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Analysis of some new conservative schemes for nonlinear Schrödinger equation with wave operator ☆

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Abstract

Some new conservative finite difference schemes are presented for an initial-boundary value problem of Schrödinger equation with wave operator. They have the advantages that there are some discrete energies which are conserved respectively. The existence of the solution of the finite difference schemes are proved by Leray–Schauder fixed point theorem. And the uniqueness, stability and convergence of difference solutions with order $O(h^2+\tau^2)$ are proved in the energy norm. Results of numerical experiment demonstrate the efficiency of the new scheme.

Keywords: Schrödinger equation; Difference scheme; Conservation; Convergence

1. Introduction

In paper [1] and its references, the following initial-boundary value problem of Schrödinger equation with wave operator is discussed:

$$u_{tt}-u_{xx}+\mathrm{i}\alpha u_t+\beta(x)q(|u|^2)u=0 \quad (X_1<x<X_\mathrm{r},0<t<T), \tag{1.1}$$

$$u(x,0)=u_0(x),u_t|_{t=0}=u_1(x) \quad (X_1\leqslant x\leqslant X_\mathrm{r}), \tag{1.2}$$

$$u|_{x=X_1}=u|_{x=X_\mathrm{r}}=0 \quad (0\leqslant t\leqslant T), \tag{1.3}$$

where $u_{tt}=\frac{\partial^2u}{\partial t^2}, u_{xx}=\frac{\partial^2u}{\partial x^2}, u_x=\frac{\partial u}{\partial x}$, $u(x,t)$ is a complex function, α is a real constants, $\beta(x)$ and $q(x)$ are real functions, and $\mathrm{i}^2=-1$.

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Computing the inner product of (1.1) with u_t and then taking the real part, we can obtain the following conservative law:

$$\|u_t\|_{L_2}^2 + \|u_x\|_{L_2}^2 + \int_{X_1}^{X_r} \beta(x)Q(|u|^2)\,\mathrm{d}x = \mathrm{Const}, \tag{1.4}$$

where $Q(s) = \int_0^s q(r)\,\mathrm{d}r$.

An implicit nonconservative finite difference scheme was proposed in [1], which needs lots of algebraic operators. An explicit conservative finite difference scheme were constructed by us in [2], but which is conditionally stable and needs another scheme to begin computing. It is known that the conservative schemes are better than the nonconservative ones for cubic nonlinear Schödinger equation. Zhang et al. [3] point out that the nonconservative schemes may easily show nonlinear blow up, and they presented a conservative scheme for nonlinear Schrödinger equation. In [4] Li and Vu-quoc said "in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation". In [5–13] the conservative finite difference schemes were used for a system of the generalized nonlinear Schrödinger equations, Regularized long wave equations, Sine–Gordon equation, Klein–Gordon equation, and Zakharov equations, respectively. Numerical results of all the schemes are very good. Thus, the purpose of this paper is to construct some new conservative difference schemes which are unconditionally stable and more accurate, and prove the convergence of difference solutions.

The paper is organized as follows. In Section 2, a new conservative schemes (i.e. Scheme A) is proposed, and the existence of difference solution is proved by Leray–Schauder fixed point theorem. In Section 3, the discrete conservative laws of the difference scheme is discussed. In Section 4, some prior estimates for numerical solutions are made. In Section 5, the convergence and stability for the new schemes are proved, and the proof of uniqueness of the difference solution is given. In Section 6, we construct some other conservative schemes and discuss there discrete conservative laws respectively. In the last section, various numerical results will be discussed.

2. Finite difference scheme and existence of difference solution

In this section, we describe a new difference schemes for problems (1.1)–(1.3). As usual, the following notations are used:

$$x_j = X_1 + jh, \quad t_n = n\tau, \qquad j = 0, 1, \ldots, J, \;\; n = 0, 1, \ldots, N = [T/\tau],$$

where $h = \frac{X_r - X_1}{J}$ and τ denote the spatial and temporal mesh sizes respectively, $u_j^n \equiv u(x_j, t_n)$, $U_j^n \sim u(x_j, t_n)$.

$$(V_j^n)_x = \frac{V_{j+1}^n - V_j^n}{h}, \qquad (V_j^n)_{\bar{x}} = \frac{V_j^n - V_{j-1}^n}{h}, \qquad (V_j^n)_t = \frac{V_j^{n+1} - V_j^n}{\tau}, \qquad (V_j^n)_{\bar{t}} = \frac{V_j^n - V_j^{n-1}}{\tau},$$

$$(V_j^n)_{\hat{t}} = \frac{1}{2}((V_j^n)_t + (V_j^n)_{\bar{t}}), \qquad (U^n, V^n) = h\sum_{j=1}^{J} U_j^n \overline{V_j^n}, \qquad \|V^n\|^2 = (V^n, V^n), \qquad \|V^n\|_\infty = \max_{1\leqslant j\leqslant J} |V_j^n|,$$

and in the paper, C denotes a general positive constant which may have different values in different occurrences.

We consider the following finite difference scheme for problems (1.1)–(1.3):

Scheme A

$$(U_j^n)_{t\bar{t}} + \frac{h^2}{12}(U_j^n)_{x\bar{x}t\bar{t}} - \frac{1}{2}(U_j^{n+1} + U_j^{n-1})_{x\bar{x}} + \mathrm{i}\alpha(U_j^n)_{\hat{t}} + \mathrm{i}\alpha\frac{h^2}{12}(U_j^n)_{x\bar{x}\hat{t}} + \beta_j \frac{Q(|U_j^{n+1}|^2) - Q(|U_j^{n-1}|^2)}{|U_j^{n+1}|^2 - |U_j^{n-1}|^2}$$
$$\times \frac{U_j^{n+1} + U_j^{n-1}}{2} = 0, \quad j = 1, 2, \ldots, J-1; \;\; n = 1, 2, \ldots, [T/\tau], \tag{2.1}$$

$$U_j^0 = u_0(x_j), \quad (U_j^0)_{\hat{t}} = u_1(x_j), \qquad j = 0, 1, 2, \ldots, J, \tag{2.2}$$

$$U_0^n = U_J^n = 0, \quad n = 0, 1, 2, \ldots, [T/\tau], \tag{2.3}$$

where $\beta_j = \beta(x_j)$. From (2.1) and (2.2), we obtain

$$\frac{2}{\tau^2}[U_j^1 - \tau u_1(x_j) - u_0(x_j)] + \frac{h^2}{6\tau^2}[U_j^1 - \tau u_1(x_j) - u_0(x_j)]_{x\bar{x}} - [U_j^1 - \tau u_1(x_j)]_{x\bar{x}} + \mathrm{i}\alpha u_1(x_j) + \frac{\mathrm{i}\alpha h^2}{12}(u_1(x_j))_{x\bar{x}}$$
$$+ \beta_j \frac{Q(|U_j^1|^2) - Q(|U_j^1 - 2\tau u_1(x_j)|^2)}{|U_j^1|^2 - |U_j^1 - 2\tau u_1(x_j)|^2}(U_j^1 - \tau u_1(x_j)) = 0. \tag{2.2'}$$

