On A Discrete-Time Collocation Method for the Nonlinear Schrödinger Equation with Wave Operator

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We consider a discrete-time orthogonal spline collocation scheme for solving Schrödinger equation with wave operator. The scheme is proposed recently by Wang et al. (J Comput Appl Math 235 (2011), 1993–2005) and is showed to have high-order convergence rate when a parameter θ in the scheme is not less than $\frac{1}{4}$. In this article, we show that the result can be extended to include $\theta \in (0, \frac{1}{4})$ under an assumption. Numerical example is given to justify the theoretical result. Numer Methods Partial Differential Eq 29: 693–705, 2013

Keywords: conserved quantity; nonlinear Schrödinger equation; orthogonal spline collocation method; wave operator

I. INTRODUCTION

The nonlinear Schrödinger equation with wave operator is introduced in Ref. [1]. In this article, we consider the corresponding initial boundary value problem:

$$\begin{gathered} u_{tt} - u_{xx} + \mathbf{i}\alpha u_t + \beta(x)q(|u^2|)u = 0, \;\; -x_L < x < x_R, \;\; 0 < t \le T, \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), \;\; x_L \le x \le x_R, \\ u|_{x_L} = u|_{x_R} = 0, \;\; 0 \le t \le T, \end{gathered} \tag{1}$$

where $u(x,t)$ is an unknown complex function, $\mathbf{i}^2 = -1$ and α is a real constant. In this article, we assume that $\beta(x) \ge 0$ and $q(x)$ is a real function such that $Q(s) = \int_0^s q(\eta)d\eta$ is non-negative for $s \in [0, \infty)$.

Finding approximate solutions of (I) has attracted quite a lot of attentions in recent years [2–7]. In Ref. [2], a nonconservative finite difference scheme is proposed to solve the problem.

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The accuracy is further improved by considering conservative schemes [5–7]. Very recently, the discrete-time orthogonal spline collocation (OSC) method is proposed to solve the problem [4]. This article is closely related to the result in Ref. [4].

In the implementation of the OSC method, the solution is approximated by piecewise polynomials. The coefficients of the polynomial are determined by requiring that the equation holds at Gauss points that are the nodes of the Gauss–Legendre quadrature. Interested reader may refer to Ref. [8] for more details on the OSC method. The OSC method has been shown to be an efficient method for solving ordinary differential equations [9], linear hyperbolic problems [10], and nonlinear parabolic problems [11]. Recently, the method has been applied to Schrödinger-type equations [12–14].

As mentioned above, Wang et al. [4] has used the OSC method to study (I). They prove that when the parameter θ in the proposed scheme (see Section III) is not less than $\frac{1}{4}$ the approximate solution tends to the exact solution with fast rate. The main contribution of this article is to show that, by considering the inverse estimate (Lemma 4) and some delicate analysis, this restriction on θ can be relaxed.

This article is organized as follows. In Section II, the method is reviewed and some lemmas are introduced. The main result is presented in Section III. In the last section, the theoretical analysis is justified by numerical experiment.

II. PRELIMINARIES

We first give a review of some notations for the collocation method. One may refer to Refs. [8, 10–12] for more details. Given a partition

$$\Lambda : x_L = x_0 < x_1 < \cdots < x_{N-1} < x_N = x_R$$

of $\bar{I} = [x_L, x_R]$, let $h_j = x_j - x_{j-1}$, $j = 1, 2, \ldots, N$ and $h = \max_j h_j$. In this article, we assume that the partition is quasiuniform, which means that

$$h_j \geq \rho h$$

for some positive constant ρ.

Let $\{t_n\}_{n=0}^{J}$ be a partition of $[0, T]$, where $t_n = n\tau$ and $\tau = T/J$. In the remaining of this article, we use C to denote a generic positive constant, which may be different from line to line.

In this article, we consider an integer $r \geq 3$ and denote $\mathcal{M}^0(\Lambda)$ the space of piecewise Hermite on I defined by

$$\mathcal{M}^0(\Lambda) = \{v \in C^1(\bar{I}) : v|_{[x_{j-1},x_j]} \in \mathcal{P}_r\} \cap \{v(x_L) = v(x_R) = 0\},$$

where $\mathcal{P}_r$ denotes the set of all polynomials of degree at most r. We use $\mathcal{R}(\mathcal{M}^0(\Lambda))$ to denotes the set of all real-valued functions in $\mathcal{M}^0(\Lambda)$.

To apply the collocation method, we introduce the Gauss points $\mathcal{G} = \{\xi_{j,k}\}_{j,k=1}^{N,r-1}$ taken as

$$\xi_{j,k} = \frac{1}{2}(x_{j-1} + x_j) + \frac{h_j}{2}\zeta_k, \tag{2}$$

where $\{\zeta\}_{k=1}^{r-1}$ denote the nodes for the $(r-1)$ Gaussian quadrature rule on the interval $[-1,1]$ with the corresponding weights $\{w_k\}_{k=1}^{r-1}$, $w_k > 0$. For two functions μ, ν defined on $\mathcal{G}$, let $(\mu,\nu)_{\mathcal{G}}$ and $\|\mu\|_{\mathcal{G}}$ be given by

$$(\mu,\nu)_{\mathcal{G}} = \sum_{j=1}^{N} h_j \sum_{k=1}^{r-1} w_k(\mu\bar{\nu})(\xi_{j,k}),$$

and

$$\|\mu\|_{\mathcal{G}} = (\mu,\mu)_{\mathcal{G}}^{1/2}.$$

We use $W_p^l(I)$ to denote the general Sobolev space. For $p=2$, we denote $W_2^l(I) = H^l(I)$, in this case, we have

$$\|\mu\|_{H^l(I)} = \left(\sum_{0\le j\le l} \left\|\frac{d^j\mu}{dx^j}\right\|_{L^2(I)}^2\right)^{1/2}.$$

We state some lemmas that are needed in the analysis of this article:

