A singular perturbation problem for an envelope equation in plasma physics.

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Abstract: We investigate the so-called Langmuir wave envelope approximation which consists in taking the limit $\omega \to \infty$ in the nonlinear plasma wave equation

$$\frac{1}{\omega^2}\partial_t^2 E_\omega - 2i\partial_t E_\omega - \Delta E_\omega = f(|E_\omega|^2)E_\omega,$$

stated for nonlinearities satisfying $|f(\rho)| \leq K\rho^\sigma$. For any finite value of $\omega > 1$, the solution E_ω with the initial data $E_\omega(x, t = 0) \in H^2(\mathbf{R}^n)$, $\partial_t E_\omega(x, t = 0) \in H^1(\mathbf{R}^n)$ is shown to exist locally in time and to be unique. Under some specific conditions including ω below a threshold value, we construct solutions E_ω that blow up in a finite time with a divergent L^2 norm; nevertheless, in the so-called subcritical case ($\sigma n < 2$), the solution defined for fixed initial data is global provided that ω should be large enough. We demonstrate the strong convergence of E_ω towards the nonlinear Schrödinger solution E reached as $\omega \to \infty$, as long as E exists. In this same limit, we finally discuss the behavior of the time derivative of E_ω and compare the blow-up times associated with E_ω and with its time-enveloped counterpart E.

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Contents

1 Introduction, notations and statement of the results.

1.1 Physical interest.

Singular phenomena as self-focusing and wave collapse in plasmas rank among the most extensively investigated topics in nonlinear physics ([1],[2],[3],[4],[5]). In a plasma medium, wave collapse enters into the description of the strong Langmuir turbulence in which electron plasma waves (also called Langmuir waves) nonlinearly couple with large-scale density fluctuations as follows: in a first stage, Langmuir waves are created by high-frequency motions of the electrons surrounding ion-density acoustic waves. They generate a longitudinal electrostatic field that oscillates rapidly with a pulsation $\tilde{\omega}$ close to the electron plasma frequency $\omega_{pe} = \sqrt{q^2 N_0/\epsilon_0 m_e}$, as given by the dispersion relation

$$\tilde{\omega}^2 = \omega_{pe}^2 (1 + 3(k\lambda_D)^2), \tag{1}$$

where k denotes the characteristic wave-number of plasma waves. Here q, m_e, ϵ_0 and N_0 respectively correspond to the electron charge and mass, to the vacuum dielectric constant and to the background electron density; λ_D is the Debye length corresponding to the elementary length of the shielding cloud formed by electrons interacting with an isolated ion. In a quasi-neutral plasma, and when no dissipative effect such as Landau damping takes place, Langmuir wave-numbers classically obey the condition $k\lambda_D << 1$. Under these conditions, considering then a non uniform density $N = N_0(1 + \delta n/N_0)$ with $\delta n/N_0 << 1$, one can formally substitute N_0 by N in the definition of ω_{pe}, so that the relation (1) now reads as

$$\tilde{\omega}^2 = \omega_{pe}^2 + 3k^2 v_{th}^2 + \omega_{pe}^2 \frac{\delta n}{N_0}, \tag{2}$$

where v_{th} denotes the thermal electron velocity $v_{th} = \lambda_D \omega_{pe}$.

After taking the inverse Fourier transform in space and time of (2), the scalar potential $\phi(x,t)$ of the Langmuir electric field $\mathbf{E} = -\nabla\phi$ is found to evolve following the Nonlinear Wave Equation (hereafter abbreviated by NLW)

$$\Delta(\partial_t^2 + \omega_{pe}^2 - 3v_{th}^2 \Delta)\phi = -\omega_{pe}^2 div(\frac{\delta n}{N_0}\nabla\phi). \tag{3}$$

In the right-hand side of (3), the perturbation $\delta n/N_0$ contains in principle both low-frequency and high-frequency fluctuations of the plasma density, namely $\delta n = \delta n_{LF} + \delta n_{HF}$. However, when regarding slowly varying motions as induced by the so-called ponderomotive force $\mathbf{F} = -(q^2/2m_e\omega_{pe}^2)\nabla|\mathbf{E}|^2$ that tends to decrease locally the electron density, δn mainly consists in a low-frequency contribution governed by the wave equation ([3])

$$(\partial_t^2 - c_s^2\Delta)\frac{\delta n_{LF}}{N_0} = c_s^2 \frac{\epsilon_0}{N_0 T_e}\Delta|\nabla\phi|^2, \tag{4}$$

where $c_s = \sqrt{(ZT_e + 3T_i)/m_i}$ is the ion-sound speed expressed in terms of the electron and ion temperatures (resp. T_e and T_i) and of the ion mass m_i. The equation set (3)-(4) contains the well-known Zakharov equations [5] that describe the nonlinear coupling between Langmuir waves with some large scale ion-acoustic perturbations. For sufficiently strong initial electrostatic energy and in the absence of any dissipative effect, this coupling leads the potential well $\delta n \approx \delta n_{LF}$ and the electric field $\mathbf{E}$ to blow up - or to collapse - at a finite time $T^* < +\infty$. Such a singular dynamics generates some highly spiky electric fields, which makes the medium highly turbulent. When neglecting the ion inertia in (4), i.e. in the static limit $\partial_t^2 \delta n_{LF} << c_s^2 \Delta \delta n_{LF}$, equations (3)-(4) reduce to the following NLW

$$(\partial_t^2 + \omega_{pe}^2 - 3v_{th}^2 \Delta)\tilde{E} = \omega_{pe}^2 \frac{\epsilon_0}{N_0 T_e} f(|\tilde{E}|^2)\tilde{E}, \tag{5}$$

with $f(|\tilde{E}|^2) = |\tilde{E}|^2$. In (5), one has set $\tilde{E} = -\nabla\phi$ where $\tilde{E}$ now denotes the scalar envelope of the electric field: by doing so, the vectorial system (3)-(4) simplifies into the scalar model (5) that constitues a good approximation of the true Zakharov equations, as shown in [6], [7], [8], [9].

