon(t) - v_0(t) = \sum_{i=1}^{5} l_\varepsilon^{(i)}(t) \tag{4.5}$$

with

$$l_\varepsilon^{(1)}(t) = (I_\varepsilon(t) - I_0(t))v_{00}, \tag{4.6}$$

$$l_\varepsilon^{(2)}(t) = I_\varepsilon(t)(v_{0\varepsilon} - v_{00}), \tag{4.7}$$

$$l_\varepsilon^{(3)}(t) = J_\varepsilon(t)v_{1\varepsilon}, \tag{4.8}$$

$$l_\varepsilon^{(4)}(t) = \int_0^t \left(iI_0(t-s) - \frac{1}{\varepsilon}J_\varepsilon(t-s)\right)f(v_0(s))ds, \tag{4.9}$$

$$l_\varepsilon^{(5)}(t) = \frac{1}{\varepsilon}\int_0^t J_\varepsilon(t-s)(f(v_0(s)) - f(v_\varepsilon(s)))ds. \tag{4.10}$$

This is the same decomposition as in Najman's paper [8]. The following results have been shown in it.

$$\lim_{\varepsilon\to 0} \|l_\varepsilon^{(i)}\|_{L_t^\infty L_x^q} = 0, \qquad i = 1, 2, 3, \tag{4.11}$$

where

$$\begin{aligned} 2 \le q < 6, &\qquad n = 3, \\ q = 2, &\qquad n = 1, 2. \end{aligned} \tag{4.12}$$

With respect to $l_\varepsilon^{(4)}$, we use the property of the solutions of the nonlinear Schrödinger equation in order to extend admissible range of nonlinear power γ. In the case of $n = 3$, we have for any $1 < \gamma < 5$, see [5]

$$v_0 \in L_t^r W_x^{1,q}, \qquad \frac{1}{r} = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right), \qquad 2 \le q < 6.$$

The embedding theorem implies

$$v_0 \in L_t^r L_x^p, \qquad \frac{1}{q} - \frac{1}{3} < \frac{1}{p} \le \frac{1}{q}.$$

From the condition $1 < 4/(\gamma - 1)$, we obtain

$$\|\nabla f(v_0)\|_{L_t^1 L_x^2} \le \|\nabla f(v_0)\|_{L_t^{4/(\gamma-1)} L_x^2}$$

From $|\nabla f(v_0)| \le c|v_0|^{\gamma-1}|\nabla v_0|$ and the Hölder inequality, we have

$$\|\nabla f(v_0)\|_{L_t^{4/(\gamma-1)} L_x^2} \le c\|v_0\|_{L_t^{r_1} L_x^{p_1}}^{\gamma-1} \|\nabla v_0\|_{L_t^{r_2} L_x^{q_2}},$$

with

$$\frac{1}{2} = \frac{\gamma - 1}{p_1} + \frac{1}{q_2}, \qquad \frac{\gamma - 1}{4} = \frac{\gamma - 1}{r_1} + \frac{1}{r_2}.$$

For each $1 < \gamma < 5$, we set $1/p_1 = 1/12$, $1/q_2 = 1/2 - (\gamma - 1)/12$, $1/r_1 = 1/8$, $1/r_2 = (\gamma - 1)/8$, and we have

$$\nabla f(v_0) \in L_t^1 L_x^2.$$

In the case of $n = 1, 2$, it is obvious that

$$f(v_0) \in L_t^1 L_x^2, \qquad 1 < \gamma < \infty.$$

Then we can apply the Theorem 1 (b) in [8], we have for q satisfying (4.12),

$$\lim_{\varepsilon \to 0} \|l_\varepsilon^{(4)}\|_{L_t^\infty L_x^q} = 0. \tag{4.13}$$

So we have to estimate only $l_\varepsilon^{(5)}$. For simplicity we denote

$$F(s) = f(v_0(s)) - f(v_\varepsilon(s)). \tag{4.14}$$

For the case of $n = 3$, we define the new norm with $0 < \delta \le 2/5$, $0 < t \le T$

$$|\!|\!| \cdot |\!|\!|_{0,t} \equiv \| \cdot \|_{L^\infty(0,t;L^2)} + \| \cdot \|_{L^{4/3\delta}(0,t;L^{2/(1-\delta)})}. \tag{4.15}$$

Since $2 < 2/(1 - \delta) < 6$, we have from (4.11) and (4.13)

$$\lim_{\varepsilon \to 0} |\!|\!| l_\varepsilon^{(i)} |\!|\!|_{0,t} = 0, \qquad i = 1, 2, 3, 4. \tag{4.16}$$

From Proposition 2, we have

$$|||l_\varepsilon^{(5)}|||_{0,t} \le C\|F\|_{L_t^{r_2'}L_x^{q_2'}}.$$

Considering (3.2) we choose $q_2' = 2/(1+\delta)$, $r_2' = 4/(4-3\delta)$. The Hölder inequality implies

$$\|F\|_{L_t^{4/(4-3\delta)}L_x^{2/(1+\delta)}} \le C\lambda\|v_\varepsilon - v_0\|_{L_t^{4/3\delta}L_x^{2/(1-\delta)}}\big(\|v_\varepsilon\|_{L_t^{2(\gamma-1)/(2-3\delta)}L_x^{(\gamma-1)/\delta}}$$
$$+ \|v_0\|_{L_t^{2(\gamma-1)/(2-3\delta)}L_x^{(\gamma-1)/\delta}}\big)^{\gamma-1}.$$