Lemma 1. ([4, 12]) *For $\mu \in \mathcal{R}(\mathcal{M}^0(\Lambda))$, there exist positive constants α_1 and α_2 such that*

$$\alpha_1\|\mu\|_{\mathcal{G}} \le \|\mu\|_{L^2(I)} \le \alpha_2\|\mu\|_{\mathcal{G}}.$$

Lemma 2. ([12]) *For $\mu,\nu \in \mathcal{R}(\mathcal{M}^0(\Lambda))$, we have*

$$(\mu_{xx},\nu)_{\mathcal{G}} = (\mu,\nu_{xx})_{\mathcal{G}},$$

$$-(\mu_{xx},\nu)_{\mathcal{G}} = \int_{x_L}^{x_R} \mu_x\nu_x dx + \eta_r\sum_{j=1}^{N}\mu_j^{(r)}\nu_j^{(r)}h_j^{2r-1},$$

where $\mu_j^{(r)}$, $\nu_j^{(r)}$ are the constant values of the rth derivative of μ, ν on $I_j = [x_j, x_{j-1}]$ and η_r is a positive constant depending on r only.

Lemma 3. ([12]) *Let $u \in \mathcal{R}(\mathcal{M}^0(\Lambda))$ such that $u \in H^{r+3}(I_j)$ for $j=1,\dots,N$, and suppose that $V \in \mathcal{R}(\mathcal{M}^0(\Lambda))$ satisfies*

$$(u_{xx}-V_{xx})(\xi_{j,k}) - (u-V)(\xi_{j,k}) = 0, \quad k=1,\dots,r-1.$$

Then,

$$\|u-V\|_{L^\infty(I)} \le Ch^{r+1}\|u\|_{H^{r+1}(I)}.$$

Here, we remark that Lemma 3 still holds for $u \in \mathcal{M}^0(\Lambda)$ by considering the real and imaginary part of u correspondingly.

The following inverse estimate plays an important role in our study. A special case has been used in Ref. [12]. We present the general case here.

Lemma 4. ([15, 16]) *Let $I_j = [x_{j-1}, x_j]$, $\rho h \le x_j - x_{j-1} \le h$ and $\mathcal{P}$ be a finite dimensional subspace of $W_p^l(I_j) \cap W_q^m(I_j)$, where $0 \le m \le l$. Then, there exists a constant C such that for all $v \in \mathcal{P}$, we have*

$$\|v\|_{W_p^l(I_j)} \le Ch^{m-l+1/p-1/q}\|v\|_{W_q^m(I_j)}.$$

III. MAIN RESULT

In this article, we consider the approximate solution $u_h^n \in \mathcal{M}^0(\Lambda)$ of (I) given by the discrete-time OSC scheme, which is introduced in Ref. [4]:

$$\left\{\frac{1}{\tau^2}\left(u_h^{n+1}-2u_h^n+u_h^{n-1}\right)-(1-2\theta)\left(u_h^n\right)_{xx}-\theta\left[\left(u_h^{n+1}\right)_{xx}+\left(u_h^{n-1}\right)_{xx}\right]\right.$$
$$\left.+\frac{\mathbf{i}\alpha}{2\tau}\left(u_h^{n+1}-u_h^{n-1}\right)+\beta\frac{Q\left(\left|u_h^{n+1}\right|^2\right)-Q\left(\left|u_h^{n-1}\right|^2\right)}{\left|u_h^{n+1}\right|^2-\left|u_h^{n-1}\right|^2}\frac{u_h^{n+1}+u_h^{n-1}}{2}\right\}(\xi_{j,k})=0, \tag{3}$$

where u_h^0, is prescribed by approximating $u_0(x)$. For u_h^1, under the assumption that the solution of (I) is sufficiently smooth, Taylor's theorem gives $u(x,\tau)=z(x)+O(\tau^3)$, where

$$z(x)=u_0(x)+\tau u_1(x)+\frac{\tau^2}{2}\left(\frac{\partial^2 u_0}{\partial x^2}-\alpha\mathbf{i}u_1-\beta q(|u_0|^2)u_0\right)(x).$$

Consequently, we can prescribe u_h^1 by approximating $z(x)$ with Hermite piecewise interpolations [4]. In this article, we assume that the partitions in the x and t directions satisfy the relation $\tau=\gamma h^{1+\delta}$, where $\delta>0$, which can be arbitrarily small and γ is positive.

We remark that in our scheme, the t derivatives are approximated by finite differences. However, when considering the approximate solution u_h^n, the notation $(u_h^n)_t$ will be used throughout the article to denote $\frac{u_h^{n+1}-u_h^n}{\tau}$ when there is no confusion from the context.

The following property for the approximate solution is obtained in Ref. [4].

Lemma 5. *Consider the solution u_h^n of (3). For $n\le J-1$, one has the discrete conservation law:*

$$\begin{aligned}E^n=&\left\|\left(u_h^n\right)_t\right\|_{\mathcal{G}}^2-(1-2\theta)\left[\left(\left(u_{h,1}^{n+1}\right)_{xx},u_{h,1}^n\right)_{\mathcal{G}}+\left(\left(u_{h,2}^{n+1}\right)_{xx},u_{h,2}^n\right)_{\mathcal{G}}\right]\\&-\theta\left[\left(\left(u_{h,1}^{n+1}\right)_{xx},u_{h,1}^{n+1}\right)_{\mathcal{G}}+\left(\left(u_{h,2}^{n+1}\right)_{xx},u_{h,2}^{n+1}\right)_{\mathcal{G}}+\left(\left(u_{h,1}^{n}\right)_{xx},u_{h,1}^{n}\right)_{\mathcal{G}}+\left(\left(u_{h,2}^{n}\right)_{xx},u_{h,2}^{n}\right)_{\mathcal{G}}\right]\\&+\frac{1}{2}\sum_{j=1}^{N}h_j\sum_{k=1}^{r-1}w_k\beta(\xi_{j,k})\left[Q\left(\left|u_h^{n+1}(\xi_{j,k})\right|^2\right)+Q\left(\left|u_h^{n}(\xi_{j,k})\right|^2\right)\right]\\=&E^{n-1}=\cdots=E^0=const,\end{aligned}\tag{4}$$

where $u_{h,1}^{n+1}$, $u_{h,2}^{n+1}$ denote the real and imaginary part of u_h^{n+1}, respectively.