As commonly used, the hypothesis of time-envelope approximation consists in inserting the substitution $\tilde{E} = 2Re(E_\omega(x,t)exp(-i\omega_{pe}t))$ into (5) and to consider $E_\omega(x,t)$ as a slowly varying function as compared with the plasma frequency ω_{pe}, namely $\partial_t E_\omega << \omega_{pe} E_\omega$. More precisely, for a narrow k-range $k\lambda_D << 1$ and for weak low-frequency fluctuations $\delta n/N_0 << 1$, it can be seen from the dispersion relation (2) that the characteristic frequency $\delta\omega$ associated with the envelope of E_ω is given by $\delta\omega = \tilde{\omega} - \omega_{pe} \approx (3/2)\omega_{pe}(k\lambda_D)^2 + \omega_{pe}\delta n/(2N_0)$. Then performing a simple rescaling on E_ω together with normalizing space coordinates in units k^{-1} and the time variable as $t \to \delta\omega t$, allows us to transform (5) into

$$(\frac{1}{\omega^2}\partial_t^2 - 2i\partial_t - \Delta)E_\omega = f(|E_\omega|^2)E_\omega, \;\; \omega^2 = \omega_{pe}/\delta\omega \tag{6}$$

in such a way that applying the envelope hypothesis on (5) is nothing else but taking the limit $\omega \to \infty$ in (6). From a physical viewpoint, even though the quantity $\delta\omega/\omega_{pe}$ never tends to zero *stricto-sensu*, the previous limit $\omega \to \infty$ simply means that this latter ratio becomes negligible in front of the remaining contributions of order unity in (6). Making the time-envelope assumption thus amounts for ignoring the second-order time derivative in (6) which formally converges towards a Nonlinear Schrödinger Equation (NLS).

This paper is devoted to the previous limit: we investigate the time behavior of E_ω when the latter obeys the complete equation (6) for $\omega > 1$ (i.e. when taking the rapid fluctuations of the Langmuir field into account), and when passing to the limit $\omega \to \infty$, for which the first term of (6) can be viewed as a singular perturbation of NLS. We will focus our attention on general nonlinearities satisfying $|f(\rho)| \leq K\rho^\sigma$ with K =const.> 0 and $\sigma \geq 1$. Those apply not only onto the former ion-static limit of the Zakharov equation, but also to a larger class of nonlinearities as the saturing ones of the form $f(\rho) = \rho - \gamma\rho^2$ $(\gamma > 0)$, $f(\rho) = 1 - e^{-\rho}$ or $f(\rho) = \rho/(1+\rho)$, which represent corrections to the cubic NLS for large wave amplitudes [10], [11].

Finally, we recall some properties of the Nonlinear Schrödinger Equation

$$(NLS) \quad (-2i\partial_t - \Delta)E = f(|E|^2)E,$$

(see [12], [13], [14], [15]), namely

*NLS is locally well-posed in $H^2(\mathbf{R}^n)$ for $n \leq 3$.

*The L^2 norm of E is time invariant.

*All the solutions are global in the subcritical case $n\sigma < 2$.

*Finite time blow-up can occur in the complementary situation $n\sigma \geq 2$ provided that the potential f in (NLS) satisfies the inequality $f(\rho)\rho \geq (1 + 2/n)\int_0^\rho f(s)ds$.

1.2 Mathematical setting.

The aim of this paper is to study the equation

$$(NLW_\omega) \ \frac{1}{\omega^2}\partial_t^2 E_\omega - 2i\partial_t E_\omega - \Delta E_\omega = f(|E_\omega|^2)E_\omega, \ x \in \mathbf{R}^n, \ t \geq 0$$

for $n \leq 3$, and particularly the limit $\omega \to +\infty$. Formally, the limit equation is

$$(NLS) \ \ -2i\partial_t E - \Delta E = f(|E|^2)E, \ x \in \mathbf{R^n}, \ t \geq 0.$$

The hypothesis on the nonlinearity is that f is a $\mathcal{C}^2$ function, and that there exists $K > 0$, $\sigma \geq 1$ such as

$$(H) \ \ |f(u)| \leq K|u|^\sigma, \ |f'(u)| \leq K|u|^{\sigma-1}, \ |f''(u)| \leq K|u|^{\sigma-2}.$$

We now introduce the semi-groups associated with the linear part of (NLW_ω): we denote by $S_0^\omega(t)E_0$ the solution to

$$\begin{cases} \frac{1}{\omega^2}\partial_t^2 E - 2i\partial_t E - \Delta E = 0, \\ E(x,0) = E_0(x), \\ \partial_t E(x,0) = 0, \end{cases}$$

by $S_1^\omega(t)E_1$ the solution to

$$\begin{cases} \frac{1}{\omega^2}\partial_t^2 E - 2i\partial_t E - \Delta E = 0, \\ E(x,0) = 0, \\ \partial_t E(x,0) = E_1(x), \end{cases}$$

and by $S(t)E_0$ the solution to

$$\begin{cases} -2i\partial_t E - \Delta E = 0, \\ E(x,0) = E_0(x). \end{cases}$$

The semi-groups $S_0^\omega(t)$, $S_1^\omega(t)$, $S(t)$ are in fact Fourier multipliers given by

$$\mathcal{F}(S_0^\omega(t))(\xi) = \frac{1+\sqrt{1+\xi^2/\omega^2}}{2\sqrt{1+\xi^2/\omega^2}} e^{i\omega^2(1-\sqrt{1+\xi^2/\omega^2})t} - \frac{1-\sqrt{1+\xi^2/\omega^2}}{2\sqrt{1+\xi^2/\omega^2}} e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t}, \tag{7}$$

$$\mathcal{F}(S_1^\omega(t))(\xi) = \frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t} - e^{i\omega^2(1-\sqrt{1+\xi^2/\omega^2})t}}{2i\omega^2\sqrt{1+\xi^2/\omega^2}}, \tag{8}$$

$$\mathcal{F}(S(t))(\xi) = e^{-i\xi^2 t/2}, \tag{9}$$

where $\mathcal{F}$ denotes the Fourier transform with respect to $x \in \mathbf{R}^n$. With these notations, E_ω satisfies (NLW_ω) if and only if E_ω satisfies the integral equation

$$(INT_\omega) \;\; E_\omega = S_0^\omega(t)E_\omega(0) + S_1^\omega(t)\partial_t E_\omega(0) + \omega^2 \int_0^t S_1^\omega(t-s) f(|E_\omega|^2)E_\omega(s)ds,$$

and E satisfies (NLS) if and only if E satisfies the following

$$(INT) \;\; E(t) = S(t)E(0) + \frac{i}{2}\int_0^t S(t-s) f(|E|^2)E(s)ds.$$

1.3 Statement of the main results.

The main results of section 2 are the following:

(Theorem 1) (NLW_ω) is locally well-posed in $H^2 \times H^1$.

