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Discrete-time orthogonal spline collocation methods for the nonlinear Schrödinger equation with wave operator

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ABSTRACT

In this paper, discrete-time orthogonal spline collocation schemes are proposed for the nonlinear Schrödinger equation with wave operator. These schemes are constructed by using orthogonal spline collocation approaches combined with finite difference methods. The conservative property, the convergence, and the stability of these methods are theoretically analyzed and also verified by extensive numerical experiments. In addition, some interesting phenomena which require further theoretical analysis are discussed numerically.

1. Introduction

The nonlinear Schrödinger (NLS) equation with wave operator was presented in [1] and its initial boundary value problem (IBVP) was studied numerically in [2]. In this paper, we consider the following IBVP of the NLSE with wave operator

$$u_{tt} - u_{xx} + \mathrm{i}\alpha u_t + \beta(x)q(|u|^2)u = 0, \qquad -x_L < x < x_R, \quad 0 < t \le T, \tag{1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad -x_L \le x \le x_R, \tag{2}$$

$$u|_{x_L} = u|_{x_R} = 0, \quad 0 \le t \le T, \tag{3}$$

where $u(x,t)$ is an unknown complex function, α is a real constant, $\beta(x)$ and $q(s)$ are real functions, and $\mathrm{i}^2 = -1$. Computing the inner product of (1) with u_t and taking the real part, one could easily obtain the following conservation law:

$$\int_{-x_L}^{x_R} [|u_t|^2 + |u_x|^2 + \beta(x)Q(|u|^2)]\mathrm{d}x = \text{const}, \tag{4}$$

where

$$Q(s) = \int_0^s q(z)\mathrm{d}z. \tag{5}$$

In [2], a finite difference scheme was formulated for problem (1)–(3); however, it was nonconservative, and its accuracy was only of order $O(\tau + h^2)$. It is desirable and natural to form numerical schemes keeping special properties of original problems, such as conservation laws [3]. Therefore, [4–6] were devoted to constructing many conservative schemes,

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and they improved the accuracy order to $O(\tau^2 + h^2)$. [7] applies the multisymplectic Fourier pseudospectral method and discusses the conservative property. The aim of this paper is to study problem (1)–(3) numerically by discrete-time orthogonal spline collocation (OSC) methods which also preserve the conservative property.

The OSC method, also known as the spline collocation method at Gauss points, was introduced by de Boor and Swartz for an ordinary differential equation [8], while collocation methods for nonlinear parabolic problems were thoroughly analyzed in [9]. Essentially, the OSC method determines an approximate solution in the form of a piecewise polynomial with unknown parameters and requires it to satisfy the differential equation exactly at Gauss points which are the nodes of the Gauss–Legendre quadrature rule. For more details, one may refer to [10] and references therein. In [11], discrete-time OSC methods were formulated for solutions of linear Schrödinger-type equations in two space variables. Semi-discrete OSC methods were applied to solve the cubic NLS equation [12], extended to the Schrödinger equation with general power nonlinearity, and also extended to the generalized NLS equation [13]. In [14], conservative OSC schemes were formulated for the coupled nonlinear Klein–Gordon–Schrödinger equation.

This paper is organized as follows. In Section 2, we introduce some preliminaries. A discrete-time OSC scheme is constructed in Section 3, and its conservative property is also discussed. In Section 4, the convergence and stability of the OSC method is analyzed, and another OSC scheme is formulated and discussed in Section 5. Section 6 is devoted to verifying the theoretical analysis by extensive numerical tests. Finally, in Section 7, some conclusions are made.

2. Preliminaries

For v a complex-valued function, we denote by v_1 and v_2 its real and imaginary parts, respectively. Given any space S of functions, let $\mathscr{R}(S)$ denotes the set of all real-valued functions in S. Given a partition

$$\Lambda : x_L = x_0 < x_1 < \cdots < x_{N-1} < x_N = x_R$$

of $\bar{\Omega} = [x_L, x_R]$, let $h_j = x_j - x_{j-1}, j = 1, 2, \ldots, N$, and $h = \max_j h_j$. A partition is said to be quasi-uniform (see [12] and references therein) if there exists a finite positive number a such that

$$\max_{0\leqslant j\leqslant N} \frac{h}{h_j} \leqslant a.$$

Let $\{t_n\}_{n=0}^{J}$ be a partition of $[0, T]$, where $t_n = n\tau$ and $\tau = T/J$. Throughout, we use C to denote a generic positive constant whose value is not necessarily the same on each occurrence.

Let $\mathcal{M}^0(\Lambda)$ be the space of the piecewise Hermite on Ω defined by

$$\mathcal{M}^0(\Lambda) = \{v \in C^1(\bar{\Omega}) : v\,|_{[x_{j-1},x_j]} \in P_r\} \cap \{v(x_L) = v(x_R) = 0\},$$

where $r \geq 3$ and P_r denotes the set of all polynomials of degree at most r.

Let $\mathcal{G} = \{\xi_{j,k}\}_{j,k=1}^{N,r-1}$ be the set of Gauss points

$$\xi_{j,k} = x_{j-1} + h_j \zeta_k,$$

where $\{\zeta_k\}_{k=1}^{r-1}$ denote the nodes for the $(r-1)$-point Gaussian quadrature rule on the interval Ω with corresponding weights $\{w_k\}_{k=1}^{r-1}$, $w_k > 0$. For μ, ν defined on $\mathcal{G}$, let $(\mu, \nu)_{\mathcal{G}}$ and $\|\mu\|_{\mathcal{G}}$ be given by

$$(\mu, \nu)_{\mathcal{G}} = \sum_{j=1}^{N} h_j \sum_{k=1}^{r-1} w_k (\mu\bar{\nu})(\xi_{j,k}),$$

and

$$\|\mu\|_{\mathcal{G}} = (\mu, \mu)_{\mathcal{G}}^{1/2}.$$

It can be shown from [9,15] that each $\mu \in \mathcal{M}^0(\Lambda)$ is uniquely defined by its values on $\mathcal{G}$. Hence, $\mathcal{M}^0(\Lambda)$ can be regarded as a Hilbert space with $(\cdot, \cdot)_{\mathcal{G}}$ as an inner product.

For l a nonnegative integer, we denote by

$$\|\mu\|_{H^l(\Omega)} = \left(\sum_{0\leqslant j\leqslant l} \left\| \frac{\partial^j \mu}{\partial x^j} \right\|_{L^2(\Omega)}^2 \right)^{1/2}$$

the norm on the usual Sobolev space $H^l(\Omega)$.

The following lemmas are required in the subsequent analysis.

Lemma 2.1 ([9,16]). *For $\nu \in \mathscr{R}(\mathcal{M}^0(\Lambda))$, there exist positive constants α_1 and $\alpha_2 = \alpha_2(\Lambda)$ such that*

$$\alpha_1 \|\nu\|_{\mathcal{G}} \leq \|\nu\|_{L^2(\Omega)} \leq \alpha_2 \|\nu\|_{\mathcal{G}}. \tag{6}$$

Lemma 2.2 *([9,12,16,17]). For $\mu, \nu \in \mathscr{R}(\mathcal{M}^0(\Lambda))$, we have*

$$-(\mu_{xx}, \nu)_{\mathcal{G}} = -(\mu, \nu_{xx})_{\mathcal{G}}, \tag{7}$$

and

$$-(\nu_{xx}, \nu)_{\mathcal{G}} \geqslant \|\nu_x\|^2_{L^2(\Omega)}. \tag{8}$$

Lemma 2.3 *([9,16]). Let $\varphi \in H^{r+3}(\Omega)$, and suppose that $\psi : [0, T] \to \mathscr{R}\left(\mathcal{M}^0(\Lambda)\right)$ satisfies*

$$(\varphi_{xx} - \psi_{xx})(\xi_{j,k}) - (\varphi - \psi)(\xi_{j,k}) = 0, \quad j = 1, 2, \ldots, N, k = 1, 2, \ldots, r-1. \tag{9}$$