To estimate the error between the exact solution and the approximate one, we need to have an uniform bound for the approximate solution. This is given in the following lemma.

Lemma 6. *Suppose that $\theta\in(0,\frac{1}{4})$, $\tau=\gamma h^{1+\delta}$, and $u_h^n\in\mathcal{M}^0(\Lambda)$ is the solution of (3). There exists a constant C such that $|u_h^n|<C$ for all h and n, provided that the initial condition admit a solution $u(x,t)\in C^{2,4}\cap L^2(H^{r+3})$.*

Proof. Notice that, all the terms of E^n in (4) are non-negative except the terms

$$-(1-2\theta)\left(\left(u_{h,1}^{n+1}\right)_{xx},u_{h,1}^n\right)_{\mathcal{G}}\quad\text{and}\quad-(1-2\theta)\left(\left(u_{h,2}^{n+1}\right)_{xx},u_{h,2}^n\right)_{\mathcal{G}}.$$

As they can be treated similarly, we first estimate $-(1-2\theta)((u_{h,1}^{n+1})_{xx}, u_{h,1}^{n})_{\mathcal{G}}$, which can be written as

$$
\begin{aligned}
&-\frac{1-2\theta}{2}\left[\left(\left(u_{h,1}^{n+1}\right)_{xx}, u_{h,1}^{n}\right)_{\mathcal{G}} + \left(\left(u_{h,1}^{n}\right)_{xx}, u_{h,1}^{n+1}\right)_{\mathcal{G}}\right] \\
&= -\frac{1-2\theta}{2}\left[\sum_{j=1}^{N} h_j \sum_{k=1}^{r-1} w_k\left(\left(u_{h,1}^{n+1}\right)_{xx}\cdot u_{h,1}^{n}\right)(\xi_{j,k}) + \sum_{j=1}^{N} h_j \sum_{k=1}^{r-1} w_k\left(\left(u_{h,1}^{n}\right)_{xx}\cdot u_{h,1}^{n+1}\right)(\xi_{j,k})\right] \\
&= -\frac{1-2\theta}{2}\sum_{j=1}^{N} h_j \sum_{k=1}^{r-1}\frac{w_k}{2}\left[\left(u_{h,1}^{n+1}+u_{h,1}^{n}\right)_{xx}\cdot\left(u_{h,1}^{n+1}+u_{h,1}^{n}\right) - \left(u_{h,1}^{n+1}-u_{h,1}^{n}\right)_{xx}\cdot\left(u_{h,1}^{n+1}-u_{h,1}^{n}\right)\right](\xi_{j,k}) \\
&= -\frac{1-2\theta}{4}\left(\left(u_{h,1}^{n+1}+u_{h,1}^{n}\right)_{xx}, u_{h,1}^{n+1}+u_{h,1}^{n}\right)_{\mathcal{G}} + \frac{1-2\theta}{4}\left(\left(u_{h,1}^{n+1}-u_{h,1}^{n}\right)_{xx}, u_{h,1}^{n+1}-u_{h,1}^{n}\right)_{\mathcal{G}}. \qquad (5)
\end{aligned}
$$

By Lemma 2, the first term in the last equality is positive and we need to estimate the second term.

As we assume that $h_j \geq \rho h$, applying Lemma 4 with $p=\infty$, $q=2$, and $m=0$ gives

$$
\left\|\frac{d^l\left(u_{h,1}^{n+1}-u_{h,1}^{n}\right)}{dx^l}\right\|_{L^\infty(I_j)} = \tau\left\|\frac{d^l\left(u_{h,1}^{n}\right)_t}{dx^l}\right\|_{L^\infty(I_j)} \leq \tau\left\|\left(u_{h,1}^{n}\right)_t\right\|_{W_\infty^l(I_j)} \leq C\tau h_j^{-l-\frac{1}{2}}\left\|\left(u_{h,1}^{n}\right)_t\right\|_{L^2(I_j)}. \qquad (6)
$$

By (6), Lemma 1, Lemma 2 and the fact that $\sum_{j=1}^{N}\|\cdot\|_{L^2(I_j)}^2 = \|\cdot\|^2$, one then has

$$
\begin{aligned}
&\left(\left(u_{h,1}^{n+1}-u_{h,1}^{n}\right)_{xx}, u_{h,1}^{n+1}-u_{h,1}^{n}\right)_{\mathcal{G}} \\
&= -\left\|\left(u_{h,1}^{n+1}-u_{h,1}^{n}\right)_x\right\|^2 - \eta_r\sum_{j=1}^{N}\left\|\frac{d^r\left(u_{h,1}^{n+1}-u_{h,1}^{n}\right)}{dx^r}\right\|_{L^\infty(I_j)}^2 h_j^{2r-1} \\
&\geq -\left\|\left(u_{h,1}^{n+1}-u_{h,1}^{n}\right)_x\right\|^2 - C\eta_r\sum_{j=1}^{N}\tau^2 h_j^{-2r-1}\left\|\left(u_{h,1}^{n}\right)_t\right\|_{L^2(I_j)}^2 h_j^{2r-1} \\
&\geq -\alpha_2^2\sum_{j=1}^{N} h_j\sum_{k=1}^{r-1} w_k\left(u_{h,1}^{n+1}-u_{h,1}^{n}\right)_x^2(\xi_{j,k}) - C\eta_r\tau^2(\rho h)^{-2}\left\|\left(u_{h,1}^{n}\right)_t\right\|^2 \\
&= -\alpha_2^2\tau^2\sum_{j=1}^{N} h_j\sum_{k=1}^{r-1} w_k\left(\left(u_{h,1}^{n}\right)_t\right)_x^2(\xi_{j,k}) - C\tau^2 h^{-2}\left\|\left(u_{h,1}^{n}\right)_t\right\|^2. \qquad (7)
\end{aligned}
$$