Now we are going to prove the existence of difference solutions U^{n+1} for the finite difference systems (2.1)–(2.3). For any mesh function ϕ which define on $[X_l, X_r]$, and $\phi|_{x=X_1} = \phi|_{x=X_r} = 0$, we define a mesh function Φ as follows:

$$[(\Phi_j - U_j^n) - (U_j^n - U_j^{n-1})] + \frac{h^2}{12}[(\Phi_j - U_j^n) - (U_j^n - U_j^{n-1})]_{x\bar{x}} - \frac{\tau^2}{2}[(\phi_j - U_j^n) - (U_j^n - U_j^{n-1})]_{x\bar{x}}$$
$$+ \frac{\mathrm{i}\alpha\tau}{2}[(\phi_j - U_j^n) + (U_j^n - U_j^{n-1})] + \frac{\mathrm{i}\alpha\tau h^2}{24}[(\phi_j - U_j^n) + (U_j^n - U_j^{n-1})]_{x\bar{x}}$$
$$+ \beta_j \frac{Q(|\phi_j|^2) - Q(|U_j^{n-1}|^2)}{|\phi_j|^2 - |U_j^{n-1}|^2}\frac{[(\phi_j - U_j^n) - (U_j^n - U_j^{n-1})]}{2} = 0, \quad j = 1,2,\ldots,J-1;\ \ n = 1,2,\ldots,[T/\tau], \tag{2.4}$$

where $\Phi|_{x=X_1} = \Phi|_{x=X_r} = 0$. It defines a mapping $\Phi = T(\phi)$ of H^1 into itself. Obviously, the mapping $T(\phi)$ is continuous for any $\phi \in H^1$. In order to obtain the existence of the solutions for the finite difference systems (2.1)–(2.3), it is sufficient to prove the uniform boundedness for all the possible fixed point Φ for the mapping λT with respect to the parameter $0 \leqslant \lambda \leqslant 1$ by Leray–Schauder fixed point theorem. Then the fixed point Φ of the mapping λT satisfy that

$$[(\Phi_j - U_j^n) - \lambda(U_j^n - U_j^{n-1})] + \frac{h^2}{12}[(\Phi_j - U_j^n) - \lambda(U_j^n - U_j^{n-1})]_{x\bar{x}} - \frac{\lambda\tau^2}{2}[(\Phi_j - U_j^n) - (U_j^n - U_j^{n-1})]_{x\bar{x}}$$
$$+ \frac{\mathrm{i}\lambda\alpha\tau}{2}[(\Phi_j - U_j^n) + (U_j^n - U_j^{n-1})] + \frac{\mathrm{i}\lambda\alpha\tau h^2}{24}[(\Phi_j - U_j^n) + (U_j^n - U_j^{n-1})]_{x\bar{x}}$$
$$+ \lambda\tau^2\beta_j \frac{Q(|\Phi_j|^2) - Q(|U_j^{n-1}|^2)}{|\Phi_j|^2 - |U_j^{n-1}|^2}\frac{[(\Phi_j - U_j^n) - (U_j^n - U_j^{n-1})]}{2} = 0, \quad j = 1,2,\ldots,J-1;\ \ n = 1,2,\ldots,[T/\tau]. \tag{2.5}$$

Computing the inner product of difference equation (2.5) with $(\Phi - U^{n-1}) = [(\Phi - U^n) + (U^n - U^{n-1})]$, and then taking the real part in the resulting formula, we obtain

$$\|\Phi - U^n\|^2 - \frac{h^2}{12}\|\Phi_x - U_x^n\|^2 + \frac{\lambda\tau^2}{2}\|\Phi_x - U_x^n\|^2 + \frac{\lambda\tau^2 h}{2}\sum_{j=1}^{J-1}\beta_j Q(|\Phi_j|^2)$$
$$\leqslant (\lambda - 1)\mathrm{Re}((U^n - U^{n-1}), \Phi - U^n) + \lambda\|U^n - U^{n-1}\|^2 \frac{(1-\lambda)h^2}{12}\mathrm{Re}((U_x^n - U_x^{n-1}), \Phi_x - U_x^n)$$
$$- \frac{\lambda h^2}{12}\|U^n - U^{n-1}\|^2 + \frac{\lambda\tau^2}{2}\|U^n - U^{n-1}\|^2 + \frac{\lambda\tau^2 h}{2}\sum_{j=1}^{J-1}\beta_j Q(|U_j^{n-1}|^2)$$
$$\leqslant \frac{1}{4}\|\Phi - U^n\|^2 + \|U^n - U^{n-1}\|^2 + \|U^n - U^{n-1}\|^2 + \frac{h^2}{24}\|(U_x^n - U_x^{n-1})\|^2 + \frac{h^2}{24}\|\Phi_x - U_x^n\|^2$$
$$+ \frac{\tau^2}{2}\|U^n - U^{n-1}\|^2 + \frac{\tau^2 h}{2}\sum_{j=1}^{J-1}\beta_j Q(|U_j^{n-1}|^2), \tag{2.6}$$

according to $\|\Phi_x - U_x^n\|^2 \leqslant \frac{4}{h^2}\|\Phi - U^n\|^2$, under the conditions of Theorem 5.1, formula (2.6) implies that

$$\|\Phi - U^n\|^2 \leqslant C_1\|U^n - U^{n-1}\|^2 + 2\tau^2 h\sum_{j=1}^{J-1}\beta_j Q(|U_j^{n-1}|^2), \tag{2.7}$$

where $C_1 = 28/3 + 2\tau^2$. It implies that

$$\|\Phi\| \leqslant [C_1\|U^n - U^{n-1}\|^2 + 2\tau^2 h\sum_{j=1}^{J-1}\beta_j Q(|U_j^{n-1}|^2)]^{1/2} + \|U^n\|. \tag{2.8}$$

This means that $\|\Phi\|$ is uniformly bounded with the parameter $0 \leqslant \lambda \leqslant 1$ respectively. Thus the solution of the finite difference systems (2.1)–(2.3). The uniqueness of difference solution will be proved in Section 5.

3. Discrete conservative laws of new scheme

To obtain the discrete conservative laws, we introduce the following lemmas:

Lemma 3.1. *For any two mesh functions U_j, V_j, there is the identity*

$$\sum_{j=1}^{J-1} U_j(V_j)_{x\bar{x}} = -\sum_{j=1}^{J-1}(U_j)_x(V_j)_x - U_0(V_0)_x + U_J(V_J)_{\bar{x}}. \tag{3.1}$$

Lemma 3.2. *For all mesh functions U_j satisfied Eq.* (2.3), *the following equalities hold*:

$$\mathrm{Re}(U_{t\bar{t}}^n, U_{\hat{t}}^n) = \frac{1}{2}(\|U_t^n\|^2)_{\bar{t}}, \tag{3.2}$$

$$\mathrm{Re}(U_{x\bar{x}}^{n+1} + U_{x\bar{x}}^{n-1}, U_{\hat{t}}^n) = -(\|U_x^n\|^2)_{\hat{t}}, \tag{3.3}$$

$$\mathrm{Re}(U_{x\bar{x}t\bar{t}}^n, U_{\hat{t}}^n) = -\frac{1}{2}(\|U_{xt}^n\|^2)_{\bar{t}}, \tag{3.4}$$

$$\mathrm{Re}(U_{x\bar{x}\hat{t}}^n, U_{\hat{t}}^n) = -\|U_{x\hat{t}}^n\|^2. \tag{3.5}$$

Proof.

$$\mathrm{Re}(U_{t\bar{t}}^n, U_{\hat{t}}^n) = \frac{1}{2\tau}\mathrm{Re}(U_t^n - U_t^{n-1}, U_t^n + U_t^{n-1}) = \frac{1}{2}(\|U_t^n\|^2)_{\bar{t}},$$

$$\begin{aligned}\mathrm{Re}(U_{x\bar{x}}^{n+1} + U_{x\bar{x}}^{n-1}, U_{\hat{t}}^n) &= \frac{1}{2\tau}\mathrm{Re}(U_{x\bar{x}}^{n+1} + U_{x\bar{x}}^{n-1}, U^{n+1} - U^{n-1})\\ &= -\frac{1}{2\tau}\mathrm{Re}(U_x^{n+1} + U_x^{n-1}, U_x^{n+1} - U_x^{n-1})\\ &= -\frac{1}{2\tau}(\|U_x^{n+1}\|^2 - \|U_x^{n-1}\|^2) = -(\|U_x^n\|^2)_{\hat{t}},\end{aligned}$$

$$\mathrm{Re}(U_{x\bar{x}t\bar{t}}^n, U_{\hat{t}}^n) = -\mathrm{Re}(U_{xt\bar{t}}^n, U_{x\hat{t}}^n) = -\frac{1}{2}(\|U_{xt}^n\|^2)_{\bar{t}},$$

$$\mathrm{Re}(U_{x\bar{x}\hat{t}}^n, U_{\hat{t}}^n) = -\mathrm{Re}(U_{x\hat{t}}^n, U_{x\hat{t}}^n) = -\|U_{x\hat{t}}^n\|^2.$$