Therefore, we have

$$\|\varphi - \psi\|_{L^2(\Omega)} \le Ch^{r+1}\|\varphi\|_{H^{r+3}(\Omega)}. \tag{10}$$

In this paper, we will make repeated use of the following inequality:

$$\alpha\beta \le \varepsilon\alpha^2 + \frac{1}{4\varepsilon}\beta^2, \quad \varepsilon > 0,$$

where $\alpha, \beta \in \mathbb{R}$. Next, we introduce several difference quotient notations:

$$\nu_t^n = \frac{\nu^{n+1} - \nu^n}{\tau}, \qquad \nu_{\bar{t}}^n = \frac{\nu^n - \nu^{n-1}}{\tau}, \qquad \nu_{t\bar{t}}^n = (\nu_t^n)_{\bar{t}}, \qquad \nu_{\hat{t}}^n = \frac{\nu^{n+1} - \nu^{n-1}}{2\tau}.$$

3. Discrete-time OSC scheme and conservative property

The continuous-time OSC approximation to the solution u of (1) is a differential map $u_h : [0, T] \to \mathcal{M}^0(\Lambda)$ such that

$$\begin{aligned}&(u_h)_{tt}(\xi_{j,k}, t) - (u_h)_{xx}(\xi_{j,k}, t) + \mathrm{i}\alpha(u_h)_t(\xi_{j,k}, t) + \beta(\xi_{j,k})q(|u_h(\xi_{j,k}, t)|^2)u_h(\xi_{j,k}, t) = 0,\\ &\quad j = 1, 2, \ldots, N,\ k = 1, 2, \ldots, r-1,\end{aligned} \tag{11}$$

for $t \in (0, T]$.

In order to work out the approximate solution of (1), one might discretize (11) in time by finite difference techniques. Some concise and effective finite difference schemes in [6] could be written together in the following class of scheme:

$$(U_j^n)_{t\bar{t}} - (1-2\theta)(U_j^n)_{x\bar{x}} - \theta(U_j^{n+1} + U_j^{n-1})_{x\bar{x}} + \mathrm{i}\alpha(U_j^n)_{\hat{t}} + \beta_j\frac{Q(|U_j^{n+1}|^2) - Q(|U_j^{n-1}|^2)}{|U_j^{n+1}|^2 - |U_j^{n-1}|^2}\frac{U_j^{n+1} + U_j^{n-1}}{2} = 0, \tag{12}$$

where $\theta \ge 0$ and U_j^n is the approximation of $u(x_j, t_n)$. Scheme (12) is just Scheme B in [6] if $\theta = 0$, and Scheme C if $\theta = 0.5$.

Now we combine (11) and (12) to construct the discrete-time OSC scheme as follows:

$$\begin{aligned}&\left\{\frac{1}{\tau^2}(u_h^{n+1} - 2u_h^n + u_h^{n-1}) - (1-2\theta)(u_h^n)_{xx} - \theta[(u_h^{n+1})_{xx} + (u_h^{n-1})_{xx}] + \frac{\mathrm{i}\alpha}{2\tau}(u_h^{n+1} - u_h^{n-1})\right.\\ &\quad \left. +\ \beta\frac{Q(|u_h^{n+1}|^2) - Q(|u_h^{n-1}|^2)}{|u_h^{n+1}|^2 - |u_h^{n-1}|^2}\frac{u_h^{n+1} + u_h^{n-1}}{2}\right\}(\xi_{j,k}) = 0,\\ &\quad j = 1, 2, \ldots, N,\ k = 1, 2, \ldots, r-1,\ n = 1, 2, \ldots, J-1,\end{aligned} \tag{13}$$

where u_h^0 and u_h^1 can be prescribed by approximating the initial conditions (2) using Hermite piecewise interpolations.

Theorem 3.1. *The OSC scheme (13) admits the following conservation law:*

$$\begin{aligned}E^n &= \|(u_h^n)_t\|^2_{\mathcal{G}} - (1-2\theta)[((u_{h,1}^{n+1})_{xx}, u_{h,1}^n)_{\mathcal{G}} + ((u_{h,2}^{n+1})_{xx}, u_{h,2}^n)_{\mathcal{G}}]\\ &\quad - \theta[((u_{h,1}^{n+1})_{xx}, u_{h,1}^{n+1})_{\mathcal{G}} + ((u_{h,2}^{n+1})_{xx}, u_{h,2}^{n+1})_{\mathcal{G}} + ((u_{h,1}^n)_{xx}, u_{h,1}^n)_{\mathcal{G}} + ((u_{h,2}^n)_{xx}, u_{h,2}^n)_{\mathcal{G}}]\\ &\quad + \frac{1}{2}\sum_{j=1}^N h_j \sum_{k=1}^{r-1} w_k\beta(\xi_{j,k})[Q(|u_h^{n+1}(\xi_{j,k})|^2) + Q(|u_h^n(\xi_{j,k})|^2)]\\ &= E^{n-1} = \cdots = E^0 = const,\end{aligned} \tag{14}$$

where $u_{h,1}^n$ and $u_{h,2}^n$ are the real and imaginary parts of u_h^n, respectively.

Proof. Computing the inner product of (13) with $(u_h^{n+1}-u_h^{n-1})$ and taking the real part, we obtain

$$I_1+I_2+I_3+I_4+I_5=0, \tag{15}$$

where

$$\begin{aligned} I_1 &= \mathrm{Re}\left\{\frac{1}{\tau^2}(u_h^{n+1}-2u_h^n+u_h^{n-1},u_h^{n+1}-u_h^{n-1})_{\mathcal{G}}\right\} \\ &= \frac{1}{\tau^2}(\|u_h^{n+1}-u_h^n\|_{\mathcal{G}}^2-\|u_h^n-u_h^{n-1}\|_{\mathcal{G}}^2)=\|(u_h^n)_t\|_{\mathcal{G}}^2-\|(u_h^{n-1})_t\|_{\mathcal{G}}^2, \\ I_2 &= \mathrm{Re}\{-(1-2\theta)((u_h^n)_{xx},u_h^{n+1}-u_h^{n-1})_{\mathcal{G}}\}, \\ I_3 &= \mathrm{Re}\{-\theta((u_h^{n+1})_{xx}+(u_h^{n-1})_{xx},u_h^{n+1}-u_h^{n-1})_{\mathcal{G}}\}, \\ I_4 &= \mathrm{Re}\left\{\frac{\mathrm{i}\alpha}{2\tau}(u_h^{n+1}-u_h^{n-1},u_h^{n+1}-u_h^{n-1})_{\mathcal{G}}\right\}=0, \\ I_5 &= \mathrm{Re}\left\{\left(\beta(x)\frac{Q(|u_h^{n+1}|^2)-Q(|u_h^{n-1}|^2)}{|u_h^{n+1}|^2-|u_h^{n-1}|^2}\frac{u_h^{n+1}+u_h^{n-1}}{2},u_h^{n+1}-u_h^{n-1}\right)_{\mathcal{G}}\right\} \\ &= \frac{1}{2}\sum_{j=1}^{N}h_j\sum_{k=1}^{r-1}w_k\beta(\xi_{j,k})[Q(|u_h^{n+1}|^2)-Q(|u_h^{n-1}|^2)](\xi_{j,k}). \end{aligned}$$

Setting $u_h^n=u_{h,1}^n+\mathrm{i}u_{h,2}^n$, we can obtain

$$I_2=-(1-2\theta)\sum_{j=1}^{N}h_j\sum_{k=1}^{r-1}w_k[(u_{h,1}^n)_{xx}(u_{h,1}^{n+1}-u_{h,1}^{n-1})+(u_{h,2}^n)_{xx}(u_{h,2}^{n+1}-u_{h,2}^{n-1})](\xi_{j,k}). \tag{16}$$

It follows from Lemma 2.2 and (16) that

$$I_2=-(1-2\theta)[((u_{h,1}^{n+1})_{xx},u_{h,1}^n)_{\mathcal{G}}+((u_{h,2}^{n+1})_{xx},u_{h,2}^n)_{\mathcal{G}}-((u_{h,1}^n)_{xx},u_{h,1}^{n-1})_{\mathcal{G}}-((u_{h,2}^n)_{xx},u_{h,2}^{n-1})_{\mathcal{G}}]. \tag{17}$$

Similarly, one could obtain

$$I_3=-\theta[((u_{h,1}^{n+1})_{xx},u_{h,1}^{n+1})_{\mathcal{G}}+((u_{h,2}^{n+1})_{xx},u_{h,2}^{n+1})_{\mathcal{G}}-((u_{h,1}^{n-1})_{xx},u_{h,1}^{n-1})_{\mathcal{G}}-((u_{h,2}^{n-1})_{xx},u_{h,2}^{n-1})_{\mathcal{G}}]. \tag{18}$$

Therefore, (14) can be easily derived from (15), (17) and (18). □

4. Convergence and stability

For verifying the convergence of scheme (13), first we need the following lemma [14].