We first assume that r is odd. Notice that $\zeta_k = -\zeta_{r-k}$. For $k \leq \frac{r-1}{2}$, by taking

$$
\tilde{h}_{j,k} = \tilde{h}_{j,r-k} = h_j|\zeta_k|,
$$

and noticing (2), the Taylor expansion of $(u_{h,1}^{n})_t$ at $\xi_{j,k}$ with step size $\tilde{h}_{j,k}$ then gives (note that $d^r(u_{h,1}^{n})_t/dx^r$ is a constant on I_j):

$$
\tilde{h}_{j,k}\left(\left(u_{h,1}^{n}\right)_t\right)_x(\xi_{j,k}) = \left(\left(u_{h,1}^{n}\right)_t(\xi_{j,r-k}) - \left(u_{h,1}^{n}\right)_t(\xi_{j,k})\right) - \sum_{\ell=2}^{r}\frac{1}{\ell\,!}\frac{d^\ell\left(u_{h,1}^{n}\right)_t(\xi_{j,k})}{dx^\ell}\tilde{h}_{j,k}^{\ell},
$$

which, by Cauchy's inequality, yields

$$\begin{aligned}\left(\left(u_{h,1}^n\right)_t\right)_x^2(\xi_{j,k}) &= \left(\frac{1}{\tilde{h}_{j,k}}\left(\left(u_{h,1}^n\right)_t(\xi_{j,r-k}) - \left(u_{h,1}^n\right)_t(\xi_{j,k})\right) - \sum_{\ell=2}^{r}\frac{1}{\ell\,!}\frac{d^\ell\left(u_{h,1}^n\right)_t(\xi_{j,k})}{dx^\ell}\tilde{h}_{j,k}^{\ell-1}\right)^2\\ &\le (r+1)\left(\frac{\left(u_{h,1}^n\right)_t^2(\xi_{j,r-k})}{\tilde{h}_{j,k}^2} + \frac{\left(u_{h,1}^n\right)_t^2(\xi_{j,k})}{\tilde{h}_{j,k}^2} + \sum_{\ell=2}^{r}\left(\frac{1}{\ell\,!}\frac{d^\ell\left(u_{h,1}^n\right)_t(\xi_{j,k})}{dx^\ell}\tilde{h}_{j,k}^{\ell-1}\right)^2\right).\end{aligned}$$

Similarly, by considering the Taylor expansion of $(u_{h,1}^n)_t$ at $\xi_{j,r-k}$ with step size $-\tilde{h}_{j,r-k} = -\tilde{h}_{j,k}$, we get

$$\begin{aligned}\left(\left(u_{h,1}^n\right)_t\right)_x^2(\xi_{j,r-k}) &= \left(\frac{1}{\tilde{h}_{j,k}}\left(-\left(u_{h,1}^n\right)_t(\xi_{j,k}) + \left(u_{h,1}^n\right)_t(\xi_{j,r-k})\right) + \sum_{\ell=2}^{r}\frac{(-1)^\ell}{\ell\,!}\frac{d^\ell\left(u_{h,1}^n\right)_t(\xi_{j,r-k})}{dx^\ell}\tilde{h}_{j,k}^{\ell-1}\right)^2\\ &\le (r+1)\left(\frac{\left(u_{h,1}^n\right)_t^2(\xi_{j,r-k})}{\tilde{h}_{j,k}^2} + \frac{\left(u_{h,1}^n\right)_t^2(\xi_{j,k})}{\tilde{h}_{j,k}^2} + \sum_{\ell=2}^{r}\left(\frac{1}{\ell\,!}\frac{d^\ell\left(u_{h,1}^n\right)_t(\xi_{j,r-k})}{dx^\ell}\tilde{h}_{j,k}^{\ell-1}\right)^2\right).\end{aligned}$$

By (6) and (7), noticing that $w_k = w_{r-k}$, if we take $\zeta = \min_k \zeta_k$, we have

$$\begin{aligned}&\left(\left(u_{h,1}^{n+1} - u_{h,1}^n\right)_{xx}, u_{h,1}^{n+1} - u_{h,1}^n\right)_{\mathcal{G}}\\ &\ge -C\tau^2h^{-2}\left\|\left(u_{h,1}^n\right)_t\right\|^2 - \alpha_2^2\tau^2\sum_{j=1}^{N}h_j\sum_{k=1}^{\frac{r-1}{2}}w_k\left[\left(\left(u_{h,1}^n\right)_t\right)_x^2(\xi_{j,k}) + \left(\left(u_{h,1}^n\right)_t\right)_x^2(\xi_{j,r-k})\right]\\ &\ge -C\tau^2h^{-2}\left\|\left(u_{h,1}^n\right)_t\right\|^2 - 2(r+1)\alpha_2^2\tau^2\sum_{j=1}^{N}h_j\sum_{k=1}^{\frac{r-1}{2}}\frac{w_k}{\tilde{h}_{j,k}^2}\left(\left(u_{h,1}^n\right)_t^2(\xi_{j,k}) + \left(u_{h,1}^n\right)_t^2(\xi_{j,r-k})\right)\\ &\quad - 2(r+1)\alpha_2^2\tau^2\sum_{j=1}^{N}h_j\sum_{k=1}^{\frac{r-1}{2}}w_k\sum_{\ell=2}^{r}\left(\frac{1}{\ell\,!}\tau h_j^{-\ell-\frac{1}{2}}\left\|\left(u_{h,1}^n\right)_t\right\|_{L^2(I_j)}\tilde{h}_{j,k}^{\ell-1}\right)^2\\ &\ge -C\tau^2h^{-2}\left\|\left(u_{h,1}^n\right)_t\right\|^2 - \frac{2(r+1)\alpha_2^2\tau^2}{\rho^2\zeta^2h^2}\sum_{j=1}^{N}h_j\sum_{k=1}^{r}w_k\left(u_{h,1}^n\right)_t^2(\xi_{j,k})\\ &\quad - 2(r+1)\alpha_2^2\tau^2\sum_{j=1}^{N}h_j\sum_{k=1}^{\frac{r-1}{2}}w_k\sum_{\ell=2}^{r}\frac{1}{(\ell\,!)^2}\tau^2h_j^{-3}\left\|\left(u_{h,1}^n\right)_t\right\|_{L^2(I_j)}^2\\ &\ge -C\tau^2h^{-2}\left\|\left(u_{h,1}^n\right)_t\right\|^2 - C\frac{\tau^2}{h^2}\left\|\left(u_{h,1}^n\right)_t\right\|^2 - C\tau^2\frac{\tau^2}{h^2}\left\|\left(u_{h,1}^n\right)_t\right\|^2, \qquad (8)\end{aligned}$$