This completes the proof of Lemma 3.2. □

Theorem 3.1. *The difference scheme* (2.1)–(2.3) *admits the following invariant*:

$$E^n = \|U_t^n\|^2 + \frac{1}{2}(\|U_x^n\|^2 + \|U_x^{n-1}\|^2) - \frac{h^2}{12}\|U_{xt}^n\|^2 + \frac{h}{2}\sum_{j=1}^{J}\beta_j[Q(|U_j^n|^2) + Q(|U_j^{n+1}|^2)] = \cdots = E^0. \tag{3.6}$$

Proof. Computing the inner product of difference equation (2.1) with $2U^n_{\hat{t}}$, and then taking the real part in the resulting formula, we obtain

$$(\|U^n_t\|^2)_{\bar{t}} + (\|U^n_x\|^2)_{\hat{t}} - \frac{h^2}{12}(\|U^n_{xt}\|^2)_{\bar{t}} + h\sum_{j=1}^{J}\beta_j[Q(|U^n_j|^2)]_{\hat{t}} = 0, \tag{3.7}$$

where Lemma 3.2 and the boundary conditions (2.3) are used. Let

$$E^n = \|U^n_t\|^2 + \frac{1}{2}(\|U^n_x\|^2 + \|U^{n+1}_x\|^2) - \frac{h^2}{12}\|U^n_{xt}\|^2 + \frac{h}{2}\sum_{j=1}^{J}\beta_j[Q(|U^n_j|^2) + Q(|U^{n+1}_j|^2)].$$

Then Eq. (3.6) is gotten from (3.7). Theorem 3.1 is now proved. □

4. Some priori estimates for the difference solution

In this section, we will estimate the difference solution. First, three lemmas are introduced from [2,14].

Lemma 4.1. *Suppose that* $u_0(x) \in H^1_0[X_1, X_r]$, $u_1(x) \in L_2$, $\beta(x) > 0$, $Q(s) > 0$, $s \in [0, +\infty]$, $\beta(x)$, $q'(s) \in C^1$, *there is the estimation for the solution of the initial-boundary value problems (1.1)–(1.3),*

$$\|u\|_{L^2} \leqslant C, \qquad \|u_x\|_{L^2} \leqslant C, \qquad \|u\|_{L^\infty} \leqslant C. \tag{4.1}$$

Lemma 4.2 (Discrete Sobolev inequality [14]). *For any discrete function* $u_h = \{u_j | j = 0,1,\ldots,J\}$ *in the real axis and for any given* $\varepsilon > 0$, *there exists a constant* K *dependent on* ε *and* n *such that*

$$\|u_h\|_\infty \leqslant \varepsilon\|(u_h)_x\| + K\|u_h\|.$$

Lemma 4.3 [14]. *Suppose that discrete function* $w(n)$ *satisfies the recurrence formula*

$$w_n - w_{n-1} \leqslant A\tau w_n + B\tau w_{n-1} + C_n\tau,$$

where A, B *and* C_n $(n = 1,\ldots,N)$ *are nonnegative constants. Then*

$$\max_{1\leqslant n\leqslant N}|w_n| \leqslant \left(w_0 + \tau\sum_{k=1}^{N}C_k\right)\mathrm{e}^{2(A+B)T},$$

where τ *is small, such that* $(A+B)\tau \leqslant \frac{N-1}{2N}$ $(N > 1)$.

Lemma 4.4. *Suppose that* $u_0(x) \in H^1_0[X_1, X_r]$, $u_1(x) \in L_2$, $\beta(x) > 0$, $Q(s) > 0$, $s \in [0, +\infty]$, $\beta(x)$, $q'(s) \in C^1$. *Then the following estimates hold*:

$$\|U^n\| \leqslant C, \qquad \|U^n_x\| \leqslant C, \qquad \|U^n\|_\infty \leqslant C. \tag{4.2}$$

Proof. Using Young's inequality, we obtain

$$\begin{aligned}
-\frac{h^2}{12}\|U^n_{xt}\|^2 &= -\frac{h^3}{12}\sum_{j=1}^{J-1}[(U^n_j)_{xt}]^2 = -\frac{h}{12}\sum_{j=1}^{J-1}[(U^n_{j+1})_t - (U^n_j)_t]^2 \\
&\geqslant -\frac{h}{6}\sum_{j=0}^{J-1}\{[(U^n_{j+1})_t]^2 + [(U^n_j)_t]^2\} = -\frac{1}{3}\|U^n_t\|^2,
\end{aligned} \tag{4.3}$$

then from (3.6) we obtain

$$\begin{aligned}
\frac{2}{3}\|U^n_t\|^2 + \frac{1}{2}(\|U^n_x\|^2 + \|U^{n+1}_x\|^2) &\leqslant \|U^n_t\|^2 + \frac{1}{2}(\|U^n_x\|^2 + \|U^{n+1}_x\|^2) - \frac{h^2}{12}\|U^n_{xt}\|^2 s \\
&\quad + \frac{h}{2}\sum_{j=1}^{J}\beta_j[Q(|U^n_j|^2) + Q(|U^{n+1}_j|^2)] = C.
\end{aligned} \tag{4.4}$$

From (4.4) we obtain

$$\|U_t^n\| \leqslant C, \qquad \|U_x^n\| \leqslant C$$

from $\|U_t^n\| \leqslant C$, we obtain (see [2]) $\|U^n\| \leqslant C$. It follows from Lemma 3.2 that

$$\|U^n\|_\infty \leqslant C.$$

Therefore Lemma 4.4 is proved. □

5. Convergence and stability of the difference scheme

Now, we consider the convergence of the difference schemes (2.1)–(2.3). First, we define the truncation error as follows:

$$\begin{aligned} Er_j^n &= (u_j^n)_{\bar{t}t} + \frac{h^2}{12}(u_j^n)_{x\bar{x}t\bar{t}} - \frac{1}{2}(u_j^{n+1} + u_j^{n-1})_{x\bar{x}} + \mathrm{i}\alpha(u_j^n)_{\hat{t}} + \mathrm{i}\alpha\frac{h^2}{12}(u_j^n)_{x\bar{x}\hat{t}} \\ &\quad + \beta_j \frac{Q(|u_j^{n+1}|^2) - Q(|u_j^{n-1}|^2)}{|u_j^{n+1}|^2 - |u_j^{n-1}|^2}\frac{u_j^{n+1} + u_j^{n-1}}{2}. \end{aligned} \tag{5.1}$$

According to Taylor's expansion, we obtain

Lemma 5.1. *Assume that $u \in C^{4,3}$, then the truncation errors of the difference schemes (2.1)–(2.3) satisfy*

$$Er_j^n = \mathrm{O}(h^2 + \tau^2). \tag{5.2}$$

Remark 5.1. For the introduction of the items of $\frac{h^2}{12}(U_j^n)_{x\bar{x}t\bar{t}}$ and $\mathrm{i}\alpha\frac{h^2}{12}(U_j^n)_{x\bar{x}\hat{t}}$ in the schemes (2.1)–(2.3), it is easy to know that the Scheme A is more accurate than the Scheme C in Section 6 which is without adding the items above by Taylor's expansion.