(Theorem 2) For fixed ω, there exist initial data for which the corresponding solutions blow up in a finite time.

(Theorem 3) For fixed initial data, in the subcritical case ($\sigma n < 2$), the solution is global provided that ω should be large enough.

The main result of section 3 is the following (Theorem 6).

Let $E_0 \in H^2(\mathbf{R}^n)$, $E_1 \in H^1(\mathbf{R}^n)$ and E_ω be the solution to (NLW_ω) with $E_\omega(0) = E_0$, $\partial_t E_\omega(0) = E_1$ and T_ω the existence time of E_ω. Let E be the solution to (NLS) with $E(0) = E_0$ and $T(E_0)$ its existence time.

Then

$$\liminf_{\omega\to\infty} T_\omega \geq T(E_0).$$

Moreover, for all $T < T(E_0)$, $E_\omega \to_{\omega\to\infty} E$ *in* $L^\infty(0,T,H^2)$ *and*

$$\partial_t E_\omega - \partial_t E - e^{2i\omega^2 t} S(-t)(E_1 - \frac{i}{2}\Delta E_0 + \frac{1}{2i} f(|E_0|^2)E_0) - e^{2i\omega^2 t} g(x,t) \to_{\omega\to\infty} 0$$

in $L^\infty(0,T,L^2)$, *where* g *is defined by*

$$\begin{cases} 2i\partial_t g - \Delta g = [f'(|E|^2)|E|^2 + f(|E|^2)][S(-t)(E_1 - \frac{i}{2}\Delta E_0 + \frac{1}{2i} f(|E_0|^2)E_0) + g], \\ g(0,x) = 0. \end{cases}$$

Remark: Note that all the results of this paper can be extended to the vectorial system (3), (4) with $\frac{\delta n}{N_0} = -\frac{\epsilon_0}{T_e N_0}|\nabla\phi|^2$ and to the equivalent of (6):

$$(NLW\nabla) \ \ (\frac{1}{\omega^2}\partial_t^2 - 2i\partial_t - \Delta)\nabla\phi = -\nabla(-\Delta)^{-1} div(|\nabla\phi|^2\nabla\phi).$$

The dispersive $(NLS\nabla)$ part of $(NLW\nabla)$ has been studied in [16] and [17]. The results of this paper have been announced in [18].

2 The local Cauchy problem and finite time blow up.

2.1 Local Cauchy problem.

The main result is the following

Theorem 1 *Let $E_0 \in H^2(\mathbf{R}^n)$, $E_1 \in H^1(\mathbf{R}^n)$, then $\exists\ T_0 > 0$ such that there exists a unique solution $E_\omega \in \mathcal{C}([0,T_0],H^2)$, $\partial_t E_\omega \in \mathcal{C}([0,T_0],H^1)$, $\partial_t^2 E_\omega \in \mathcal{C}([0,T_0],L^2)$ satisfying (NLW_ω) with $E_\omega(0) = E_0$ and $\partial_t E_\omega(0) = E_1$. $|E_\omega|_{L^\infty(0,T_0,H^2)}$ is moreover bounded, and this bound as well as T_0 only depend on $|E_0|_{H^2}$ and $\frac{|E_1|_{H^1}}{\omega}$.*

Proof: During the proof, K, K' and C will denote some positive constants that can change from one line to another one. We first remark that, by (7) and (8), one has

$$|\mathcal{F}(S_0^\omega(t))|_{L^\infty} \leq 2, \quad |\mathcal{F}(S_1^\omega(t))|_{L^\infty} \leq \frac{1}{\omega^2}, \quad ||\xi|\mathcal{F}(S_1^\omega(t))|_{L^\infty} \leq \frac{1}{\omega}.$$

This implies that

$$\forall \phi,\ \forall s \quad |S_1^\omega(t)\phi|_{H^s} \leq \frac{C}{\omega}|\phi|_{H^{s-1}},\ |S_0^\omega(t)\phi|_{H^s} \leq 2|\phi|_{H^s}. \tag{10}$$

In order to prove Theorem 1, we use a classical fixed point method (see [19]). Let us first introduce the following functionnal:

$$\mathcal{T}_\omega(E) = S_0^\omega(t)E_0 + S_1^\omega(t)E_1 + \omega^2 \int_0^t S_1^\omega(t-s) f(|E|^2)E(s)ds. \tag{11}$$

Using (10), we get

$$|\mathcal{T}_\omega(E)|_{L^\infty(0,T,H^2)} \leq 2|E_0|_{H^2} + \frac{C}{\omega}|E_1|_{H^1} + \int_0^t |f(|E|^2)E|_{L^\infty(0,T,H^2)}ds. \tag{12}$$

$H^2(\mathbf{R}^n)$ being an algebra for $n \leq 3$, (12) and (H) lead to

$$|\mathcal{T}_\omega(E)|_{L^\infty(0,T,H^2)} \leq 2|E_0|_{H^2} + \frac{C}{\omega}|E_1|_{H^1} + KT|E|^{2\sigma+1}_{L^\infty(0,T,H^2)}. \tag{13}$$

Let $R = 2(2|E_0|_{H^2} + \frac{C}{\omega}|E_1|_{H^1})$ and $B_R = \{\phi \in L^\infty(0,T,H^2)\ /\ |\phi|_{L^\infty(0,T,H^2)} \leq R\}$, we have thus proved the

Lemma 1 *If $KTR^{2\sigma+1} \leq \frac{R}{2}$, then $\mathcal{T}_\omega$ maps B_R into itself.*