Lemma 4.1. *Let $v\in\mathscr{R}(\mathcal{M}^0(\Lambda))$. If $\theta>1/2$, then*

$$R_\tau\le Q_\tau; \tag{19}$$

else, if $0\le\theta\le 1/2$, then

$$\hat{R}_\tau\le Q_\tau, \tag{20}$$

where

$$\begin{aligned} Q_\tau &= \|v_t^{n-1}\|_{\mathcal{G}}^2-(1-2\theta)(v_{xx}^n,v^{n-1})_{\mathcal{G}}, \\ R_\tau &= \|v_t^{n-1}\|_{\mathcal{G}}^2-\frac{1}{2}(1-2\theta)[(v_{xx}^n,v^n)_{\mathcal{G}}+(v_{xx}^{n-1},v^{n-1})_{\mathcal{G}}], \\ \hat{R}_\tau &= \|v_t^{n-1}\|_{\mathcal{G}}^2+\frac{1}{2}(1-2\theta)[(v_{xx}^n,v^n)_{\mathcal{G}}+(v_{xx}^{n-1},v^{n-1})_{\mathcal{G}}]. \end{aligned}$$

Theorem 4.1. *Suppose that $\theta\ge 1/4$, $q(s)\in C^1$, $u(x,t)\in C^{2,4}\cap L^2(H^{r+3})$ is the solution of (1), and $\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial t^2}\in L^2(H^{r+3})$, while $u_h^n\in\mathcal{M}^0(\Lambda)$ $(n=0,1,\ldots,J)$ is the solution of (13). If $W:[0,T]\to\mathcal{M}^0(\Lambda)$ is defined by (9), and $\|(u_h^0-W^0)_t\|_{L^2(\Omega)}^2$, $\|u_h^0-W^0\|_{H^1(\Omega)}^2$ and $\|u_h^1-W^1\|_{H^1(\Omega)}^2$ are $O(\tau^2+h^{r+1})^2$, then, for τ and h sufficiently small, we have*

$$\max_{1\le n\le J}\|u^n-u_h^n\|_{L^2(\Omega)}=O(\tau^2+h^{r+1}). \tag{21}$$

Proof. Substituting $u^n(x) = u(x, n\tau)$ into (13), it follows from Taylor's theorem that

$$\left\{\frac{1}{\tau^2}(u^{n+1} - 2u^n + u^{n-1}) - (1-2\theta)u^n_{xx} - \theta(u^{n+1}_{xx} + u^{n-1}_{xx}) + \frac{\mathrm{i}\alpha}{2\tau}(u^{n+1} - u^{n-1}) + \beta\frac{Q(|u^{n+1}|^2) - Q(|u^{n-1}|^2)}{|u^{n+1}|^2 - |u^{n-1}|^2}\,\frac{u^{n+1} + u^{n-1}}{2}\right\}(\xi_{j,k}) = \sigma^n(\xi_{j,k}), \tag{22}$$

where $\sigma^n(\xi_{j,k}) = O(\tau^2)$. Let $\hat{e}^n = u^n - W^n$, $e^n = u^n_h - W^n$; then $u^n - u^n_h = \hat{e}^n - e^n$. Setting

$$G(u^n) = \frac{Q(|u^{n+1}|^2) - Q(|u^{n-1}|^2)}{|u^{n+1}|^2 - |u^{n-1}|^2},$$

one may get from (13) and (22) that

$$\begin{aligned}&\left\{\frac{1}{\tau^2}(e^{n+1} - 2e^n + e^{n-1}) - (1-2\theta)e^n_{xx} - \theta(e^{n+1}_{xx} + e^{n-1}_{xx}) + \frac{\mathrm{i}\alpha}{2\tau}(e^{n+1} - e^{n-1})\right\}(\xi_{j,k})\\ &= \left\{\frac{1}{\tau^2}(\hat{e}^{n+1} - 2\hat{e}^n + \hat{e}^{n-1}) - (1-2\theta)\hat{e}^n_{xx} - \theta(\hat{e}^{n+1}_{xx} + \hat{e}^{n-1}_{xx}) + \frac{\mathrm{i}\alpha}{2\tau}(\hat{e}^{n+1} - \hat{e}^{n-1}) - \sigma^n\right.\\ &\quad \left. + \beta G(u^n)\frac{\hat{e}^{n+1} + \hat{e}^{n-1} - e^{n+1} - e^{n-1}}{2} + \beta(G(u^n) - G(u^n_h))\frac{u^{n+1}_h + u^{n-1}_h}{2}\right\}(\xi_{j,k}).\end{aligned} \tag{23}$$

Computing the inner product of (23) with $(e^{n+1} - e^{n-1})$ and taking the real part, similar to the proof of Theorem 3.1, we have

$$\begin{aligned}&\|e^n_t\|^2_{\mathcal{G}} - \|e^{n-1}_t\|^2_{\mathcal{G}} - (1-2\theta)[((e^{n+1}_1)_{xx}, e^n_1)_{\mathcal{G}} + ((e^{n+1}_2)_{xx}, e^n_2)_{\mathcal{G}} - ((e^n_1)_{xx}, e^{n-1}_1)_{\mathcal{G}} - ((e^n_2)_{xx}, e^{n-1}_2)_{\mathcal{G}}]\\ &\quad - \theta[((e^{n+1}_1)_{xx}, e^{n+1}_1)_{\mathcal{G}} + ((e^{n+1}_2)_{xx}, e^{n+1}_2)_{\mathcal{G}} - ((e^{n-1}_1)_{xx}, e^{n-1}_1)_{\mathcal{G}} - ((e^{n-1}_2)_{xx}, e^{n-1}_2)_{\mathcal{G}}] = II_1 + II_2 + II_3,\end{aligned} \tag{24}$$