Theorem 5.1. *Assume that $u_0(x) \in H_0^1[X_1, X_r]$, $u_1(x) \in L_2$, $\beta(x) > 0$, $Q(s) > 0$, $s \in [0, +\infty]$, $\beta(x)$, $q'(s) \in C^1$ and $u \in C^{4,3}$, then the solution of the difference problem (2.1)–(2.3) is unique.*

Proof. Let $\epsilon^n = W^n - U^n$, where W^n and U^n are two solutions of the difference Scheme A with initial value W^0, W^1 and U^0, U^1 respectively, and W^0, W^1, U^0, U^1 satisfy

$$\|W^0\|_\infty \leqslant K, \qquad \|W^1\|_\infty \leqslant K, \qquad \|U^0\|_\infty \leqslant K, \qquad \|U^1\|_\infty \leqslant K.$$

By Lemma 4.4, there exists a constant $C(K)$, such that

$$\|W^n\|_\infty \leqslant C(K), \qquad \|U^n\|_\infty \leqslant C(K).$$

Then ϵ^n satisfies that

$$\begin{aligned} &(\epsilon_j^n)_{\bar{t}t} + \frac{h^2}{12}(\epsilon_j^n)_{x\bar{x}t\bar{t}} - \frac{1}{2}(\epsilon_j^{n+1} + \epsilon_j^{n-1})_{x\bar{x}} + \mathrm{i}\alpha(\epsilon_j^n)_{\hat{t}} + \mathrm{i}\alpha\frac{h^2}{12}(\epsilon_j^n)_{x\bar{x}\hat{t}} \\ &\quad + \beta_j G(W_j^n)\frac{W_j^{n+1} + W_j^{n-1}}{2} - \beta_j G(U_j^n)\frac{U_j^{n+1} + U_j^{n-1}}{2} \\ &= (\epsilon_j^n)_{\bar{t}t} + \frac{h^2}{12}(\epsilon_j^n)_{x\bar{x}t\bar{t}} - \frac{1}{2}(\epsilon_j^{n+1} + \epsilon_j^{n-1})_{x\bar{x}} + \mathrm{i}\alpha(\epsilon_j^n)_{\hat{t}} + \mathrm{i}\alpha\frac{h^2}{12}(e_j^n)_{x\bar{x}\hat{t}} \\ &\quad + \beta_j G(W_j^n)\frac{\epsilon_j^{n+1} + \epsilon_j^{n-1}}{2} + \beta_j[G(W_j^n) - G(U_j^n)]\frac{U_j^{n+1} + U_j^{n-1}}{2}, \end{aligned} \tag{5.3}$$

where

$$G(U_j^n) = \frac{Q(|U_j^{n+1}|^2) - Q(|U_j^{n-1}|^2)}{|U_j^{n+1}|^2 - |U_j^{n-1}|^2}.$$

Computing the inner product of (5.3) with $2\epsilon_{\hat{t}}^n$, and then taking the real part, we obtain

$$(\|\epsilon_t^n\|^2)_{\bar{t}} + (\|\epsilon_x^n\|^2)_{\hat{t}} - \frac{h^2}{12}(\|\epsilon_{xt}^n\|^2)_{\bar{t}} + \mathrm{Re}\left\{h\sum_{j=1}^{J-1}\beta_j G(W_j^n)(\epsilon_j^{n+1}+\epsilon_j^{n-1})\bar{\epsilon}_{\hat{t}}^n\right\}$$
$$+\mathrm{Re}\left\{h\sum_{j=1}^{J-1}\beta_j[G(W_j^n)-G(U_j^n)](U_j^{n+1}+U_j^{n-1})\bar{\epsilon}_{\hat{t}}^n\right\} = 0. \tag{5.4}$$

Similarly to the proof in [2], we can obtain that

$$\mathrm{Re}\left\{h\sum_{j=1}^{J-1}\beta_j G(W_j^n)(\epsilon_j^{n+1}+\epsilon_j^{n-1})\bar{\epsilon}_{\hat{t}}^n\right\} \leqslant C(\|\epsilon^{n-1}\|^2+\|\epsilon^{n+1}\|^2+\|\epsilon_t^n\|^2+\|\epsilon_t^{n-1}\|^2), \tag{5.5}$$

$$\mathrm{Re}\left\{h\sum_{j=1}^{J-1}\beta_j[G(W_j^n)-G(U_j^n)](U_j^{n+1}+U_j^{n-1})\bar{\epsilon}_{\hat{t}}^n\right\} \leqslant C(\|\epsilon^{n-1}\|^2+\|\epsilon^{n+1}\|^2+\|\epsilon_t^n\|^2+\|\epsilon_t^{n-1}\|^2), \tag{5.6}$$

and

$$\|\epsilon^n\|_t^2 \leqslant C(\|\epsilon_t^{n-1}\|^2+\|\epsilon^n\|^2+\|\epsilon^{n-1}\|^2). \tag{5.7}$$

Substituting (5.5) and (5.6) into (5.4), then adding the results with the formula (5.7), we get

$$\|\epsilon_t^n\|^2-\|\epsilon_t^{n-1}\|^2+\frac{1}{2}(\|\epsilon_x^{n+1}\|^2-\|\epsilon_x^{n-1}\|^2)+\|\epsilon^n\|^2-\|\epsilon^{n-1}\|^2-\frac{h^2}{12}\|\epsilon_{xt}^n\|^2+\frac{h^2}{12}\|\epsilon_{xt}^{n-1}\|^2$$
$$\leqslant \tau C(\|\epsilon_t^{n-1}\|^2+\|\epsilon_t^n\|^2+\|\epsilon_x^{n-1}\|^2+\|\epsilon^n\|^2+\|\epsilon^{n+1}\|^2), \tag{5.8}$$

and let

$$Q^n = \|\epsilon_t^n\|^2+\frac{1}{2}(\|\epsilon_x^n\|^2+\|\epsilon_x^{n+1}\|^2)+\|\epsilon^n\|^2-\frac{h^2}{12}\|\epsilon_{xt}^n\|^2,$$

it is easy to see that

$$Q^n \geqslant \frac{2}{3}\|\epsilon_t^n\|^2+\frac{1}{2}(\|\epsilon_x^n\|^2+\|\epsilon_x^{n+1}\|^2)+\|\epsilon^n\|^2, \tag{5.9}$$

then we get

$$Q^n - Q^{n-1} \leqslant C\tau(Q^n - Q^{n-1}). \tag{5.10}$$

It follows from Lemma 4.3 that

$$Q^n \leqslant Q^0\exp(CT). \tag{5.11}$$

According to $Q^0 = 0$, we obtain

$$Q^n \leqslant 0. \tag{5.12}$$

From (5.9) and (5.12), using Lemma 4.2, we get $\|\epsilon\|_\infty = 0$, i.e. $W^n = U^n$. Therefore the proof of Theorem 5.1 is completed. □

Theorem 5.2. *Under the conditions of Theorem 5.1, the solution of the difference problem (2.1)–(2.3) converges to the solution of problem (1.1)–(1.3) with order* $\mathrm{O}(h^2+\tau^2)$ *by the* $\|\cdot\|_\infty$ *norm.*

Proof. Let

$$e_j^n = u_j^n - U_j^n.$$

Subtracting (2.1) from (5.1), we obtain

$$\begin{aligned}Er_j^n &= (e_j^n)_{\bar{t}t} + \frac{h^2}{12}(e_j^n)_{x\bar{x}t\bar{t}} - \frac{1}{2}(e_j^{n+1} + e_j^{n-1})_{x\bar{x}} + \mathrm{i}\alpha(e_j^n)_{\hat{t}} + \mathrm{i}\alpha\frac{h^2}{12}(e_j^n)_{x\bar{x}\hat{t}} \\ &\quad + \beta_j G(u_j^n)\frac{u_j^{n+1} + u_j^{n-1}}{2} - \beta_j G(U_j^n)\frac{U_j^{n+1} + U_j^{n-1}}{2} \\ &= (e_j^n)_{\bar{t}t} + \frac{h^2}{12}(e_j^n)_{x\bar{x}t\bar{t}} - \frac{1}{2}(e_j^{n+1} + e_j^{n-1})_{x\bar{x}} + \mathrm{i}\alpha(e_j^n)_{\hat{t}} + \mathrm{i}\alpha\frac{h^2}{12}(e_j^n)_{x\bar{x}\hat{t}} \\ &\quad + \beta_j G(u_j^n)\frac{e_j^{n+1} + e_j^{n-1}}{2} + \beta_j[G(u_j^n) - G(U_j^n)]\frac{U_j^{n+1} + U_j^{n-1}}{2}.\end{aligned} \tag{5.13}$$