Now, $\forall E,\ F \in B_R$, we have

$$|\mathcal{T}_\omega(E) - \mathcal{T}_\omega(F)|_{L^\infty(0,T,H^2)} \leq KTR^{2\sigma}|E-F|_{L^\infty(0,T,H^2)},$$

so that since $KTR^{2\sigma} < 1$, $\mathcal{T}_\omega$ is a contraction in B_R. This yields the existence of a unique solution to (NLW_ω). It is clear that the existence time T_0 only depends on $|E_0|_{H^2}$ and $\frac{|E_1|_{H^1}}{\omega}$. The results $\partial_t E_\omega \in \mathcal{C}([0,T_0],H^1)$ and $\partial_t^2 E_\omega \in \mathcal{C}([0,T_0],L^2)$ easily follow from (INT_ω). ■

(NLW_ω) has some invariants: we first define $F(x) = \int_0^x f(t)dt$, and indeed enounce the

Proposition 1 *The following quantities are constant on* $[0, T_0]$.

$$Q_\omega = \int |E_\omega|^2 - \frac{1}{\omega^2} \int Im(\bar{E}_\omega \partial_t E_\omega). \tag{14}$$

$$\mathcal{E}_\omega = \frac{1}{\omega^2} \int |\partial_t E_\omega|^2 + \int |\nabla E_\omega|^2 - \int F(|E_\omega|^2), \tag{15}$$

where, as in what follows henceforth, the symbol of the undefinite integral $\int$ written with no other specification denotes a *space*-integration in $\mathbf{R}^n$.
Proof: The invariants (14) and (15) follow from a straightforward calculation which consists in multiplying (NLW_ω) by $\bar{E}_\omega$ and $\partial_t \bar{E}_\omega$ and in space-integrating the imaginary and real parts of the results respectively. ■

2.2 Finite time blow-up.

We apply Levine's concavity methods [20] onto our mathematical setting.

Theorem 2 *Suppose that the nonlinearity satisfies*

$$(H1) \;\; f(u)u \geq (1 + 2\alpha)F(u) \;\; \text{for } \alpha > 0$$

and let

$$\mathcal{E} = \frac{\omega^2}{2}\mathcal{E}_\omega - \frac{\omega^4}{1+2\alpha} Q_\omega$$

where $\mathcal{E}_\omega$ *and* Q_ω *are defined by (14) and (15), and let* $I_\omega(t) = \frac{1}{2}\int |E_\omega|^2$.

i)If

$$\frac{1}{\omega^2} > \frac{1}{\omega_{lim}^2} = \frac{8I_\omega(0) - \frac{2\alpha}{1+2\alpha} Q_\omega}{4\sqrt{\alpha} I'_\omega(0) - \alpha \mathcal{E}_\omega}$$

and if one of the following conditions holds
$(C1)$ $\mathcal{E} < 0$,
$(C2)$ $\mathcal{E} = 0$, $I'_\omega(0) > 0$,
then there exists $T^- < +\infty$ *such as*

$$\lim_{t \to T^-} |E_\omega(t)|_{H^2} = +\infty.$$

ii)If $\mathcal{E} > 0$,
$(C3)$ $I'_\omega(0) > 2\omega^2 \sqrt{\mathcal{E} I_\omega(0)/\omega^4 + I_\omega^2(0)/\alpha} > 0$,
and if the solution exists globally, then

$$\lim_{t \to \infty} \int |E_\omega|^2 = +\infty.$$

Proof: We multiply (NLW_ω) by $\bar{E}_\omega$ and retain the real part of the result, we get

$$\frac{1}{2\omega^2}\partial_t^2\int|E_\omega|^2 \geq \frac{2}{\omega^2}(1+\alpha)\int|\partial_t E_\omega|^2 - 2\omega^2\int|E_\omega|^2 - \frac{2}{\omega^2}(1+2\alpha)\mathcal{E}. \quad (16)$$

If we multiply (16) by $I_\omega(t)$, using Cauchy-Schwartz inequality, we get

$$I_\omega I_\omega'' \geq (1+\alpha)(I_\omega')^2 - 4\omega^4 I_\omega^2 - 2I_\omega(1+2\alpha)\mathcal{E}, \quad (17)$$

since

$$2I_\omega(1+\alpha)\int|\partial_t E_\omega|^2 \geq (1+\alpha)(\int|\bar{E}_\omega\partial_t E_\omega|)^2$$
$$\geq (1+\alpha)|\int Re(\bar{E}_\omega\partial_t E_\omega)|^2 = (1+\alpha)(I_\omega')^2.$$

i) Case $\mathcal{E}\leq 0$:
Following Levine [20], we introduce $H(t) = I_\omega(t) - \mathcal{E}\times(t+\tau)^2$, $\tau > 0$ to be fixed later on. Using (17), this leads to

$$HH'' - (1+\alpha)(H')^2 \geq -\frac{(1+\alpha)\mathcal{E}}{I_\omega}(I_\omega'\times(t+\tau) - 2I_\omega)^2 - 4\omega^4 I_\omega H. \quad (18)$$

Since $\mathcal{E}\leq 0$, (18) simply reduces to

$$HH'' - (1+\alpha)(H')^2 \geq -4\omega^4 H(H+\mathcal{E}\times(t-\tau)^2) \geq -4H^2\omega^4,$$

or equivalently

$$-\frac{1}{\alpha}H^{\alpha+2}(H^{-\alpha})'' \geq -4H^2\omega^4,$$

which is equivalent to

$$J'' \leq 4\alpha\omega^4 J$$

with $J(t)\equiv H^{-\alpha}(t)$. It follows that

$$J(t) \leq J(0)\{ch(2\sqrt{\alpha}\omega^2 t) + \frac{J'(0)}{2\sqrt{\alpha}\omega^2 J(0)}sh(2\sqrt{\alpha}\omega^2 t)\}.$$

In terms of $H(t)$, this last inequality means

$$H(t) \geq H(0)\{ch(2\sqrt{\alpha}\omega^2 t) - \frac{\sqrt{\alpha}H'(0)}{2H(0)\omega^2}sh(2\sqrt{\alpha}\omega^2 t)\}^{-1/\alpha}.$$

$(C1)$ If $\mathcal{E} < 0$, one can choose τ sufficiently large in order to ensure $H'(0) > 0$, and the above inequality then implies $H(t)\to\infty$ when

$$t \to T^- \leq T_*(\tau) = \frac{1}{2\sqrt{\alpha}\omega^2}argth\Big\{\frac{2\omega^2}{\sqrt{\alpha}}(\frac{I_\omega(0)-\mathcal{E}\tau^2}{I_\omega'(0)-2\mathcal{E}\tau})\Big\},$$

provided that this expression makes sense.