We take δ such that $2 \le (\gamma-1)/\delta \le 6$ for each $1 < \gamma \le 17/5$, and then

$$\|v_\varepsilon\|_{L_t^{2(\gamma-1)/(2-3\delta)}L_x^{(\gamma-1)/\delta}} + \|v_0\|_{L_t^{2(\gamma-1)/(2-3\delta)}L_x^{(\gamma-1)/\delta}}$$
$$\le t^{(2-3\delta)/2(\gamma-1)}\|v_\varepsilon\|_{L_t^\infty L_x^{(\gamma-1)/\delta}} + t^{(2-3\delta)/2(\gamma-1)}\|v_0\|_{L_t^\infty L^{(\gamma-1)/\delta}}$$
$$\le Ct^{(2-3\delta)/2(\gamma-1)}.$$

Therefore we obtain

$$|||l_\varepsilon^{(5)}|||_{0,t} \le Ct^{(2-3\delta)/2}\|v_\varepsilon - v_0\|_{L_t^{4/3\delta}L_x^{2/(1-\delta)}} \tag{4.17}$$
$$\le Ct^{(2-3\delta)/2}|||v_\varepsilon - v_0|||_{0,t}.$$

From this and (4.16), we have

$$|||v_\varepsilon - v_0|||_{0,t} \le c_\varepsilon + Ct^{(2-3\delta)/2}|||v_\varepsilon - v_0|||_{0,t},$$

where

$$\lim_{\varepsilon\to 0} c_\varepsilon = 0.$$

This implies that for sufficiently small T_0

$$\lim_{\varepsilon\to 0}|||v_\varepsilon - v_0|||_{0,T_0} = 0.$$

Previous estimate can be repeated on the time interval $[T_0, 2T_0]$. So we have

$$\lim_{\varepsilon\to 0}|||v_\varepsilon - v_0|||_{T_0,2T_0} = 0.$$

Repeating this procedure, we obtain eventually

$$\lim_{\varepsilon\to 0}|||v_\varepsilon - v_0|||_{0,T} = 0. \tag{4.18}$$

Thus we have with $1 < \gamma \le 17/5$

$$\lim_{\varepsilon\to 0}\|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} = 0. \tag{4.19}$$

This completes the proof for the case $n = 3$.

We consider the case $n = 1, 2$. We utilize Proposition 2 again to obtain

$$\|l_\varepsilon^{(5)}\|_{L_t^\infty L_x^2} \le C\|F\|_{L_t^{r'} L_x^{q'}}.$$

The Hölder inequality with

$$\frac{1}{q'} = \frac{1}{2} + \frac{\gamma - 1}{s} \tag{4.20}$$

implies

$$\begin{aligned}\|l_\varepsilon^{(5)}\|_{L^\infty(0,t;L^2)} &\le C\| \, |v_\varepsilon|^{\gamma-1} v_\varepsilon - |v_0|^{\gamma-1} v_0\|_{L^{r'}(0,t;L^{q'})} \\ &\le C\|v_\varepsilon - v_0\|_{L^\infty(0,t;L^2)} (\|v_\varepsilon\|_{L^{(\gamma-1)r'}(0,t;L^s)} + \|v_0\|_{L^{(\gamma-1)r'}(0,t;L^s)})^{\gamma-1} \\ &\le Ct^{1/r'} \|v_\varepsilon - v_0\|_{L^\infty(0,t;L^2)}.\end{aligned}$$

The argument from (4.17) to (4.18) gives

$$\lim_{\varepsilon \to 0} \|v_\varepsilon - v_0\|_{L^\infty(0,T;L^2)} = 0. \tag{4.21}$$

Considering (4.3) and (4.20), we are allowed

$$1 < \gamma < \infty.$$

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References

[1] Brezis, H., Gallouet, T., Nonlinear Schrödinger evolution equations, Nonlinear Analysis, Theory, Methods, Application, **4** (1980), 677–681.

[2] Fattorini, H. O., *Second Order Linear Differential Equations in Banach Spaces*, North–Holland, Amsterdam, 1985.

[3] Ginibre, J., Velo, G., Time decay of finite energy solutions of the non linear Klein–Gordon and Schrödinger equations, Ann. Inst. Henri Poincaré, Physique théorique, **43**, n4 (1985), 399–442.

[4] Ginibre, J., Velo, G., Smoothing properties and retarded estimates for some dispersive evolution equations, Commun. Math. Phys., **144** (1992), 163–188.

[5] Kato, T., On nonlinear Schrödinger equations, Ann. Inst. H. Poincaré, Physique théorique, **46** (1987), 113–129.

[6] Lions, J. L., *Quelques Méthodes de Résolution des Problémes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[7] Marshall, B., Strauss, W., Wainger, S., L^p–L^q estimates for the Klein–Gordon equation, J. Math. pures Appl., **59** (1980), 417–440.
[8] Najman, B., The nonrelativistic limit of the nonlinear Klein–Gordon equation, Nonlinear Analysis, Theory, Methods, Application, Vol. 15, **3** (1990), 217–228.
[9] Pecher, H., Nonlinear small data scattering for the wave and Klein–Gordon equation, Mathematische Zeitschrift, **185** (1984), 261–270.
[10] Schiff, L. I., Nonlinear meson theory of nuclear forces, Phys. Rev., **84** (1951), 1–9.
[11] Tsutsumi, M., Nonrelativistic approximation of nonlinear Klein–Gordon equation in two space dimension, Nonlinear Analysis, Theory, Methods, Application, **8** (1984), 637–643.
[12] Yajima, K., Existence of solutions for Schrödinger evolution equations, Commun. Math. Phys., **110** (1987), 415–426.

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