Computing the inner product of (5.13) with $2e_{\hat{t}}^n$, and then taking the real part, we obtain

$$\begin{aligned}2\mathrm{Re}(Er^n, e_{\hat{t}}^n) &= (\|e_t^n\|^2)_{\bar{t}} + (\|e_x^n\|^2)_{\hat{t}} - \frac{h^2}{12}(\|e_{xt}^n\|^2)_{\bar{t}} + \mathrm{Re}\left\{h\sum_{j=1}^{J-1}\beta_j G(u_j^n)(e_j^{n+1} + e_j^{n-1})\bar{e}_{\hat{t}}^n\right\} \\ &\quad + \mathrm{Re}\left\{h\sum_{j=1}^{J-1}\beta_j[G(u_j^n) - G(U_j^n)](U_j^{n+1} + U_j^{n-1})\bar{e}_{\hat{t}}^n\right\}.\end{aligned} \tag{5.14}$$

Similarly to the proof in [2], we obtain

$$2\mathrm{Re}(Er^n, e_{\hat{t}}^n) \leqslant \|Er^n\|^2 + \|e_t^{n-1}\|^2 + \|e_t^n\|^2, \tag{5.15}$$

$$\mathrm{Re}\left\{h\sum_{j=1}^{J-1}\beta_j G(u_j^n)(e_j^{n+1} + e_j^{n-1})\bar{e}_{\hat{t}}^n\right\} \leqslant C(\|e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e_t^n\|^2 + \|e_t^{n-1}\|^2), \tag{5.16}$$

$$\mathrm{Re}\left\{h\sum_{j=1}^{J-1}\beta_j[G(u_j^n) - G(U_j^n)](U_j^{n+1} + U_j^{n-1})\bar{e}_{\hat{t}}^n\right\} \leqslant C(\|e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e_t^n\|^2 + \|e_t^{n-1}\|^2), \tag{5.17}$$

and

$$\|e^n\|_t^2 \leqslant C(\|e_t^{n-1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2). \tag{5.18}$$

Substituting (5.15)–(5.17) into (5.14), then adding the results with (5.18), we get

$$\begin{aligned}&\|e_t^n\|^2 - \|e_t^{n-1}\|^2 + \frac{1}{2}(\|e_x^{n+1}\|^2 - \|e_x^{n-1}\|^2) + \|e^n\|^2 - \|e^{n-1}\|^2 - \frac{h^2}{12}\|e_{xt}^n\|^2 + \frac{h^2}{12}\|e_{xt}^{n-1}\|^2 \\ &\quad \leqslant \tau C(\|Er^n\|^2 + \|e_t^{n-1}\|^2 + \|e_t^n\|^2 + \|e_x^{n-1}\|^2 + \|e^n\|^2 + \|e^{n+1}\|^2),\end{aligned} \tag{5.19}$$

let

$$B^n = \|e_t^n\|^2 + \frac{1}{2}(\|e_x^n\|^2 + \|e_x^{n+1}\|^2) + \|e^n\|^2 - \frac{h^2}{12}\|e_{xt}^n\|^2,$$

it is easy to see that

$$B^n \geqslant \frac{2}{3}\|e_t^n\|^2 + \frac{1}{2}(\|e_x^n\|^2 + \|e_x^{n+1}\|^2) + \|e^n\|^2, \tag{5.20}$$

then we get

$$B^n - B^{n-1} \leqslant \tau\|Er^n\|^2 + C\tau(B^n - B^{n-1}). \tag{5.21}$$

It follows from Lemma 4.3 that

$$B^n \leqslant \left(B^0 + \tau\sum_{k=1}^{n}\|Er^k\|^2\right)\exp(CT) \leqslant C(B^0 + (h^2 + \tau^2)^2). \tag{5.22}$$

According to

$$B^0 = \mathrm{O}(h^2+\tau^2)^2,$$

we get

$$B^n \leqslant C(h^2+\tau^2)^2. \tag{5.23}$$

It follows from the definition of B^n and inequality (5.20) that

$$\|e_x^n\| \leqslant \mathrm{O}(h^2+\tau^2), \qquad \|e_t^n\| \leqslant \mathrm{O}(h^2+\tau^2), \qquad \|e^n\| \leqslant \mathrm{O}(h^2+\tau^2).$$

Thanks for Lemma 4.2, we obtain

$$\|e^n\|_\infty \leqslant \mathrm{O}(h^2+\tau^2).$$

Then the convergence is proved. □

Similarly, we can prove the stability of the difference solution, i.e.

Theorem 5.3. *Under the conditions of Theorem 5.1, the solution of the difference schemes (2.1)–(2.3) is unconditionally stable for initial data by $\|\cdot\|_\infty$ norm.*

6. Some other conservative finite difference schemes

In this section, we will construct some other new conservative finite difference schemes for the problems (1.1)–(1.3), and discuss there discrete conservative laws respectively.

Scheme B

$$(U_j^n)_{\bar{t}t} - (U_j^n)_{x\bar{x}} + \mathrm{i}\alpha(U_j^n)_{\hat{t}} + \beta_j \frac{Q(|U_j^{n+1}|^2) - Q(|U_j^{n-1}|^2)}{|U_j^{n+1}|^2 - |U_j^{n-1}|^2}\,\frac{U_j^{n+1}+U_j^{n-1}}{2} = 0,$$

$$j = 1,2,\ldots,J-1;\quad n = 1,2,\ldots,[T/\tau], \tag{6.1}$$

$$U_j^0 = u_0(x_j), \quad (U_j^0)_{\hat{t}} = u_1(x_j), \qquad j = 0,1,2,\ldots,J, \tag{6.2}$$

$$U_0^n = U_J^n = 0, \quad n = 0,1,2,\ldots,[T/\tau]. \tag{6.3}$$

Theorem 6.1. *Scheme B admits the following invariant*:

$$E^n = \|U_t^n\|^2 + \frac{1}{2}(\|U_x^n\|^2 + \|U_x^{n+1}\|^2) - \frac{\tau^2}{2}\|U_{xt}^n\|^2 + \frac{h}{2}\sum_{j=1}^{J}\beta_j[Q(|U_j^n|^2) + Q(|U_j^{n+1}|^2)] = \cdots = E^0. \tag{6.4}$$

Scheme C

$$(U_j^n)_{\bar{t}t} - \frac{1}{2}(U_j^{n+1} + U_j^{n-1})_{x\bar{x}} + \mathrm{i}\alpha(U_j^n)_{\hat{t}} + \beta_j \frac{Q(|U_j^{n+1}|^2) - Q(|U_j^{n-1}|^2)}{|U_j^{n+1}|^2 - |U_j^{n-1}|^2}\,\frac{U_j^{n+1}+U_j^{n-1}}{2} = 0,$$

$$j = 1,2,\ldots,J-1;\quad n = 1,2,\ldots,[T/\tau], \tag{6.5}$$

$$U_j^0 = u_0(x_j), \quad (U_j^0)_{\hat{t}} = u_1(x_j), \qquad j = 0,1,2,\ldots,J, \tag{6.6}$$

$$U_0^n = U_J^n = 0, \quad n = 0,1,2,\ldots,[T/\tau]. \tag{6.7}$$

Theorem 6.2. *Scheme C admits the following invariant*:

$$E^n = \|U_t^n\|^2 + \frac{1}{2}(\|U_x^n\|^2 + \|U_x^{n-1}\|^2) + \frac{h}{2}\sum_{j=1}^{J}\beta_j[Q(|U_j^n|^2) + Q(|U_j^{n+1}|^2)] = \cdots = E^0. \tag{6.8}$$

Scheme D

$$(U_j^n)_{t\bar{t}} - (U_j^n)_{x\bar{x}} + \mathrm{i}\alpha(U_j^n)_{\hat{t}} + \beta_j \frac{Q\left(\frac{|U_j^{n+1}|^2+|U_j^n|^2}{2}\right) - Q\left(\frac{|U_j^n|^2+|U_j^{n-1}|^2}{2}\right)}{|U_j^{n+1}|^2 - |U_j^{n-1}|^2}(U_j^{n+1} + U_j^{n-1}) = 0,$$

$$j = 1,2,\ldots,J-1; \quad n = 1,2,\ldots,[T/\tau], \tag{6.9}$$

$$U_j^0 = u_0(x_j), \quad (U_j^0)_{\hat{t}} = u_1(x_j), \qquad j = 0,1,2,\ldots,J, \tag{6.10}$$

$$U_0^n = U_J^n = 0, \quad n = 0,1,2,\ldots,[T/\tau]. \tag{6.11}$$

Theorem 6.3. *Scheme D admits the following invariant*:

$$E^n = \|U_t^n\|^2 + \frac{1}{2}(\|U_x^n\|^2 + \|U_x^{n+1}\|^2) - \frac{\tau^2}{2}\|U_{xt}^n\|^2 + h\sum_{j=1}^{J}\beta_j Q\left(\frac{|U_j^{n+1}|^2 + |U_j^n|^2}{2}\right) = \cdots = E^0. \tag{6.12}$$