where

$$\begin{aligned}II_1 &= \mathrm{Re}\left\{\left(\frac{1}{\tau^2}(\hat{e}^{n+1} - 2\hat{e}^n + \hat{e}^{n-1}) - (1-2\theta)\hat{e}^n_{xx} - \theta(\hat{e}^{n+1}_{xx} + \hat{e}^{n-1}_{xx}) + \frac{\mathrm{i}\alpha}{2\tau}(\hat{e}^{n+1} - \hat{e}^{n-1}) - \sigma^n, e^{n+1} - e^{n-1}\right)_{\mathcal{G}}\right\}\\ &= \tau|(\hat{e}^n_{t\bar{t}} - (1-2\theta)\hat{e}^n_{xx} - \theta(\hat{e}^{n+1}_{xx} + \hat{e}^{n-1}_{xx}) + \frac{\mathrm{i}\alpha}{2}(\hat{e}^n_t + \hat{e}^{n-1}_t) - \sigma^n, e^n_t + e^{n-1}_t)_{\mathcal{G}}|\\ &\le C\tau(\|\hat{e}^n_{t\bar{t}}\|^2_{\mathcal{G}} + \|\hat{e}^{n+1}_{xx}\|^2_{\mathcal{G}} + \|\hat{e}^n_{xx}\|^2_{\mathcal{G}} + \|\hat{e}^{n-1}_{xx}\|^2_{\mathcal{G}} + \|\hat{e}^n_t\|^2_{\mathcal{G}} + \|\hat{e}^{n-1}_t\|^2_{\mathcal{G}} + \|\sigma^n\|^2_{\mathcal{G}} + \|e^n_t\|^2_{\mathcal{G}} + \|e^{n-1}_t\|^2_{\mathcal{G}}),\\ II_2 &= \mathrm{Re}\left\{\left(\beta G(u^n)\frac{\hat{e}^{n+1} + \hat{e}^{n-1} - e^{n+1} - e^{n-1}}{2}, e^{n+1} - e^{n-1}\right)_{\mathcal{G}}\right\}\\ &\le C\tau(\|\hat{e}^{n+1}\|^2_{\mathcal{G}} + \|\hat{e}^{n-1}\|^2_{\mathcal{G}} + \|e^{n+1}\|^2_{\mathcal{G}} + \|e^{n-1}\|^2_{\mathcal{G}} + \|e^n_t\|^2_{\mathcal{G}} + \|e^{n-1}_t\|^2_{\mathcal{G}}),\\ II_3 &= \mathrm{Re}\left\{\left(\beta(G(u^n) - G(u^n_h))\frac{u^{n+1}_h + u^{n-1}_h}{2}, e^{n+1} - e^{n-1}\right)_{\mathcal{G}}\right\}\\ &\le \frac{\tau}{2}\left|\sum_{j=1}^{N} h_j \sum_{k=1}^{r-1} w_k\beta(\xi_{j,k})[G(u^n(\xi_{j,k})) - G(u^n_h(\xi_{j,k}))][(u^{n+1}_h + u^{n-1}_h)(\bar{e}^n_t + \bar{e}^{n-1}_t)](\xi_{j,k})\right|.\end{aligned}$$

According to the definition of W^n, one can easily obtain $\hat{e}^n_{xx} = \hat{e}^n$. So

$$II_1 \le C\tau\big(\|\hat{e}^n_{t\bar{t}}\|^2_{\mathcal{G}} + \|\hat{e}^{n+1}\|^2_{\mathcal{G}} + \|\hat{e}^n\|^2_{\mathcal{G}} + \|\hat{e}^{n-1}\|^2_{\mathcal{G}} + \|\hat{e}^n_t\|^2_{\mathcal{G}} + \|\hat{e}^{n-1}_t\|^2_{\mathcal{G}} + \|\sigma^n\|^2_{\mathcal{G}} + \|e^n_t\|^2_{\mathcal{G}} + \|e^{n-1}_t\|^2_{\mathcal{G}}\big). \tag{25}$$

Since $q(s) \in C^1$ and also thanks to the continuity of u^n_h, it is easy to prove that

$$II_3 \le C\tau(\|\hat{e}^{n+1}\|^2_{\mathcal{G}} + \|e^{n+1}\|^2_{\mathcal{G}} + \|\hat{e}^{n-1}\|^2_{\mathcal{G}} + \|e^{n-1}\|^2_{\mathcal{G}} + \|e^n_t\|^2_{\mathcal{G}} + \|e^{n-1}_t\|^2_{\mathcal{G}}). \tag{26}$$

Additionally, one may prove that

$$\|e^{n+1}\|^2_{\mathcal{G}} - \|e^{n-1}\|^2_{\mathcal{G}} \le C\tau(\|e^{n+1}\|^2_{\mathcal{G}} + \|e^{n-1}\|^2_{\mathcal{G}} + \|e^n_t\|^2_{\mathcal{G}} + \|e^{n-1}_t\|^2_{\mathcal{G}}). \tag{27}$$

Therefore, it follows from (24)–(27) that

$$\begin{aligned}&\|e_t^n\|_{\mathcal{G}}^2-\|e_t^{n-1}\|_{\mathcal{G}}^2+\|e^{n+1}\|_{\mathcal{G}}^2-\|e^{n-1}\|_{\mathcal{G}}^2\\&\quad-(1-2\theta)[((e_1^{n+1})_{xx},e_1^n)_{\mathcal{G}}+((e_2^{n+1})_{xx},e_2^n)_{\mathcal{G}}-((e_1^n)_{xx},e_1^{n-1})_{\mathcal{G}}-((e_2^n)_{xx},e_2^{n-1})_{\mathcal{G}}]\\&\quad-\theta[((e_1^{n+1})_{xx},e_1^{n+1})_{\mathcal{G}}+((e_2^{n+1})_{xx},e_2^{n+1})_{\mathcal{G}}-((e_1^{n-1})_{xx},e_1^{n-1})_{\mathcal{G}}-((e_2^{n-1})_{xx},e_2^{n-1})_{\mathcal{G}}]\\&\le C\tau(\|\hat{e}_{t\bar{t}}^n\|_{\mathcal{G}}^2+\|\hat{e}^{n+1}\|_{\mathcal{G}}^2+\|\hat{e}^n\|_{\mathcal{G}}^2+\|\hat{e}^{n-1}\|_{\mathcal{G}}^2+\|\hat{e}_t^n\|_{\mathcal{G}}^2+\|\hat{e}_t^{n-1}\|_{\mathcal{G}}^2+\|\sigma^n\|_{\mathcal{G}}^2+\|e^{n+1}\|_{\mathcal{G}}^2+\|e^{n-1}\|_{\mathcal{G}}^2\\&\quad+\|e_t^n\|_{\mathcal{G}}^2+\|e_t^{n-1}\|_{\mathcal{G}}^2).\end{aligned}\tag{28}$$

Applying Lemma 2.3, we have

$$\begin{aligned}\|\hat{e}_t^n\|_{\mathcal{G}}&=\left\|\frac{1}{\tau}\int_0^\tau\frac{\partial\hat{e}}{\partial t}(n\tau+s)\mathrm{d}s\right\|_{\mathcal{G}}\le\frac{1}{\tau}\int_0^\tau\left\|\frac{\partial\hat{e}}{\partial t}(n\tau+s)\right\|_{\mathcal{G}}\mathrm{d}s\\&\le Ch^{r+1}\left(\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(H^{r+3})}+\|u\|_{L^\infty(H^{r+2})}\right)\le Ch^{r+1},\end{aligned}\tag{29}$$

and

$$\begin{aligned}\|\hat{e}_{t\bar{t}}^n\|_{\mathcal{G}}&=\left\|\frac{1}{\tau^2}\int_{-\tau}^\tau(\tau-|s|)\frac{\partial^2\hat{e}}{\partial t^2}(n\tau+s)\,\mathrm{d}s\right\|_{\mathcal{G}}\le\frac{1}{\tau}\int_{-\tau}^\tau\left\|\frac{\partial^2\hat{e}}{\partial t^2}(n\tau+s)\right\|_{\mathcal{G}}\mathrm{d}s\\&\le Ch^{r+1}\left(\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^\infty(H^{r+3})}+\left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(H^{r+2})}+\|u\|_{L^\infty(H^{r+2})}\right)\le Ch^{r+1}.\end{aligned}\tag{30}$$