$(C2)$ If $\mathcal{E}=0$, the initial condition $I'_\omega(0)>0$ ensures $H'(0)>0$ and we get the same conclusion as previously with $T^*=T^*(\tau=0)$.

Note that in the case of $(C1)$, we can find the value of $\tau\in\,]\frac{I'_\omega(0)}{2\mathcal{E}},+\infty[$ for which $T_*(\tau)$ is the smallest as possible, we find

$$\min_\tau T_*(\tau)=\frac{1}{2\sqrt{\alpha}\omega^2}argth\Big\{\frac{2\omega^2}{\sqrt{\alpha}}\big(\frac{2I_\omega(0)}{I'_\omega(0)+\sqrt{I'_\omega(0)^2-4\mathcal{E}I_\omega(0)}}\big)\Big\}$$

and the largest ω for which this expression makes sense is given by

$$\frac{1}{\omega_{lim}^2}=\frac{8I_\omega(0)-\frac{2\alpha}{1+2\alpha}Q_\omega}{4\sqrt{\alpha}I'_\omega(0)-\alpha\mathcal{E}_\omega},$$

which achieves the proof of i).

ii) Case $\mathcal{E}>0$.
We multiply (17) by $-\alpha I_\omega^{-(\alpha+2)}(I_\omega^{-\alpha})'$, and we obtain as long as I'_ω remains positive:

$$\frac{1}{2}\partial_t((I_\omega^{-\alpha})')^2\geq 2\alpha(1+2\alpha)\mathcal{E}I_\omega^{-(\alpha+1)}(I_\omega^{-\alpha})'+4\alpha\omega^4I_\omega^{-\alpha}(I_\omega^{-\alpha})'. \tag{19}$$

Let $J=I_\omega^{-\alpha}$, (19) can then be rewritten as

$$\frac{1}{2}\partial_t(J')^2\geq 2\alpha^2\mathcal{E}\partial_t(J^{2+1/\alpha})+4\alpha\omega^4\partial_t(J^2). \tag{20}$$

Integrating (20) yields

$$\begin{aligned}(J'-\sqrt{4\alpha^2\mathcal{E}J^{2+1/\alpha}+4\alpha\omega^4J^2})(J'+\sqrt{4\alpha^2\mathcal{E}J^{2+1/\alpha}+4\alpha\omega^4J^2})\\ \geq(J'(0)^2-4\alpha^2\mathcal{E}J^{2+1/\alpha}(0)-4\alpha\omega^4J^2(0))\\ \geq\frac{\alpha^2}{I_\omega^{2\alpha+2}(0)}(I'_\omega(0)^2-4\mathcal{E}I_\omega(0)-\frac{4}{\alpha}\omega^4I_\omega(0)^2)>0\end{aligned}$$

by $(C3)$. It follows that J' (and hence I'_ω) cannot vanish as long as the solution exists and that for every $t\geq0$, one gets

$$J'+\sqrt{4\alpha^2\mathcal{E}J^{2+1/\alpha}+4\alpha\omega^4J^2}<0$$

or

$$I'_\omega>\frac{1}{\alpha}\sqrt{4\alpha^2\mathcal{E}I_\omega+4\alpha\omega^4I_\omega^2}=\frac{2\omega^2}{\sqrt{\alpha}}\sqrt{I_\omega}\sqrt{\gamma+I_\omega}$$

with $\gamma=\frac{\alpha\mathcal{E}}{\omega^4}>0$.
This implies

$$\sqrt{I_\omega}+\sqrt{\gamma+I_\omega}\geq(\sqrt{I_\omega(0)}+\sqrt{\gamma+I_\omega(0)})e^{\frac{\omega^2}{\sqrt{\alpha}}t}\to\infty$$

when $t\to\infty$. ■

2.3 Global existence in the subcritical case for large ω.

Blow-up results of the preceding section are valid for all space dimensions including the subcritical case $\sigma n < 2$, provided that ω should not be too large, i.e. $\omega < \omega_{lim}$. Indeed, in the opposite range, we have the

Theorem 3 *If f satisfies $|f(|E|^2)| \leq K|E|^{2\sigma}$ with $\sigma n < 2$, $E_0 \in H^2(\mathbf{R}^n)$, $E_1 \in H^1(\mathbf{R}^n)$, then if ω is sufficiently large, the solution to (NLW_ω) with initial data $E_\omega(0) = E_0$ and $\partial_t E_\omega(0) = E_1$ exists globally and $E_\omega \in L^\infty(0, +\infty, H^2)$.*

Proof: We multiply (NLW_ω) by $\bar{E}_\omega$ and take the real part:

$$\frac{1}{2\omega^2}\partial_t^2 \int |E_\omega|^2 - \frac{1}{\omega^2}\int |\partial_t E_\omega|^2 + 2Im \int \partial_t E_\omega \bar{E}_\omega + \int |\nabla E_\omega|^2 - \int f(|E_\omega|^2)|E_\omega|^2 = 0.$$

Using the expressions of Q_ω and $\mathcal{E}_\omega$, we obtain

$$\begin{aligned} &\tfrac{1}{2\omega^2}\partial_t^2 \textstyle\int |E_\omega|^2 + 2\omega^2 \int |E_\omega|^2 - 2\omega^2 Q_\omega - \mathcal{E}_\omega + 2\int |\nabla E_\omega|^2 \\ &= \textstyle\int f(|E_\omega|^2)|E_\omega|^2 + \int F(|E_\omega|^2). \end{aligned} \tag{21}$$