Scheme E

$$(U_j^n)_{t\bar{t}} - \frac{1}{2}(U_j^{n+1} + U_j^{n-1})_{x\bar{x}} + \mathrm{i}\alpha(U_j^n)_{\hat{t}} + \beta_j \frac{Q\left(\frac{|U_j^{n+1}|^2+|U_j^n|^2}{2}\right) - Q\left(\frac{|U_j^n|^2+|U_j^{n-1}|^2}{2}\right)}{|U_j^{n+1}|^2 - |U_j^{n-1}|^2}(U_j^{n+1} + U_j^{n-1}) = 0,$$

$$j = 1,2,\ldots,J-1; \quad n = 1,2,\ldots,[T/\tau], \tag{6.13}$$

$$U_j^0 = u_0(x_j), \quad (U_j^0)_{\hat{t}} = u_1(x_j), \qquad j = 0,1,2,\ldots,J, \tag{6.14}$$

$$U_0^n = U_J^n = 0, \quad n = 0,1,2,\ldots,[T/\tau]. \tag{6.15}$$

Theorem 6.4. *Scheme E admits the following invariant*:

$$E^n = \|U_t^n\|^2 + \frac{1}{2}(\|U_x^n\|^2 + \|U_x^{n+1}\|^2) + h\sum_{j=1}^{J}\beta_j Q\left(\frac{|U_j^{n+1}|^2 + |U_j^n|^2}{2}\right) = \cdots = E^0. \tag{6.16}$$

Scheme F

$$(U_j^n)_{t\bar{t}} + \frac{h^2}{12}(U_j^n)_{x\bar{x}t\bar{t}} - (U_j^n)_{x\bar{x}} + \mathrm{i}\alpha(U_j^n)_{\hat{t}} + \mathrm{i}\alpha\frac{h^2}{12}(U_j^n)_{x\bar{x}\hat{t}} + \beta_j \frac{Q(|U_j^{n+1}|^2) - Q(|U_j^{n-1}|^2)}{|U_j^{n+1}|^2 - |U_j^{n-1}|^2}\,\frac{U_j^{n+1} + U_j^{n-1}}{2} = 0,$$

$$j = 1,2,\ldots,J-1; \quad n = 1,2,\ldots,[T/\tau], \tag{6.17}$$

$$U_j^0 = u_0(x_j), \quad (U_j^0)_{\hat{t}} = u_1(x_j), \qquad j = 0,1,2,\ldots,J, \tag{6.18}$$

$$U_0^n = U_J^n = 0, \quad n = 0,1,2,\ldots,[T/\tau]. \tag{6.19}$$

Theorem 6.5. *Scheme F admits the following invariant*:

$$E^n = \|U_t^n\|^2 + \frac{1}{2}(\|U_x^n\|^2 + \|U_x^{n+1}\|^2) - \frac{\tau^2}{2}\|U_{xt}^n\|^2 - \frac{h^2}{12}\|U_{xt}^n\|^2 + \frac{h}{2}\sum_{j=1}^{J}\beta_j[Q(|U_j^n|^2) + Q(|U_j^{n+1}|^2)]$$

$$= \cdots = E^0. \tag{6.20}$$

Scheme G

$$(U_j^n)_{t\bar{t}} + \frac{h^2}{12}(U_j^n)_{x\bar{x}t\bar{t}} - (U_j^n)_{x\bar{x}} + \mathrm{i}\alpha(U_j^n)_{\hat{t}} + \mathrm{i}\alpha\frac{h^2}{12}(U_j^n)_{x\bar{x}\hat{t}} + \beta_j \frac{Q\left(\frac{|U_j^{n+1}|^2+|U_j^n|^2}{2}\right) - Q\left(\frac{|U_j^n|^2+|U_j^{n-1}|^2}{2}\right)}{|U_j^{n+1}|^2 - |U_j^{n-1}|^2}$$

$$\times (U_j^{n+1} + U_j^{n-1}) = 0, \quad j = 1,2,\ldots,J-1; \quad n = 1,2,\ldots,[T/\tau], \tag{6.21}$$

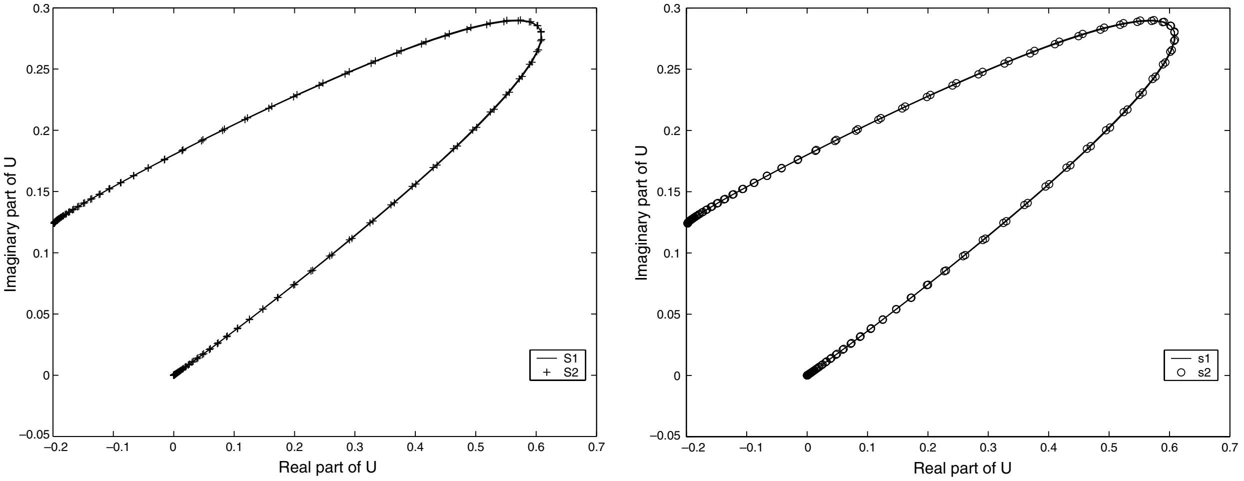

Fig. 1. Comparison of phasic picture of U computed by two schemes: (left) $h = \tau = 0.02$, $-X_l = X_r = 40$, $t = 1$ and (right) $h = \tau = 0.01$, $-X_l = X_r = 40$, $t = 1$.