It follows from Lemmas 2.1, 2.3 and (28)–(30) that

$$\begin{aligned}&\|e_t^n\|_{\mathcal{G}}^2-\|e_t^{n-1}\|_{\mathcal{G}}^2+\|e^{n+1}\|_{\mathcal{G}}^2-\|e^{n-1}\|_{\mathcal{G}}^2\\&\quad-(1-2\theta)[((e_1^{n+1})_{xx},e_1^n)_{\mathcal{G}}+((e_2^{n+1})_{xx},e_2^n)_{\mathcal{G}}-((e_1^n)_{xx},e_1^{n-1})_{\mathcal{G}}-((e_2^n)_{xx},e_2^{n-1})_{\mathcal{G}}]\\&\quad-\theta[((e_1^{n+1})_{xx},e_1^{n+1})_{\mathcal{G}}+((e_2^{n+1})_{xx},e_2^{n+1})_{\mathcal{G}}-((e_1^{n-1})_{xx},e_1^{n-1})_{\mathcal{G}}-((e_2^{n-1})_{xx},e_2^{n-1})_{\mathcal{G}}]\\&\le C\tau(\|e^{n+1}\|_{\mathcal{G}}^2+\|e^{n-1}\|_{\mathcal{G}}^2+\|e_t^n\|_{\mathcal{G}}^2+\|e_t^{n-1}\|_{\mathcal{G}}^2+O(\tau^2+h^{r+1})^2).\end{aligned}\tag{31}$$

Note that $\|e_t^n\|_{\mathcal{G}}^2=\|(e_1^n)_t\|_{\mathcal{G}}^2+\|(e_2^n)_t\|_{\mathcal{G}}^2$, where e_1^n and e_2^n are real and imaginary parts of e^n, respectively. If $\theta>1/2$, applying Lemmas 4.1 and 2.2, we have

$$\begin{aligned}&\|(e_1^n)_t\|_{\mathcal{G}}^2-(1-2\theta)((e_1^{n+1})_{xx},e_1^n)_{\mathcal{G}}-\theta[((e_1^{n+1})_{xx},e_1^{n+1})_{\mathcal{G}}+((e_1^n)_{xx},e_1^n)_{\mathcal{G}}]\\&\ge\|(e_1^n)_t\|_{\mathcal{G}}^2-\frac{1}{2}[((e_1^{n+1})_{xx},e_1^{n+1})_{\mathcal{G}}+((e_1^n)_{xx},e_1^n)_{\mathcal{G}}]\\&\ge\|(e_1^n)_t\|_{\mathcal{G}}^2+\frac{1}{2}(\|(e_1^{n+1})_x\|_{\mathcal{G}}^2+\|(e_1^n)_x\|_{\mathcal{G}}^2)\ge\|(e_1^n)_t\|_{\mathcal{G}}^2\ge 0.\end{aligned}\tag{32}$$

In like manner, if $1/4\le\theta\le 1/2$, one could get

$$\begin{aligned}&\|(e_1^n)_t\|_{\mathcal{G}}^2-(1-2\theta)((e_1^{n+1})_{xx},e_1^n)_{\mathcal{G}}-\theta[((e_1^{n+1})_{xx},e_1^{n+1})_{\mathcal{G}}+((e_1^n)_{xx},e_1^n)_{\mathcal{G}}]\\&\ge\|(e_1^n)_t\|_{\mathcal{G}}^2+\frac{1}{2}(4\theta-1)(\|(e_1^{n+1})_x\|_{\mathcal{G}}^2+\|(e_1^n)_x\|_{\mathcal{G}}^2)\ge\|(e_1^n)_t\|_{\mathcal{G}}^2\ge 0.\end{aligned}\tag{33}$$

Similarly, (32) and (33) also hold for e_2^n. Let

$$\begin{aligned}\omega^n&=\|e_t^n\|_{\mathcal{G}}^2+\|e^{n+1}\|_{\mathcal{G}}^2+\|e^n\|_{\mathcal{G}}^2-(1-2\theta)[((e_1^{n+1})_{xx},e_1^n)_{\mathcal{G}}+((e_2^{n+1})_{xx},e_2^n)_{\mathcal{G}}]\\&\quad-\theta[((e_1^{n+1})_{xx},e_1^{n+1})_{\mathcal{G}}+((e_2^{n+1})_{xx},e_2^{n+1})_{\mathcal{G}}+((e_1^n)_{xx},e_1^n)_{\mathcal{G}}+((e_2^n)_{xx},e_2^n)_{\mathcal{G}}].\end{aligned}\tag{34}$$

Thus, if $\theta\ge 1/4$, one can obtain from (31)–(34) that

$$\omega^n-\omega^{n-1}\le C\tau(\omega^n+\omega^{n-1})+C\tau(\tau^2+h^{r+1})^2.\tag{35}$$

Therefore, if follows from the discrete Gronwall inequality (see [18] and references therein) that

$$\max_{1\le n\leqslant J-1}\omega^n\le\left(\omega^0+\tau\sum_{k=1}^{J-1}C(\tau^2+h^4)^2\right)\exp(C(J-1)\tau)\le C(\omega^0+(\tau^2+h^4)^2).\tag{36}$$

Using Lemma 2.1 in [12], we get $\omega^0 = O(\tau^2 + h^{r+1})^2$, so

$$\max_{1\le n\le J} \|e^n\|_{\mathcal{G}} = O(\tau^2 + h^{r+1}).$$

Therefore, the triangle inequality, Lemmas 2.1 and 2.3 yield the desired result (21). □

Theorem 4.2. *Assume that $\theta \ge 1/4$ and that the conditions of Theorem 4.1 are satisfied. Then scheme (13) is unconditionally stable.*

Proof. Let $\eta^n(x)$ be the error of $u_h^n(x)$, and set $\tilde{u}_h^n(x) = u_h^n(x) - \eta^n(x)$. Then we have

$$\left\{\frac{1}{\tau^2}(\eta^{n+1} - 2\eta^n + \eta^{n-1}) - (1-2\theta)\eta_{xx}^n - \theta(\eta_{xx}^{n+1} + \eta_{xx}^{n-1}) + \frac{\mathrm{i}\alpha}{2\tau}(\eta^{n+1} - \eta^{n-1})\right\}(\xi_{j,k})$$
$$= \left\{\beta\frac{Q(|\tilde{u}_h^{n+1}|^2) - Q(|\tilde{u}_h^{n-1}|^2)}{|\tilde{u}_h^{n+1}|^2 - |\tilde{u}_h^{n-1}|^2}\frac{\tilde{u}_h^{n+1} + \tilde{u}_h^{n-1}}{2} - \beta\frac{Q(|u_h^{n+1}|^2) - Q(|u_h^{n-1}|^2)}{|u_h^{n+1}|^2 - |u_h^{n-1}|^2}\frac{u_h^{n+1} + u_h^{n-1}}{2}\right\}(\xi_{j,k}). \quad (37)$$

Computing the inner product of (37) with $(\eta^{n+1} - \eta^{n-1})$ and taking the real part, by a proof similar to that of Theorem 4.1, one can obtain

$$\max_{1\le n\le J} \|\eta^n\|_{\mathcal{G}} \le C\tilde{\omega}^0,$$

where

$$\tilde{\omega}^0 = \|\eta_t^0\|_{\mathcal{G}}^2 + \|\eta^1\|_{\mathcal{G}}^2 + \|\eta^0\|_{\mathcal{G}}^2 - (1-2\theta)[((\eta_1^1)_{xx}, \eta_1^0)_{\mathcal{G}} + ((\eta_2^1)_{xx}, \eta_2^0)_{\mathcal{G}}]$$
$$- \theta[((\eta_1^1)_{xx}, \eta_1^1)_{\mathcal{G}} + ((\eta_2^1)_{xx}, \eta_2^1)_{\mathcal{G}} + ((\eta_1^0)_{xx}, \eta_1^0)_{\mathcal{G}} + ((\eta_2^0)_{xx}, \eta_2^0)_{\mathcal{G}}],$$

and η_1^n and η_2^n are the real and imaginary parts of η^n, $n = 0, 1$.