where, in the last line, we have used the fact $\sum_{j=1}^{N}\|\cdot\|_{L^2(I_j)}^2 = \|\cdot\|^2$.

For r being even, we can consider $((u^n_{h,1})_t)_x(\xi_{j,1}),\dots,((u^n_{h,1})_t)_x(\xi_{j,(r-2)/2})$, $((u^n_{h,1})_t)_x(\xi_{j,(r+2)/2}),\dots,((u^n_{h,1})_t)_x(\xi_{j,r-1})$ as in (8). For the term $((u^n_{h,1})_t)_x(\xi_{j,r/2})$, notice that

$$\begin{aligned}\big((u^n_{h,1})_t\big)_x(\xi_{j,r/2}) &= \frac{1}{\tilde{h}_{j,1}}\big(-\big(u^n_{h,1}\big)_t(\xi_{j,1})+\big(u^n_{h,1}\big)_t(\xi_{j,r-1})\big)\\ &\quad+\sum_{\ell=2}^{r}\frac{(-1)^\ell-1}{2\ell\,!}\frac{d^\ell\big(u^n_{h,1}\big)_t(\xi_{j,r/2})}{dx^\ell}(\tilde{h}_{j,1}/2)^{\ell-1}.\end{aligned}$$

With a slight modification of the above argument, by taking the generic constant larger, we can get the same bound as in (8).

Putting the estimate (8) back to (5), we have

$$-(1-2\theta)\big(\big(u^{n+1}_{h,1}\big)_{xx},u^n_{h,1}\big)_{\mathcal{G}} \ge -C\tau^2h^{-2}\big\|\big(u^n_{h,1}\big)_t\big\|^2 - C\frac{\tau^2}{h^2}\big\|\big(u^n_{h,1}\big)_t\big\|^2 - C\tau^2\frac{\tau^2}{h^2}\big\|\big(u^n_{h,1}\big)_t\big\|^2.$$

Similarly, we can show that

$$-(1-2\theta)\big(\big(u^{n+1}_{h,2}\big)_{xx},u^n_{h,2}\big)_{\mathcal{G}} \ge -C\tau^2h^{-2}\big\|\big(u^n_{h,2}\big)_t\big\|^2 - C\frac{\tau^2}{h^2}\big\|\big(u^n_{h,2}\big)_t\big\|^2 - C\tau^2\frac{\tau^2}{h^2}\big\|\big(u^n_{h,2}\big)_t\big\|^2.$$

By the assumption on u, one can get that E^0 is uniformly bounded. Note that $\tau=\gamma h^{1+\delta}$ implies τ^2h^{-2} can be arbitrarily small as h tends to zero. Therefore, the discrete conservation law (4) and these estimates (note that $\beta(x)\ge 0$) imply that, for h sufficiently small,

$$E^0=E^n\ge \big\|\big(u^n_h\big)_t\big\|^2_{\mathcal{G}}-(1-2\theta)\big[\big(\big(u^{n+1}_{h,1}\big)_{xx},u^n_{h,1}\big)_{\mathcal{G}}+\big(\big(u^{n+1}_{h,2}\big)_{xx},u^n_{h,2}\big)_{\mathcal{G}}\big]\ge\frac{\alpha_1^2}{2}\big\|\big(u^n_h\big)_t\big\|^2,$$

which gives an uniform bound for $\|(u^n_h)_t\|$.

On the other hand,

$$\big(u^{n+1}_h,u^{n+1}_h\big)^2_{\mathcal{G}}-\big(u^{n-1}_h,u^{n-1}_h\big)^2_{\mathcal{G}}=\big(u^{n+1}_h,u^{n+1}_h\big)^2_{\mathcal{G}}-\big(u^{n+1}_h,u^{n-1}_h\big)^2_{\mathcal{G}}+\big(u^{n+1}_h,u^{n-1}_h\big)^2_{\mathcal{G}}-\big(u^{n-1}_h,u^{n-1}_h\big)^2_{\mathcal{G}}.$$

By the elementary inequality

$$ab\le\epsilon a^2+\frac{1}{4\epsilon}b^2,\quad \text{for } \epsilon>0,$$

we have

$$\begin{aligned}\big(u^{n+1}_h,u^{n+1}_h\big)^2_{\mathcal{G}}-\big(u^{n+1}_h,u^{n-1}_h\big)^2_{\mathcal{G}}&=\sum_{j=1}^{N}h_j\sum_{k=1}^{r-1}w_k\big(u^{n+1}_h\cdot\big(u^{n+1}_h-u^n_h+u^n_h-u^{n-1}_h\big)\big)(\xi_{j,k})\\ &\le C\tau\big\|u^{n+1}_h\big\|^2_{\mathcal{G}}+\frac{C}{\tau}\big\|u^{n+1}_h-u^n_h\big\|^2_{\mathcal{G}}+\frac{C}{\tau}\big\|u^n_h-u^{n-1}_h\big\|^2_{\mathcal{G}}\\ &=C\tau\big(\big\|u^{n+1}_h\big\|^2_{\mathcal{G}}+\|(u^n_h)_t\|^2_{\mathcal{G}}+\big\|\big(u^{n-1}_h\big)_t\big\|^2_{\mathcal{G}}\big)\end{aligned}$$

and similarly,

$$\left(u_h^{n+1}, u_h^{n-1}\right)_{\mathcal{G}}^2 - \left(u_h^{n-1}, u_h^{n-1}\right)_{\mathcal{G}}^2 \leq C\tau\left(\left\|u_h^{n-1}\right\|_{\mathcal{G}}^2 + \left\|\left(u_h^n\right)_t\right\|_{\mathcal{G}}^2 + \left\|\left(u_h^{n-1}\right)_t\right\|_{\mathcal{G}}^2\right).$$