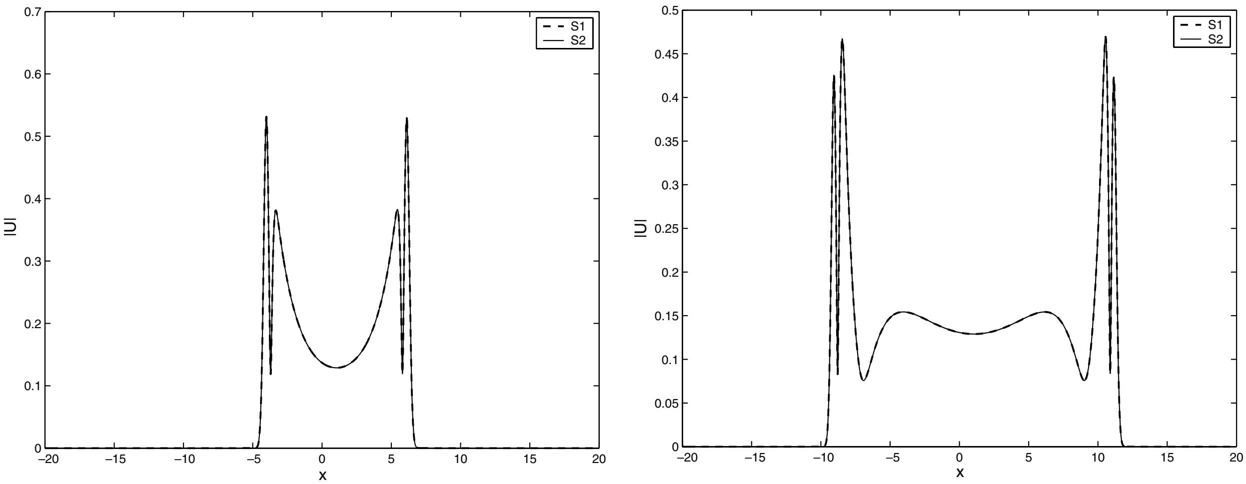

Fig. 2. Comparison of $|U|$ computed by two schemes: (left) $h = \tau = 0.01$, $-X_l = X_r = 20$, $t = 5$ and (right) $h = \tau = 0.01$, $-X_l = X_r = 20$, $t = 10$.

Table 1
Value E^n of two schemes at difference time

t_n	$\tau = h = 0.01$		$\tau = h = 0.02$	
	Scheme 2	Scheme 1	Scheme 2	Scheme 1
0.1	9.12172399599931	9.12172399599931	9.11504169913676	9.11504169913676
0.2	9.12172399599925	9.12172399599925	9.11504169913672	9.11504169913672
0.3	9.12172399599931	9.12172399599931	9.11504169913676	9.11504169913676
0.4	9.12172399599931	9.12172399599931	9.11504169913676	9.11504169913676
0.5	9.12172399599930	9.12172399599930	9.11504169913673	9.11504169913673
0.6	9.12172399599933	9.12172399599933	9.11504169913675	9.11504169913675
0.7	9.12172399599940	9.12172399599940	9.11504169913674	9.11504169913674
0.8	9.12172399599936	9.12172399599936	9.11504169913674	9.11504169913674
0.9	9.12172399599936	9.12172399599936	9.11504169913673	9.11504169913673
1.0	9.12172399599936	9.12172399599936	9.11504169913672	9.11504169913672

$$U_j^0 = u_0(x_j), \quad (U_j^0)_{\hat{t}} = u_1(x_j), \qquad j = 0,1,2,\ldots,J, \tag{6.22}$$

$$U_0^n = U_J^n = 0, \quad n = 0,1,2,\ldots,[T/\tau]. \tag{6.23}$$

Theorem 6.6. *Scheme G admits the following invariant*:

$$E^n = \|U_t^n\|^2 + \frac{1}{2}(\|U_x^n\|^2 + \|U_x^{n+1}\|^2) - \frac{\tau^2}{2}\|U_{xt}^n\|^2 - \frac{h^2}{12}\|U_{xt}^n\|^2 + h\sum_{j=1}^{J}\beta_j Q\left(\frac{|U_j^{n+1}|^2 + |U_j^n|^2}{2}\right) = \cdots = E^0. \tag{6.24}$$

Scheme H

$$(U_j^n)_{t\bar{t}} + \frac{h^2}{12}(U_j^n)_{x\bar{x}t\bar{t}} - \frac{1}{2}(U_j^{n+1} + U_j^{n-1})_{x\bar{x}} + \mathrm{i}\alpha(U_j^n)_{\hat{t}} + \mathrm{i}\alpha\frac{h^2}{12}(U_j^n)_{x\bar{x}\hat{t}}$$
$$+ \beta_j \frac{Q\left(\frac{|U_j^{n+1}|^2+|U_j^n|^2}{2}\right) - Q\left(\frac{|U_j^n|^2+|U_j^{n-1}|^2}{2}\right)}{|U_j^{n+1}|^2 - |U_j^{n-1}|^2}(U_j^{n+1} + U_j^{n-1}) = 0, \quad j = 1,2,\ldots,J-1; \ n = 1,2,\ldots,[T/\tau], \tag{6.25}$$

$$U_j^0 = u_0(x_j), \quad (U_j^0)_{\hat{t}} = u_1(x_j), \qquad j = 0,1,2,\ldots,J, \tag{6.26}$$

$$U_0^n = U_J^n = 0, \quad n = 0,1,2,\ldots,[T/\tau]. \tag{6.27}$$

Theorem 6.7. *Scheme H admits the following invariant*:

$$E^n = \|U_t^n\|^2 + \frac{1}{2}(\|U_x^n\|^2 + \|U_x^{n+1}\|^2) - \frac{h^2}{12}\|U_{xt}^n\|^2 + h\sum_{j=1}^{J}\beta_j Q\left(\frac{|U_j^{n+1}|^2 + |U_j^n|^2}{2}\right) = \cdots = E^0. \tag{6.28}$$

Similarly, we can prove the existence and uniqueness, the stability and convergence of difference solution of Schemes B–H. Finally, we can easily prove that all lemmas and theorems in this paper hold for the periodic initial-value problem for Schrödinger equation with wave operator.

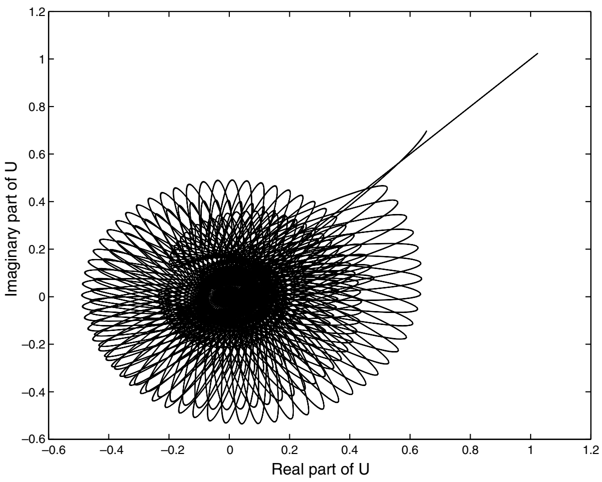

Fig. 3. Phasic picture of U, $h = \tau = 0.02$, $-X_1 = X_r = 40$, $t \in [0, 20]$.