According to [19,20] and references therein, this theorem expresses the generalized stability of the numerical scheme. □

5. Another scheme

In order to test the OSC scheme proposed in this paper, we adopt the following form of Eq. (1) [4,6,7]:

$$u_{tt} - u_{xx} + \mathrm{i}u_t + b|u|^2u = 0, \quad (38)$$

where b is a constant. Consequently, the corresponding OSC scheme might be written as

$$\left\{\frac{1}{\tau^2}(u_h^{n+1} - 2u_h^n + u_h^{n-1}) - (1-2\theta)(u_h^n)_{xx} - \theta[(u_h^{n+1})_{xx} + (u_h^{n-1})_{xx}] + \frac{\mathrm{i}}{2\tau}(u_h^{n+1} - u_h^{n-1})\right.$$
$$\left. + \frac{b}{4}(|u_h^{n+1}|^2 + |u_h^{n-1}|^2)(u_h^{n+1} + u_h^{n-1})\right\}(\xi_{j,k}) = 0,$$
$$j = 1, 2, \ldots, N,\ k = 1, 2, \ldots, r-1,\ n = 1, 2, \ldots, J-1. \quad (39)$$

Since (39) is a nonlinear scheme, one should utilize some iterative method to solve it, such as

$$\left\{\frac{1}{\tau^2}(u_h^{n+1(s+1)} - 2u_h^n + u_h^{n-1}) - (1-2\theta)(u_h^n)_{xx} - \theta[(u_h^{n+1(s+1)})_{xx} + (u_h^{n-1})_{xx}] + \frac{\mathrm{i}}{2\tau}(u_h^{n+1(s+1)} - u_h^{n-1})\right.$$
$$\left. + \frac{b}{4}(|u_h^{n+1(s)}|^2 + |u_h^{n-1}|^2)(u_h^{n+1(s+1)} + u_h^{n-1})\right\}(\xi_{j,k}) = 0,$$
$$j = 1, 2, \ldots, N, k = 1, 2, \ldots, r-1, n = 1, 2, \ldots, J-1, \quad (40)$$

where s denotes the s-th iteration at a given time step, and the iteration continues until the condition

$$\max_{0\le j\le N} |u_h^{n+1(s+1)}(x_j) - u_h^{n+1(s)}(x_j)| < 10^{-6} \quad (41)$$

is reached. For each time step, the initial iterative function $u_h^{n+1(0)}$ can be chosen as $u_h^{n+1(0)} = u_h^n$. Applying Taylor's theorem, one could get from (2) and (38) that

$$u(x, \tau) = z(x) + O(\tau^3),$$
$$z(x) = u_0(x) + \tau u_1(x) + \frac{\tau^2}{2}\left(\frac{\partial^2 u_0}{\partial x^2} - \mathrm{i}u_1 - b|u_0|^2u_0\right)(x).$$

Consequently, u_h^0 and u_h^1 can be prescribed by approximating $u_0(x)$ and $z(x)$ using Hermite piecewise cubic interpolations, respectively.

It is well known that the iterative scheme requires more computational time; therefore, we construct another OSC scheme especially for (38), as follows:

$$\left\{\frac{1}{\tau^2}(u_h^{n+1}-2u_h^n+u_h^{n-1})-(1-2\theta)(u_h^n)_{xx}-\theta[(u_h^{n+1})_{xx}+(u_h^{n-1})_{xx}]+\frac{\mathrm{i}}{2\tau}(u_h^{n+1}-u_h^{n-1})\right.$$
$$\left.+\ \frac{b}{2}|u_h^n|^2(u_h^{n+1}+u_h^{n-1})\right\}(\xi_{j,k})=0,\quad j=1,2,\ldots,N,k=1,2,\ldots,r-1,n=1,2,\ldots,J-1. \tag{42}$$

It is a linear scheme, so one could work it out directly without any annoying iteration. Moreover, it is gratifying that this procedure also keeps the conservative property.

Theorem 5.1. *The OSC scheme (42) admits the following conservation law:*

$$\begin{aligned}E^n &= \|(u_h^n)_t\|_{\mathcal{G}}^2-(1-2\theta)[((u_{h,1}^{n+1})_{xx},u_{h,1}^n)_{\mathcal{G}}+((u_{h,2}^{n+1})_{xx},u_{h,2}^n)_{\mathcal{G}}]-\theta[((u_{h,1}^{n+1})_{xx},u_{h,1}^{n+1})_{\mathcal{G}}+((u_{h,2}^{n+1})_{xx},u_{h,2}^{n+1})_{\mathcal{G}}\\&\quad+((u_{h,1}^n)_{xx},u_{h,1}^n)_{\mathcal{G}}+((u_{h,2}^n)_{xx},u_{h,2}^n)_{\mathcal{G}}]+\frac{b}{2}\sum_{j=1}^{N}h_j\sum_{k=1}^{r-1}w_k|u_h^{n+1}(\xi_{j,k})|^2|u_h^n(\xi_{j,k})|^2\\&=E^{n-1}=\cdots=E^0=const,\end{aligned} \tag{43}$$

where $u_{h,1}^n$ and $u_{h,2}^n$ are the real and imaginary parts of u_h^n, respectively.

Theorem 5.2. *Suppose that $\theta\geq 1/4, u(x,t)\in C^{2,4}\cap L^2(H^{r+3})$ is the solution of (1), and $\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial t^2}\in L^2(H^{r+3})$, while $u_h^n\in\mathcal{M}^0(\Lambda)$ $(n=0,1,\ldots,J)$ is the solution of (42). If $W:[0,T]\to\mathcal{M}^0(\Lambda)$ is defined by (9), and $\|(u_h^0-W^0)_t\|_{L^2(\Omega)}^2$, $\|u_h^0-W^0\|_{H^1(\Omega)}^2$ and $\|u_h^1-W^1\|_{H^1(\Omega)}^2$ are $O(\tau^2+h^{r+1})^2$, then, for τ and h sufficiently small, we have*

$$\max_{1\leq n\leq J}\|u^n-u_h^n\|_{L^2(\Omega)}=O(\tau^2+h^{r+1}). \tag{44}$$

Theorem 5.3. *Assume that $\theta\geq 1/4$ and that the conditions of Theorem 5.2 are satisfied. Then scheme (42) is unconditionally stable.*

The proofs of Theorems 5.1–5.3 are quite similar to those of Theorems 3.1, 4.1 and 4.2, respectively, and are omitted.

6. Numerical experiments

In this section, some numerical tests are carried out to verify the performance of our schemes. Generally, it is sufficient to chose $r=3$. Thus, the Gauss points can be given as

$$\xi_{j,k}=x_{j-1}+h_j\zeta_k,\quad j=1,2,\ldots,N,k=1,2,$$

where

$$\zeta_1=(3-\sqrt{3})/6,\qquad \zeta_2=(3+\sqrt{3})/6.$$

Let $\{\phi_j\}_{j=1}^{2N}$ be basis functions of $\mathscr{R}(\mathcal{M}^0(\Lambda))$, so one may write

$$u_h^n(x)=\sum_{j=1}^{2N}\hat{u}_j^n\phi_j^n(x),\quad n=0,1,2,\ldots,J, \tag{45}$$

where $\hat{u}_j^n$ $(j=1,2,\ldots,2N,n=0,1,2,\ldots,J)$ are unknown coefficients which should be worked out. Setting

$$\vec{u}^{\,n}=[\hat{u}_1^n,\hat{u}_2^n,\ldots,\hat{u}_{2N}^n]^\top,$$

and substituting (45) into (40), one could obtain

$$A\vec{u}^{\,n+1(s+1)}=\vec{d}^{\,n+1(s)}, \tag{46}$$

where $\vec{d}^{\,n+1(s)}$ is a vector of $u_h^{n+1(s)}$, u_h^n and u_h^{n-1}. The matrix A has a special structure commonly known as almost block diagonal (ABD; see [15] and references therein), so the system of algebraic equations (46) could be solved by using the COLROW algorithm [21–23]. Scheme (42) could be treated in a similar way.