By means of the following Gagliardo-Nirenberg's inequality (see [21]):

$$|u|_{L^{2\sigma+2}(\mathbf{R}^n)} \leq C|\nabla u|_{L^2(\mathbf{R}^n)}^{\frac{n\sigma}{2\sigma+2}} |u|_{L^2(\mathbf{R}^n)}^{\frac{(2-n)\sigma+2}{2\sigma+2}} \quad \forall u \in H^1(\mathbf{R}^n),$$

(21) can be re-expressed as

$$\begin{aligned} &\tfrac{1}{2\omega^2}\partial_t^2 \textstyle\int |E_\omega|^2 + 2\omega^2 \int |E_\omega|^2 - 2\omega^2 Q_\omega - \mathcal{E}_\omega + 2\int |\nabla E_\omega|^2 \\ &\leq K|\nabla E_\omega|_{L^2(\mathbf{R}^n)}^{n\sigma} |E_\omega|_{L^2(\mathbf{R}^n)}^{(2-n)\sigma+2}. \end{aligned} \tag{22}$$

We now apply Young's inequality on the right-hand side of (22) with the exponents $\frac{2}{n\sigma}$ and $\frac{2}{2-n\sigma}$, which gives

$$\frac{1}{2\omega^2}\partial_t^2 \int |E_\omega|^2 + 2\omega^2 \int |E_\omega|^2 - 2\omega^2 Q_\omega - \mathcal{E}_\omega \leq K'|E_\omega|_{L^2(\mathbf{R}^n)}^{\frac{4\sigma}{2-n\sigma}+2}. \tag{23}$$

Let $0 < \eta < 2$, we take ω sufficiently large such as

$$K'|E_0|_{L^2(\mathbf{R}^n)}^{\frac{4\sigma}{2-n\sigma}} \leq \omega^{2-\eta}. \tag{24}$$

On a time interval $[0, \tilde{T}]$, one can ensure

$$K'|E_\omega|_{L^2(\mathbf{R}^n)}^{\frac{4\sigma}{2-n\sigma}} \leq \omega^{2-\eta} 2^{\frac{4\sigma}{2-n\sigma}} \leq \omega^2, \tag{25}$$

if ω is sufficiently large. (23) then implies

$$\partial_t^2 \int |E_\omega|^2 + 2\omega^4 \int |E_\omega|^2 \leq 4\omega^4 Q_\omega + 2\omega^2 \mathcal{E}_\omega.$$

It follows that

$$\int |E_\omega|^2 \leq 2Q_\omega + \frac{1}{\omega^2}\mathcal{E}_\omega$$

$$+(\int |E_0|^2 - 2Q_\omega - \frac{1}{\omega^2}\mathcal{E}_\omega)\cos(\sqrt{2}\omega^2 t) + \frac{\sqrt{2}}{\omega^2}\int Re(\bar{E}_0 E_1)\sin(\sqrt{2}\omega^2 t)$$

on $[0, \tilde{T}]$, and we finally obtain

$$\int |E_\omega|^2 \leq 4\int |E_0|^2 \tag{26}$$

if ω is sufficiently large. In this case, (25) remains valid as long as E_ω exists and (26) also holds. Therefore, one gets

$$|E_\omega|^2_{L^\infty(0,\infty,L^2)} \leq 4\int |E_0|^2$$

for large ω. The Gagliardo-Nirenberg's inequality besides implies that the energy (15) controls $\int |\nabla E_\omega|^2$ and $\int \frac{1}{\omega^2}|\partial_t E_\omega|^2$. If we multiply (NLW_ω) by $\Delta\partial_t \bar{E}_\omega$, we can furthermore control the quantities $\int |\Delta E_\omega|^2$ and $\int \frac{1}{\omega^2}|\partial_t \nabla E_\omega|^2$. Theorem 3 thus follows from Theorem 1. ■

3 Convergence to the nonlinear Schrödinger equation.

The aim of this section consists in investigating the limit $\omega \to \infty$ in (NLW_ω).

3.1 Convergence of E_ω.

The result is the following.

Theorem 4 *Let $E_0^\omega \to_{\omega\to\infty} E_0$ in $H^2(\mathbf{R}^n)$, $E_1^\omega,\ E_1'^\omega \in H^1(\mathbf{R}^n)$ such that $E_1^\omega,\ E_1'^\omega$ are bounded in $L^2(\mathbf{R}^n)$ and $\frac{1}{\omega}\nabla E_1^\omega \to_{\omega\to\infty} 0$, $\frac{1}{\omega}\nabla E_1'^\omega \to_{\omega\to\infty} 0$ in $L^2(\mathbf{R}^n)$. Let $\lambda \in \mathbf{R}$.*
There exists a time T_1 depending only on $|E_0|_{H^2}$ such that solution E_ω to (NLW_ω) with the initial data

$$E_\omega(0) = E_0^\omega,\ \ \partial_t E_\omega(0) = E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega$$

exists on $[0, T_1]$ if ω is sufficiently large, and the solution E to (NLS) with the initial datum $E(0) = E_0$ exists on the same interval. Furthermore E_ω converges to E as $\omega \to \infty$ in $L^\infty(0, T_1, H^2)$.

Remark: The necessity of taking the above-defined initial data on $\partial_t E_\omega$ will be explained in the next section (Theorem 5).