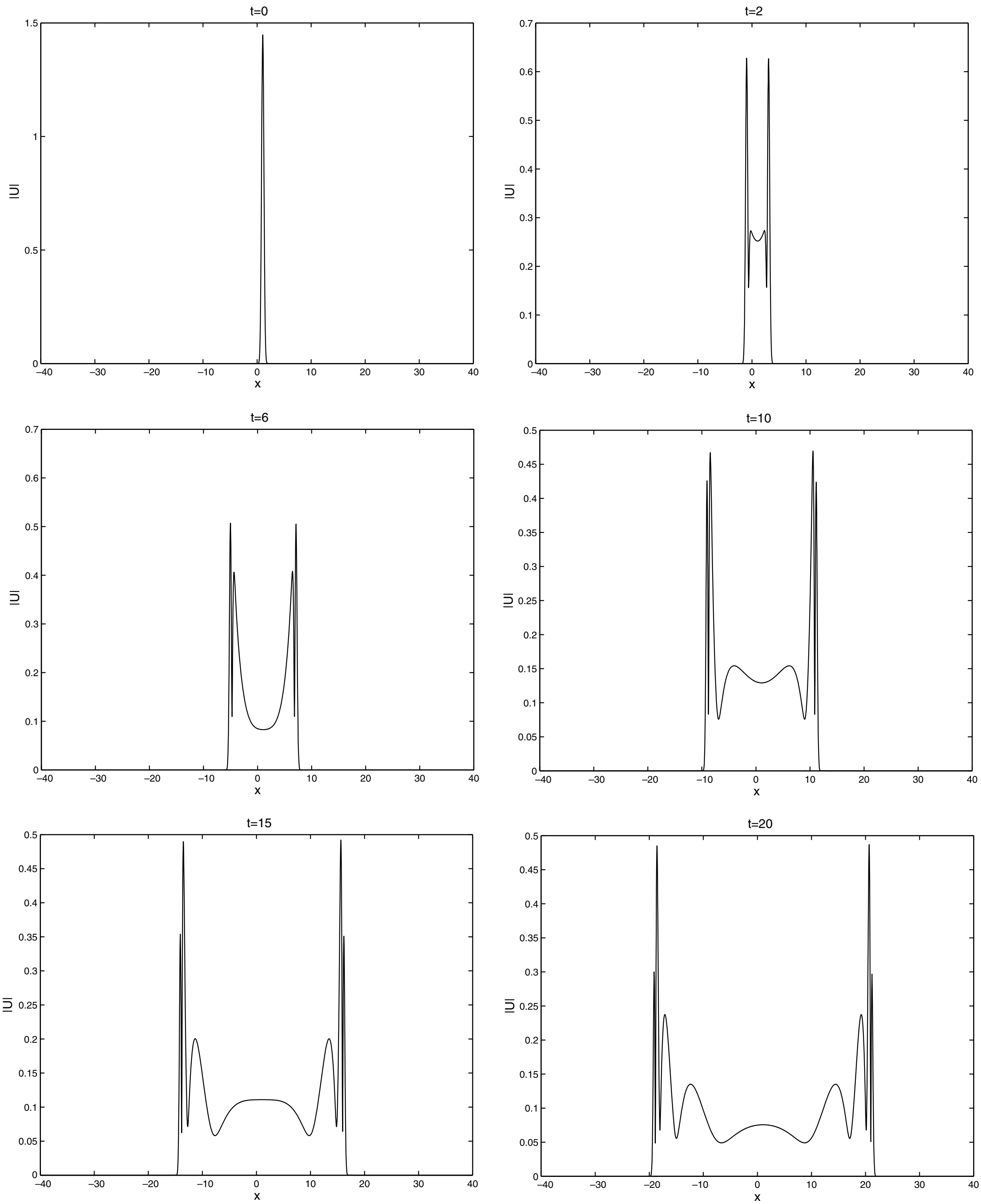

Fig. 4. Movement of soliton $|U|$ with $h = \tau = 0.02$, $-X_l = X_r = 40$.

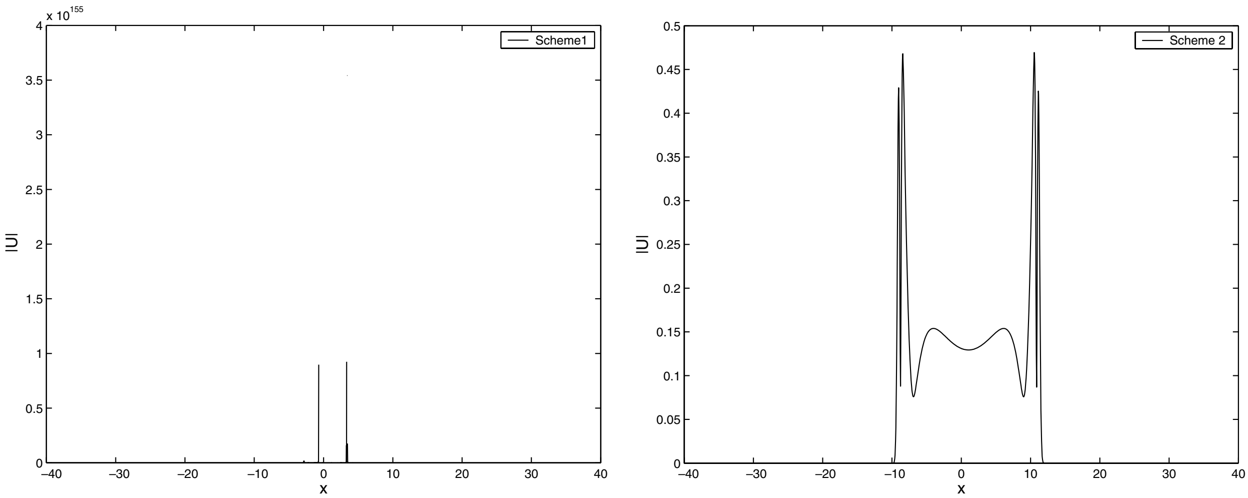

Fig. 5. Value $|U|$ computed by two schemes with $h = 0.02$, $\tau = 0.04$, $-X_l = X_r = 40$, $t = 10$.

7. Numerical experiment

In this section, we just consider Scheme A and the scheme in paper [2]. In computations, we chose the parameters as

$$\alpha = \beta = 1, \qquad u_0(x) = (1+i)x\mathrm{e}^{-10(1-x)^2}, \qquad u_1(x) = 0.$$

and let $q(|u|^2) = |u|^2$ as an example. We note the scheme in paper [2] as Scheme 1 (S1), and Scheme A as Scheme 2 (S2).

It is clear from Figs. 1 and 2 that the two schemes both are good in computation when the ratio $\lambda \leqslant 1$, and they almost have the same accuracy. It is easy to see from Table 1 that both of the two schemes are well conservative, thus both of them can be used to computing for a long time. In Fig. 4, we give the movement of the soliton $|U|$, and in Fig. 3, we give the phasic picture of U. The major advantage of scheme 2 is its unconditional stability which is numerically proved by Fig. 5. Obviously, the scheme in paper [2] is unstable, but the Scheme A can get a reasonable result when the ratio $\lambda > 1$. Thus, from the numerical experiments the Scheme A is usable. Similarly, we can show that Schemes B–H are also stable by the numerical experiments.

References

[1] G. Bo-ling, L. Hua-Xang, On the problem of numerical calculation for a class of the system of nonlinear Schrödinger equations with wave operator, J. Numer. Methods Comput. Appl. 4 (1983) 176–182.

[2] Luming Zhang, Qianshun Chang, A conservative numerical scheme for a class of nonlinear Schrödinger with wave operator, Appl. Math. Comput. 145 (2003) 603–612.

[3] F. Zhang, V.M. Peréz-Ggarcía, L. Vázquez, Numerical simulation of nonlinear Schrödinger equation system: a new conservative scheme, Appl. Math. Comput. 71 (1995) 165–177.

[4] S. Li, Vu-quoc, Finite difference calculus invariant structure of a class of algorithms for the nonlinear Klein–Gordon equation, SIAM Numer. Anal. 32 (1995) 1839–1864.

[5] Q. Chang, L. Xu, A numerical method for a system of generalized nonlinear Schrödinger equation, J. Comput. Math. 4 (1986) 191–199.

[6] Q. Chang, E. Jia, W. Sun, Difference schemes for solving the generalized nonlinear Schrödinger equation, J. Comput. Phys. 148 (1999) 397–415.

[7] Q. Chang, G. Wang, B. Guo, Conservative scheme for a model of nonlinear dispersive waves and its solitary waves induced by boundary notion, J. Comput. Phys. 93 (1991) 360–375.

[8] Luming Zhang, Qianshun Chang, A new difference scheme for regularized long-wave equation, Numer. Comput. Computer Appl., A Chin. J. 4 (2000) 247–254.

[9] F. Zhang, L. Vázquez, Two energy conserving numerical schemes for the Sine-Gordon equation, Appl. Math. Comput. 45 (1991) 17–30.

[10] Ben-yu Guo, P.J. Paseual, M.J. Rodriguez, L. Vázquez, Numerical solution of the Sine-Gordon equation, Appl. Math. Comput. 18 (1986) 1–14.
[11] Y.S. Wong, Q. Chang, L. Gong, An initial-boundary value problem of a nonlinear Klein–Gordon equation, Appl. Math. Comput. 84 (1997) 77–93.
[12] Qianshun Chang, Hong Jiang, A conservative scheme for the Zakharov equations, J. Comput. Phys. 113 (1994) 309–319.
[13] Qianshun Chang, Boling Guo, Hong Jiang, Finite difference method for generalized Zakharov equations, Math. Comput. 64 (1995) 537–553.
[14] Y. Zhou, Application of Discrete Functional Analysis to the Finite Difference Methods, International Academic Publishers, Beijing, 1990.