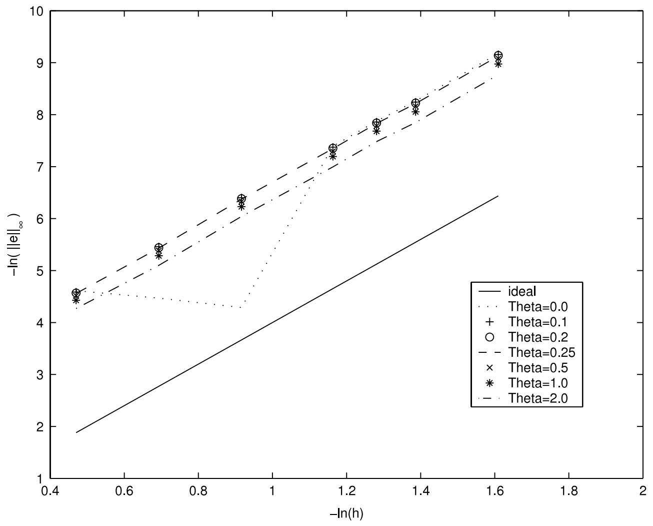

Fig. 1. The curves of convergence order of scheme (13) with various θ.

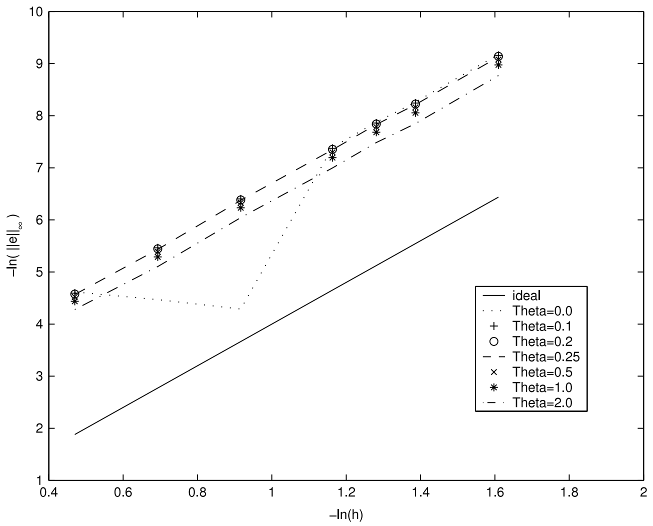

Fig. 2. The curves of convergence order of scheme (42) with various θ.

Throughout the computation, we set

$$\|e\|_\infty = \max_{1\le n\le J} \|e^n\|_\infty = \max_{0\le j\le N,1\le n\le J} |u_h^n(x_j) - u(x_j, t_n)|,$$

and $s \le 10$ for scheme (40).

Consider the soliton solution of Eq. (38) with $b = -2$ as follows [7]:

$$u(x, t) = A\,\mathrm{sech}(Kx)e^{\mathrm{i}\Omega t},$$

where

$$A = |K|, \qquad \Omega = \frac{1}{2}(-1 \pm \sqrt{1-4K^2}).$$

Consequently, the initial condition (2) might be given as

$$u_0(x) = A\,\mathrm{sech}(Kx), \qquad u_1(x) = \mathrm{i}\Omega A\,\mathrm{sech}(Kx).$$

Hereafter, we take $K = 1/4$ and $\Omega = -1/2 - \sqrt{3}/4$.

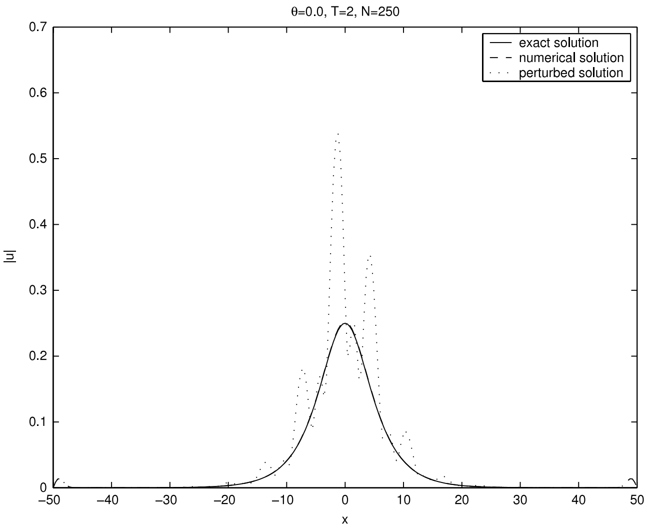

Fig. 3. $|u|$ of scheme (13) with $\theta = 0.0$, $N = 250$ at $T = 2$.

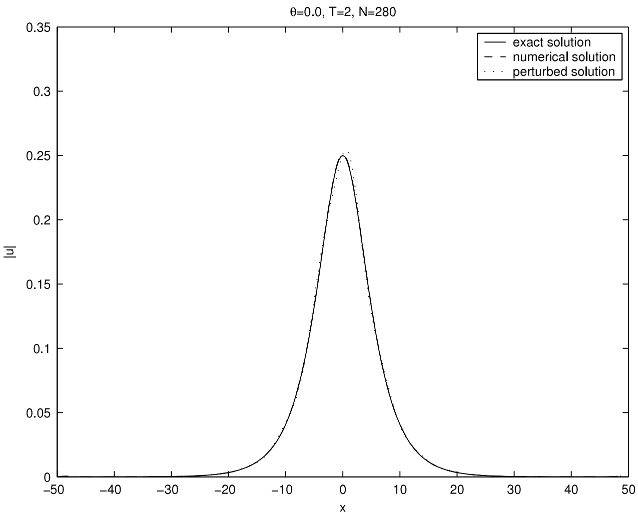

Fig. 4. $|u|$ of scheme (13) with $\theta = 0.0$, $N = 280$ at $T = 2$.

6.1. Convergence

First, we verify the convergence orders of scheme (13) and (42) which are stated in Theorems 4.1 and 5.2. We choose $x_L = -50$, $x_R = 50$, $T = 2$, $\tau = h^2$, $h = (x_R - x_L)/N$ and $N = 160,\ 200, 250, 320, 360, 400$ and 500.

Figs. 1 and 2, respectively, plot the curves of convergence orders of schemes (13) and (42) with different values of θ, where the solid lines are used for reference; their slopes are exactly 4. One can conclude from Fig. 1 that scheme (13) is convergent of order $O(\tau^2 + h^4)$ when $\theta \geq 0.1$, and when $\theta = 0$, this scheme is convergent of that fixed order if $N \geq 320$. The same conclusion could be made for scheme (42) from Fig. 2.

We are amazed at this phenomenon because the theoretical analysis declares that these schemes should be convergent of order $O(\tau^2 + h^4)$ if $\theta \geq 1/4$. Up to now, the only cause we can think of is that the condition $\theta \geq 1/4$ in Theorems 4.1 and 5.2 might be too severe and unnecessary. We expect that there is some way to improve the theoretical analysis and to find a more reasonable condition instead of the severe one.

6.2. Stability

In this subsection, we attempt to investigate the stability of the two OSC schemes. So we slightly perturb the initial function $u_h^0(x)$ as follows:

$$\tilde{u}_h^0(x) = u_0(x)\left[1 + \frac{\sin(x)}{1000}\right]. \tag{47}$$

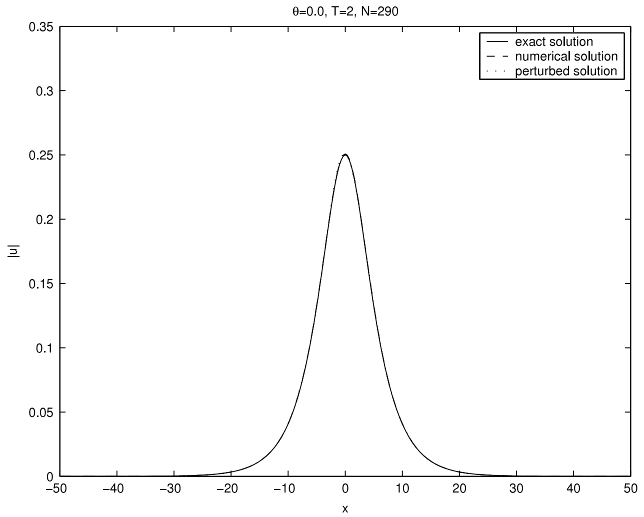

Fig. 5. $|u|$ of scheme (13) with $\theta = 0.0$, $N = 290$ at $T = 2$.