Thus,

$$\left\|u_h^{n+1}\right\|_{\mathcal{G}}^2 - \left\|u_h^{n-1}\right\|_{\mathcal{G}}^2 \leq C\tau\left(\left\|u_h^{n+1}\right\|_{\mathcal{G}}^2 + \left\|u_h^{n-1}\right\|_{\mathcal{G}}^2\right) + C\tau\left(\left\|\left(u_h^n\right)_t\right\|_{\mathcal{G}}^2 + \left\|\left(u_h^{n-1}\right)_t\right\|_{\mathcal{G}}^2\right).$$

Discrete Gronwall's inequality [17] then gives

$$\max_{1\leq n\leq J-1} \left\|u_h^{n+1}\right\|_{\mathcal{G}}^2 \leq \left(\left\|u_h^1\right\|_{\mathcal{G}}^2 + C\tau\sum_{n=1}^{J-1}\left(\left\|\left(u_h^n\right)_t\right\|_{\mathcal{G}}^2 + \left\|\left(u_h^{n-1}\right)_t\right\|_{\mathcal{G}}^2\right)\right)\exp(C(J-1)\tau).$$

The uniform bound of $\|(u_h^n)_t\|$ now yields an uniform bound of $\|u_h^n\|_{\mathcal{G}}^2$.

By Lemma 2 and the discrete conservation law (4) again, we have

$$E^0 = E^n \geq \frac{\alpha_1^2}{2}\left\|\left(u_h^n\right)_t\right\|^2 + \theta\left(\left\|\left(u_h^{n+1}\right)_x\right\|^2 + \left\|\left(u_h^n\right)_x\right\|^2\right) \geq \theta\left(\left\|\left(u_h^{n+1}\right)_x\right\|^2 + \left\|\left(u_h^n\right)_x\right\|^2\right).$$

As $\theta > 0$, one can thus get that $\|(u_h^n)_x\|^2$ is also bounded uniformly. Sobolev's theorem can then be applied to give an uniform bound for $|u_h^n|$. ■

Remark 1. In Ref. [4], the fact that u_h^n is uniform bounded is stated, without proof, as a consequence of continuity. We give a proof here not only for the completeness of our presentation but also for the sake that the technique in this proof constitute the main ingredient for the proof of our main result.

To estimate the error between the exact solution and the approximate one, we introduce the following function for comparison. For the solution u of (I), we define $W^n \in \mathcal{R}(\mathcal{M}^0(\Lambda))$ to satisfy

$$\left(u_{xx}^n - W_{xx}^n\right)(\xi_{j,k}) - (u^n - W^n)(\xi_{j,k}) = 0, \quad j = 1,2,\ldots,N, \;\; k = 1,2,\ldots,r-1, \tag{9}$$

where $u^n = u(x, n\tau)$.

We have the following main result of this article.

Theorem 1. *Suppose that $\theta \in (0, \frac{1}{4})$, $\beta(x) > 0$, $q(s) \in C^1$, $Q(x) > 0$, $u(x,t) \in C^{2,4} \cap L^2(H^{r+3})$ is the solution of (I), and $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2} \in L^2(H^{r+3})$, while $u_h^n \in \mathcal{M}^0(\Lambda)$ $(n = 0,1,\ldots,J)$ is the solution of (3) with $\tau = \gamma h^{1+\delta}$. If $W^n \in \mathcal{M}^0(\Lambda)$ is defined by (9), and $\|(u_h^0 - W^0)_t\|_{L^2(I)}^2$, $\|u_h^0 - W^0\|_{H^1(I)}^2$, and $\|u_h^1 - W^1\|_{H^1(I)}^2$ are bounded by Ch^{2r+2}, then for h sufficiently small, we have*

$$\max_{1\leq n\leq J} \left\|u^n - u_h^n\right\|_{L^\infty} \leq C(\tau^2 + h^{r+1}).$$

Proof. Let $\hat{e}^n = u^n - W^n, e^n = u_h^n - W^n$. With the uniform bound of $|u_h^n|$, following the proof of [4], one can get that

$$\begin{aligned}
&\|(e^n)_t\|_{\mathcal{G}}^2 - \|(e^{n-1})_t\|_{\mathcal{G}}^2 + \|e^{n+1}\|_{\mathcal{G}}^2 - \|e^{n-1}\|_{\mathcal{G}}^2 \\
&- (1-2\theta)\big[\big((e_1^{n+1})_{xx}, e_1^n\big)_{\mathcal{G}} + \big((e_2^{n+1})_{xx}, e_2^n\big)_{\mathcal{G}} - \big((e_1^n)_{xx}, e_1^{n-1}\big)_{\mathcal{G}} - \big((e_2^n)_{xx}, e_2^{n-1}\big)_{\mathcal{G}}\big] \\
&- \theta\big[\big((e_1^{n+1})_{xx}, e_1^{n+1}\big)_{\mathcal{G}} + \big((e_2^{n+1})_{xx}, e_2^{n+1}\big)_{\mathcal{G}} + \big((e_1^{n-1})_{xx}, e_1^{n-1}\big)_{\mathcal{G}} + \big((e_2^{n-1})_{xx}, e_2^{n-1}\big)_{\mathcal{G}}\big] \\
\leq& C\tau\big(\|e^{n+1}\|_{\mathcal{G}}^2 + \|e^{n-1}\|_{\mathcal{G}}^2 + \|e_t^n\|_{\mathcal{G}}^2 + \|e_t^{n-1}\|_{\mathcal{G}}^2\big) + C\tau(\tau^2 + h^{r+1})^2.
\end{aligned}$$