Proof: For the existence part, we apply Theorem 1. We remark that the time T_0 given by Theorem 1 depends only on $|E_0^\omega|_{H^2}$, $|E_1^\omega|_{H^1}/\omega$ and $|E_1'^\omega|_{H^1}/\omega$; these quantities are respectively close to $|E_0|_{H^2}$ and 0 if ω is sufficiently large. This ensures an existence time interval common to (NLW_ω) and to (NLS).
To prove the limit $E_\omega \to_{\omega\to\infty} E$, we rewrite the corresponding integral equations (INT_ω) and (INT):

$$(INT_\omega)\ \ E_\omega = S_0^\omega(t)E_0^\omega + S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega) + \omega^2\int_0^t S_1^\omega(t-s)f(|E_\omega|^2)E_\omega(s)ds,$$

$$(INT)\ \ E(t) = S(t)E_0 + \frac{i}{2}\int_0^t S(t-s)f(|E|^2)E(s)ds.$$

Substracting (INT) from (INT_ω) yields

$$\begin{aligned} E_\omega - E = S_0^\omega(t)E_0^\omega - S(t)E_0 + S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega) \\ +\omega^2 \textstyle\int_0^t S_1^\omega(t-s)f(|E_\omega|^2)E_\omega(s)ds - \frac{i}{2}\int_0^t S(t-s)f(|E|^2)E(s)ds. \end{aligned} \tag{27}$$

We obtain from (27):

$$|E_\omega - E|_{L^\infty(0,T,H^2)} \le |S_0^\omega(t)E_0^\omega - S(t)E_0|_{L^\infty(0,T,H^2)}$$

$$+|S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega)|_{L^\infty(0,T,H^2)}$$

$$+|\textstyle\int_0^t \omega^2 S_1^\omega(t-s)(f(|E_\omega|^2)E_\omega(s) - f(|E|^2)E(s))ds|_{L^\infty(0,T,H^2)} \qquad (28)$$

$$+|\textstyle\int_0^t(\omega^2 S_1^\omega(t-s) - \frac{i}{2}S(t-s))f(|E|^2)E(s)ds|_{L^\infty(0,T,H^2)},$$

$$= J_1 + J_2 + J_3 + J_4.$$

Note that

$$J_3 \le K(|E_\omega|^{2\sigma}_{L^\infty(0,T,H^2)} + |E|^{2\sigma}_{L^\infty(0,T,H^2)})T|E_\omega - E|_{L^\infty(0,T,H^2)},$$

$$\le K'(|E_0|_{H^2})T|E_\omega - E|_{L^\infty(0,T,H^2)}$$

if ω is sufficiently large.
Therefore, taking T_1 such that $K'(|E_0|_{H^2})T_1 \le 1/2$, one gets from (28)

$$|E_\omega - E|_{L^\infty(0,T,H^2)} \le 2(J_1 + J_2 + J_4), \qquad (29)$$

for large ω. We now prove the

Lemma 2 *i)*$S_0^\omega(t)E_0^\omega \to_{\omega\to\infty} S(t)E_0$ *in* $L^\infty(0,T_0,H^2)$ *strongly.*

ii)$S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega) \to_{\omega\to\infty} 0$ *in* $L^\infty(0,T_0,H^2)$ *strongly.*

iii)$\int_0^t(\omega^2 S_1^\omega(t-s) - \frac{i}{2}S(t-s))f(|E|^2)E(s)ds \to_{\omega\to\infty} 0$ *in* $L^\infty(0,T_0,H^2)$ *strongly.*

Proof: i) Since $|\mathcal{F}(S_0^\omega(t))|_{L^\infty} \le 2$, we see that

$$|S_0^\omega(t)E_0^\omega - S_0^\omega(t)E_0|_{L^\infty(0,T_0,H^2)} \to_{\omega\to\infty} 0,$$

hence it is sufficient to prove that

$$|S_0^\omega(t)E_0 - S(t)E_0|_{L^\infty(0,T_0,H^2)} \to_{\omega\to\infty} 0.$$

Let $D_\omega(t)$ be the following quantity:

$$D_\omega(t) = |(1+|\xi|^4)^{1/2}\mathcal{F}(S_0^\omega(t) - S(t))\mathcal{F}(E_0)|^2_{L^2(d\xi)}.$$

Then

$$D_\omega(t) = \int_{\mathbf{R}^n} |\frac{1+\sqrt{1+\xi^2/\omega^2}}{2\sqrt{1+\xi^2/\omega^2}}e^{i\omega^2(1-\sqrt{1+\xi^2/\omega^2})t} - \frac{1-\sqrt{1+\xi^2/\omega^2}}{2\sqrt{1+\xi^2/\omega^2}}e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t}$$

$$-e^{-it\xi^2/2}|^2(1+|\xi|^4)|\mathcal{F}(E_0)|^2 d\xi.$$

Since $E_0 \in H^2$, $(1+|\xi|^4)|\mathcal{F}(E_0)|^2 \in L^1(d\xi)$, hence for all $\epsilon > 0$, there exists $k < \infty$ such that

$$\int_{|\xi|\geq k} (1+|\xi|^4)|\mathcal{F}(E_0)|^2 d\xi \leq \epsilon/3.$$

Hence

$$D_\omega(t) \leq \int_{|\xi|<k} |\frac{1+\sqrt{1+\xi^2/\omega^2}}{2\sqrt{1+\xi^2/\omega^2}} e^{i\omega^2(1-\sqrt{1+\xi^2/\omega^2})t} - \frac{1-\sqrt{1+\xi^2/\omega^2}}{2\sqrt{1+\xi^2/\omega^2}} e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t}$$

$$-e^{-it\xi^2/2}|^2(1+|\xi|^4)|\mathcal{F}(E_0)|^2 d\xi + \epsilon$$

Now, we remark that for $|\xi| < k$

$$\begin{cases} |\sqrt{1+\xi^2/\omega^2}-1| < |\sqrt{1+k^2/\omega^2}-1|, \\ |\omega^2(\sqrt{1+\xi^2/\omega^2}-1-\xi^2/(2\omega^2))t| < \omega^2(\sqrt{1+k^2/\omega^2}-1-k^2/(2\omega^2))T_0. \end{cases} \tag{30}$$

Hence

$$\limsup_{\omega\to\infty} |D_\omega(t)|_{L^\infty(0,T_0)} \leq \epsilon, \quad \forall \epsilon > 0$$

and i) of lemma 2 follows.

ii) Using $|\mathcal{F}(S_1^\omega(t))|_{L^\infty} \leq \frac{1}{\omega^2}$ and $||\xi|\mathcal{F}(S_1^\omega(t))|_{L^\infty} \leq \frac{1}{\omega}$, we get

$$|S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega)|_{H^1} \leq \frac{K}{\omega},$$

since E_1^ω and $E_1'^\omega$ are bounded in L^2. On the other hand, one has

$$|\nabla S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega)|_{H^1} \leq \frac{K'}{\omega}|\nabla E_1^\omega + e^{i\omega^2\lambda}\nabla E_1'^\omega|_{L^2} \to_{\omega\to\infty} 0,$$

since $\nabla E_1^\omega/\omega$, $\nabla E_1'^\omega/\omega$ tend to 0 in L^2. ii) of lemma 2 therefore follows.