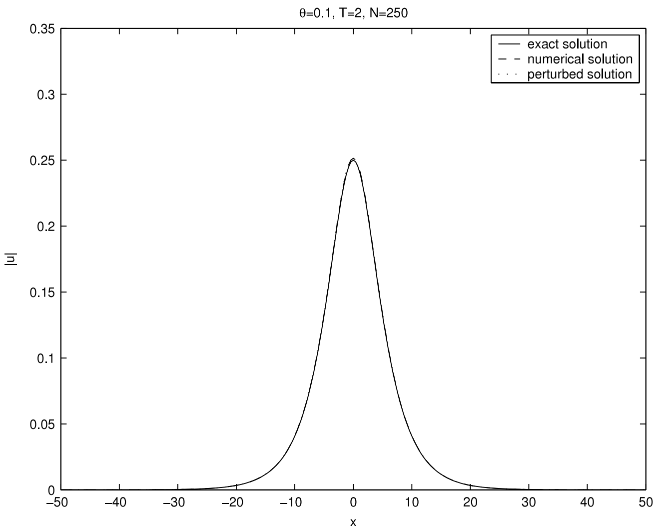

Fig. 6. $|u|$ of scheme (13) with $\theta = 0.1$, $N = 250$ at $T = 2$.

Numerical solutions are calculated by the OSC schemes proposed in this paper with u_h^0 and $\tilde{u}_h^0$, respectively. Then the numerical solution u_h^n and the perturbed one $\tilde{u}_h^n$ are compared. The problem is solved for $x \in [-50, 50]$ and $t \in [0, 2]$, and we choose $\tau = h^2$.

Figs. 3–6 plot the modulus of the exact, the numerical, and the perturbed numerical solutions at $T = 2$, and the approximate solutions are computed by scheme (13) with $\theta = 0.0$ and 0.1. The plots with $\theta = 0.2, 0.25, 0.5, 1.0, 2.0$ and $N = 250$ at $T = 2$ are similar to Fig. 6. Therefore, we may conclude from these figures that scheme (13) with $\theta = 0.0$ should be unstable when τ/h is greater than 100/290, and this scheme with $\theta \geq 0.1$ might be stable. Similar plots could be given by scheme (42), so the same conclusion can be made. Again the phenomenon when $\theta < 1/4$ is outside the theoretical analysis, i.e., Theorems 4.2 and 5.3.

6.3. Long-term computation

Hereafter, we compute the numerical solution within $[x_L, x_R] = [-50, 50]$ until $T = 10$, and we choose $h = 0.2$ and $\tau = 0.04$. Comparisons of numerical results of schemes (13) and (42) with various θ are listed in Tables 1 and 2, respectively, where $\mathrm{d}E = |E^n - E^0|/|E^0|$. It can be seen from these tables that schemes (13) and (42) are efficient for long-term computation either for $\theta \geq 1/4$ or for $\theta < 1/4$, and the linear scheme (42) takes much less time than the nonlinear one (13), though they have the same accuracy. Each scheme preserves its conserved quantity perfectly.

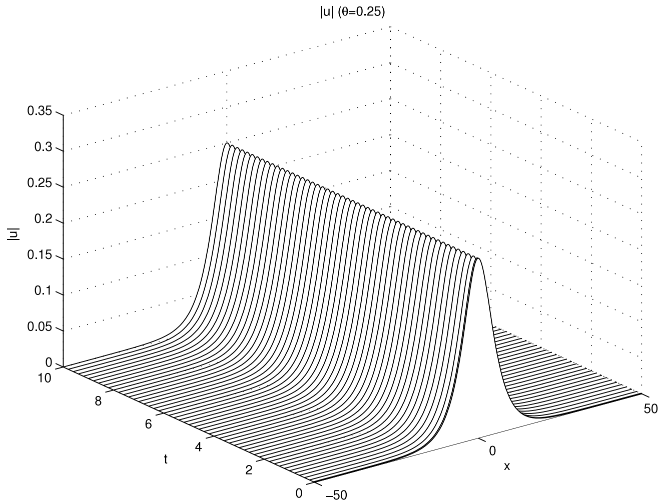

Fig. 7. The solitary wave $|u|$ of scheme (13) with $\theta = 0.25$.

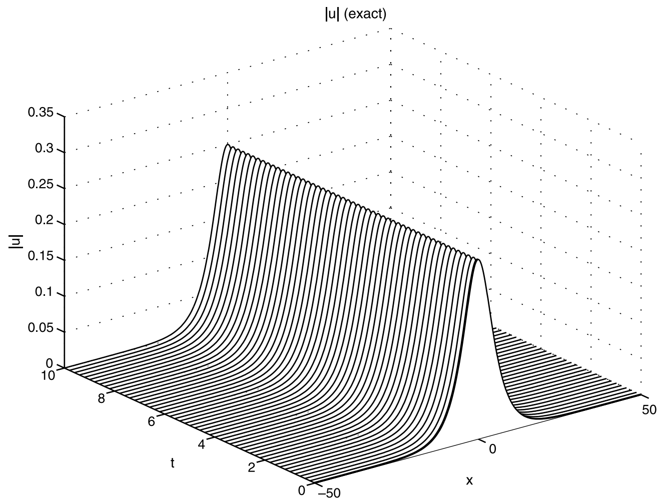

Fig. 8. The solitary wave $|u|$ of the exact solution.

Table 1
Comparisons of scheme (13) with various θ.

	$\theta = 0.0$	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 1.0$	$\theta = 2.0$
$\|e\|_\infty$	3.9670e−4	4.1466e−4	4.3264e−4	4.4165e−4	4.8683e−4	5.7783e−4	7.6216e−4
dE	3.1393e−9	2.9922e−9	2.8675e−9	2.8118e−9	2.5831e−9	2.2823e−9	1.9772e−9
CPU(s)	801.515	801.735	809.266	806.515	804.360	807.468	812.109

Table 2
Comparisons of scheme (42) with various θ.

	$\theta = 0.0$	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.25$	$\theta = 0.5$	$\theta = 1.0$	$\theta = 2.0$
$\|e\|_\infty$	3.9674e−4	4.1469e−4	4.3268e−4	4.4169e−4	4.8687e−4	5.7787e−4	7.6222e−4
dE	3.1399e−9	2.9929e−9	2.8681e−9	2.8125e−9	2.5839e−9	2.2831e−9	1.9784e−9
CPU(s)	459.500	447.281	449.343	450.110	449.562	463.171	463.968

The solitary wave computed by scheme (13) with $\theta = 0.25$ is plotted in Fig. 7, and the exact soliton is shown in Fig. 8. The plots given by schemes (13) and (42) with different values of θ are quite similar.

7. Conclusion

In this paper, discrete-time OSC schemes have been constructed for the nonlinear Schrödinger equation with wave operator. The conservation, convergence, and stability of these methods have been analyzed. Finally, a great many numerical tests have been carried out to confirm the theoretical results.

It is worth mentioning that the present theoretical analysis could only predict the state when $\theta \geq 1/4$; see also [14,16]. However, in our numerical experiments, the phenomenon when $\theta < 1/4$ is fascinating. When $\theta = 0.1$ and 0.2, the numerical exhibition is as good as that when $\theta \geq 1/4$, so we believe that the theoretical results might be generalized to $\theta > 0$. And when $\theta = 0$, the situation is more complicated. Therefore, further strict theoretical work is necessary.

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