We thus get that

$$\omega^n - \omega^{n-1} \leq C\tau(\omega^n + \omega^{n-1}) + C\tau(\tau^2 + h^{r+1})^2,$$

where

$$\begin{aligned}
\omega^n =& \|(e^n)_t\|_{\mathcal{G}}^2 + \|e^{n+1}\|_{\mathcal{G}}^2 + \|e^n\|_{\mathcal{G}}^2 - (1-2\theta)\big[\big((e_1^{n+1})_{xx}, e_1^n\big)_{\mathcal{G}} + \big((e_2^{n+1})_{xx}, e_2^n\big)_{\mathcal{G}}\big] \\
&- \theta\big[\big((e_1^{n+1})_{xx}, e_1^{n+1}\big)_{\mathcal{G}} + \big((e_2^{n+1})_{xx}, e_2^{n+1}\big)_{\mathcal{G}} + \big((e_1^n)_{xx}, e_1^n\big)_{\mathcal{G}} + \big((e_2^n)_{xx}, e_2^n\big)_{\mathcal{G}}\big].
\end{aligned}$$

As in Ref. [4], we can now apply the discrete Gronwall inequality to get that

$$\max_{1 \leq n \leq J-1} \omega^n \leq C(\tau^2 + h^{r+1})^2. \tag{10}$$

As mentioned in Remark 1, the main point in this proof is that, using the arguments similar to that we applied to $\|(u_h^n)_t\|_{\mathcal{G}}^2$ in Lemma 6, we can obtain the following:

$$\|(e^n)_t\|_{\mathcal{G}}^2 - (1-2\theta)\big[\big((e_1^{n+1})_{xx}, e_1^n\big)_{\mathcal{G}} + \big((e_2^{n+1})_{xx}, e_2^n\big)_{\mathcal{G}}\big] \geq \frac{\alpha_1^2}{2}\|(e^n)_t\|^2 \geq 0,$$

under the assumption that $\tau = \gamma h^{1+\delta}$ and h is sufficiently small.

As $\theta > 0$, we can thus conclude from (10) that

$$\max_{1 \leq n \leq J-1} \left\{\|e^{n+1}\|^2 + \left\|e_x^{n+1}\right\|^2 + \|e^n\|^2 + \left\|e_x^n\right\|^2\right\} \leq C(\tau^2 + h^{r+1})^2.$$

Sobolev's theorem then implies $\max_{1 \leq n \leq J} \|e^n\|_{L^\infty} \leq C(\tau^2 + h^{r+1})^2$.

By Lemma 3, we have $\|u - W\|_{L^\infty} \leq h^{r+1}\|u\|_{H^{r+1}}$. These all together yield

$$\max_{1 \leq n \leq J} \left\|u^n - u_h^n\right\|_{L^\infty} \leq C(\tau^2 + h^{r+1}).$$

■

IV. NUMERICAL EXPERIMENTS

In this section, we test the OSC scheme for Eq. (I) in the following form

$$u_{tt} - u_{xx} + \mathbf{i}u_t + b|u|^2u = 0. \tag{11}$$

We refer to Ref. [4] for the details of the implementation of the scheme.

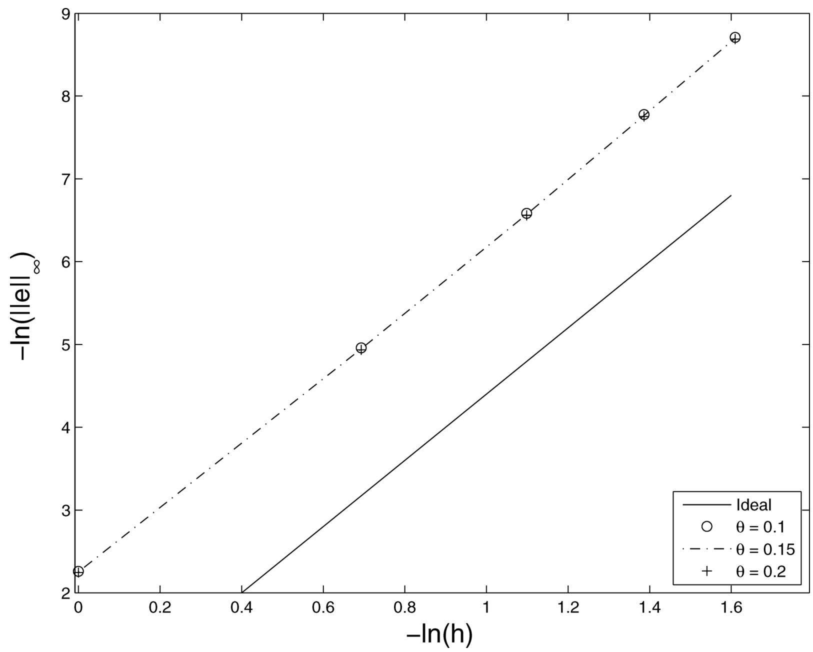

FIG. 1. The curves of convergence order of scheme (3) with various θ.

In the following, numerical test is carried out to verify the performance of the scheme (3). Generally, it is enough to choose $r = 3$. Thus, the Gauss points can be given as

$$\xi_{j,k} = \frac{1}{2}(x_{j-1} + x_j) + \frac{h_j}{2}\zeta_k, \quad j = 1, 2, \ldots, N, \;\; k = 1, 2,$$

where

$$\zeta_1 = -\frac{\sqrt{3}}{3}, \quad \zeta_2 = \frac{\sqrt{3}}{3}.$$

Throughout the following computations, we set

$$||e||_\infty = \max_{1 \le n \le J} ||e^n||_\infty = \max_{0 \le j \le N, 1 \le n \le J} \left| u_h^n(x_j) - u(x_j, t_n) \right|, \tag{12}$$

for scheme (3).

Now, we compute the numerical solution of (11) for $b = 2$ in the domain $[x_L, x_R]$ where, $x_L = -50$ and $x_R = 50$, with initial conditions

$$u(x, 0) = u_0(x) = K \operatorname{sech}(Kx), \quad u_t(x, 0) = u_1(x) = \mathbf{i}\Omega K \operatorname{sech}(Kx)$$

where $K = 1/4$ and $\Omega = -1/2 - \sqrt{3}/4$.

To verify the convergence order of scheme (3), which is stated in Theorem 3.1, we choose $\tau = h^2$, $h = (x_R - x_L)/N$ and compute the numerical solution at $t = 2$ for $N = 100$, 200, 300, 400, 500, and 800. The numerical solution for $N = 800$ is treated as the "exact" solution in computing the error $||e||_\infty$ in (12). Figure 1 plots the curves of convergence order of scheme (3)

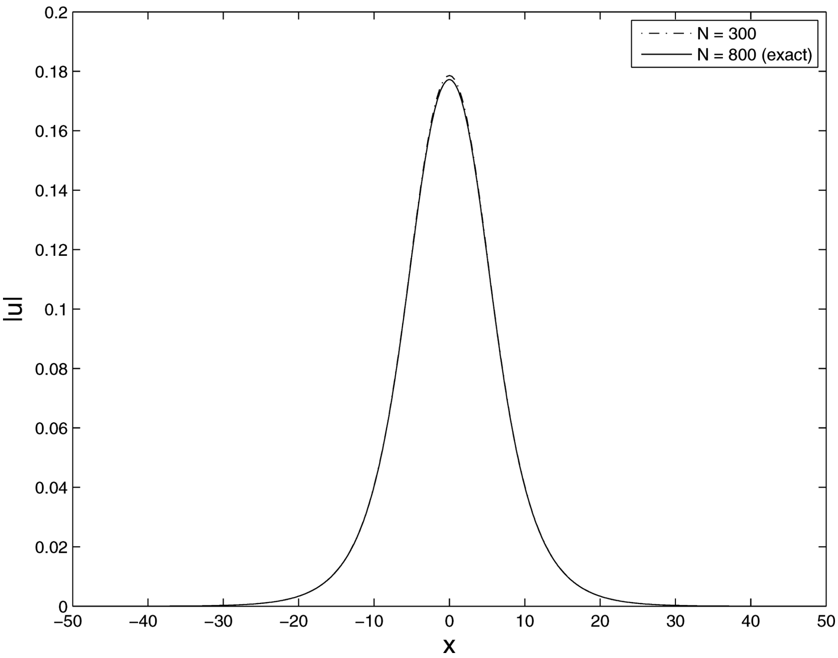

FIG. 2. $|u|$ of scheme (3) at $t = 2$ with $\theta = 0.15$.

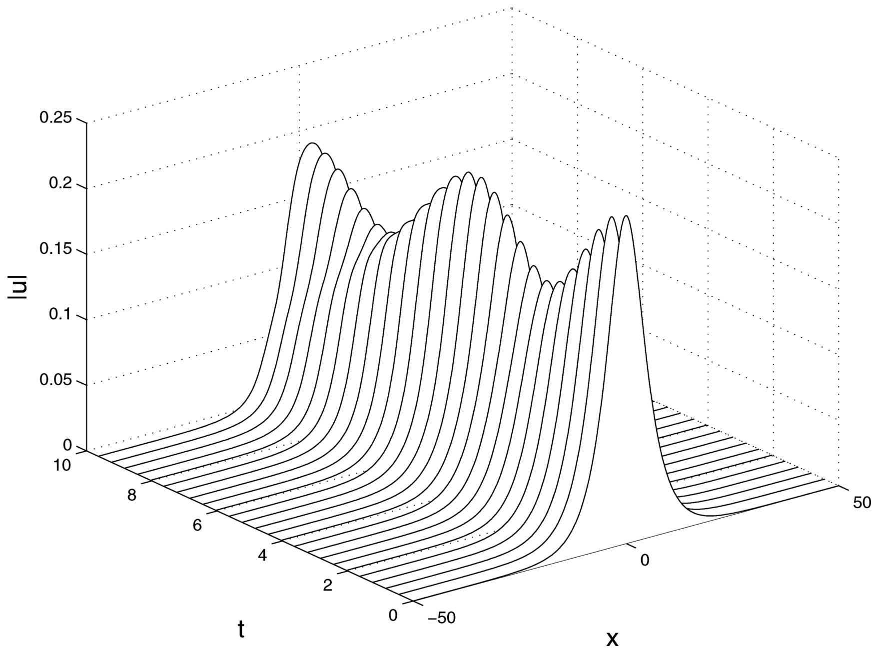

FIG. 3. $|u|$ of scheme (3) for $0 \le t \le 10$ with $\theta = 0.15$ and $N = 500$.

with different values of θ, where the solid lines are used for reference whose slope is exactly 4. One can conclude from Fig. 1 that scheme (3) is convergent of order $O(\tau^2 + h^4) = O(h^4)$ for $\theta = 0.1$, 0.15, and 0.2. Figures 2 and 3 plot the modulus of the numerical solutions from scheme (3) with $\theta = 0.15$ at $t = 2$, and for $0 \le t \le 10$, respectively.

V. CONCLUSIONS

In this article, we have considered an OSC scheme introduced in Ref. [4] for the Schrödinger equation with wave operator (I). Under the conditions $\beta(x) \ge 0$ and $\tau = \gamma h^{1+\delta}$ for a positive number δ, which can be arbitrarily small, we show that the method is convergent of order $O(\tau^2 + h^{r+1})$ even when the parameter θ involved in the scheme lies in $(0, \frac{1}{4})$. This result partly solves the problem proposed in Ref. [4]. However, the framework used in this article still cannot be straightly applied to the case that $\theta = 0$. As mentioned in Ref. [4], "when $\theta = 0$, the situation is more complicated." It is interesting to get the whole picture of the convergence of the scheme corresponding to different values of θ, and this will be our future study.

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