iii) We have

$$(1+\xi^4)^{1/2}\int_0^t \mathcal{F}((\omega^2 S_1^\omega(t-s) - \frac{i}{2}S(t-s))f(|E|^2)E(s))ds$$

$$= \int_0^t (1+\xi^4)^{1/2}\frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}}\mathcal{F}(f(|E|^2)E(s))ds$$

$$+\int_0^t (i\frac{e^{i\omega^2(1-\sqrt{1+\xi^2/\omega^2})(t-s)}}{2\sqrt{1+\xi^2/\omega^2}} - \frac{i}{2}e^{-i\xi^2(t-s)/2})(1+\xi^4)^{1/2}\mathcal{F}(f(|E|^2)E(s))ds,$$

$$= f_1(\xi,t) + f_2(\xi,t).$$

By means of inequalities (30), $f_2(\xi,t) \to_{\omega\to\infty} 0$ in $L^\infty(0,T_0,L^2(d\xi))$.
For $f_1(\xi,t)$, we construct a piecewise-constant in time approximation of $(1+\xi^4)^{1/2}\mathcal{F}(f(|E|^2)E(s))$, namely $k_n(\xi,t)$ defined as follows

$$k_n(\xi,s) = (1+\xi^4)^{1/2}\mathcal{F}(f(|E|^2)E(\frac{kT_0}{n})) \text{ for } \frac{kT_0}{n} \le s < \frac{(k+1)T_0}{n}.$$

Then, since $E \in \mathcal{C}([0,T_0],H^2)$, $k_n(\xi,s)$ satisfies

$$k_n(\xi,s) \to_{n\to\infty} (1+\xi^4)^{1/2}\mathcal{F}(f(|E|^2)E(s)) \text{ in } L^\infty(0,T_0,L^1(d\xi)). \tag{31}$$

Denoting by $[X]$ the integer part of the real number X, we have

$$\int_0^t \frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}} k_n(\xi,s)ds$$

$$= \sum_{k=0}^{[tn/T_0]-1} \int_{\frac{kT_0}{n}}^{\frac{(k+1)T_0}{n}} (1+\xi^4)^{1/2} \frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}} \mathcal{F}(f(|E|^2)E(\frac{kT_0}{n}))ds$$

$$+ \int_{[tn/T_0]T_0/n}^t (1+\xi^4)^{1/2} \frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}} \mathcal{F}(f(|E|^2)E([tn/T_0]T_0/n))ds,$$

$$= \sum_{k=0}^{[tn/T_0]-1} -(1+\xi^4)^{1/2} \frac{\mathcal{F}(f(|E|^2)E(\frac{kT_0}{n}))}{2i\sqrt{1+\xi^2/\omega^2}} \frac{1}{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})} [e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}]_{s=\frac{kT_0}{n}}^{s=\frac{(k+1)T_0}{n}}$$

$$- \frac{\mathcal{F}(f(|E|^2)E([tn/T_0]T_0/n))}{2i\sqrt{1+\xi^2/\omega^2}} \frac{(1+\xi^4)^{1/2}}{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})} [e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}]_{s=[tn/T_0]T_0/n}^{t}$$

It follows that

$$|\int_0^t \frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}} k_n(\xi,s)ds|_{L^1(d\xi)} \le \frac{2n}{\omega^2} |f(|E|^2)E|^2_{L^\infty(0,T_0,H^2)}. \tag{32}$$

The estimates (31) and (32) show that $f_1(\xi,t) \to_{\omega\to\infty} 0$ in $L^\infty(0,T_0,L^2(d\xi))$ and lemma 2 is proved. ■

Inequality (29) shows, with lemma 2, that $E_\omega \to_{\omega\to\infty} E$ in $L^\infty(0,T_1,H^2)$ and Theorem 4 is proved. ■

3.2 Behavior of $\partial_t E_\omega$.

In this section, we are concerned with the behavior of $\partial_t E_\omega$ when $\omega \to \infty$. The result is the following:

Theorem 5 *Let* $E_0^\omega \to_{\omega\to\infty} E_0$ *in* $H^2(\mathbf{R}^n)$, E_1^ω, $E_1'^\omega \in H^1(\mathbf{R}^n)$ *such that* $E_1^\omega \to_{\omega\to\infty} E_1$, $E_1'^\omega \to_{\omega\to\infty} E_1'$ *in* $L^2(\mathbf{R}^n)$ *and* $\frac{1}{\omega}\nabla E_1^\omega \to_{\omega\to\infty} 0$, $\frac{1}{\omega}\nabla E_1'^\omega \to_{\omega\to\infty} 0$ *in* $L^2(\mathbf{R}^n)$.
Let E_ω *be the solution to* (NLW_ω) *with initial data*

$$E_\omega(0) = E_0^\omega, \;\; \partial_t E_\omega(0) = E_1^\omega + e^{i\omega^2\lambda} E_1'^\omega.$$

Let g *be the solution to*

$$\begin{cases} 2i\partial_t g - \Delta g = (f'(|E|^2)|E|^2 + f(|E|^2))\{S(-t)(E_1 + e^{i\omega^2\lambda}E_1' \\ \qquad -\frac{i}{2}\Delta E_0 + \frac{1}{2i} f(|E_0|^2)E_0) + g\}, \\ g(0,x) = 0. \end{cases}$$

Then

$$\partial_t E_\omega - \partial_t E - e^{2i\omega^2 t} S(-t)\{E_1 + e^{i\omega^2\lambda}E_1' - \frac{i}{2}\Delta E_0 + \frac{1}{2i} f(|E_0|^2)E_0\} - e^{2i\omega^2 t} g(x,t) \to_{\omega\to\infty} 0$$

in $L^\infty(0, T_2, L^2)$ *for* T_2 *depending only on* $|E_0|_{H^2}$.

Remark: The choice of the initial data exposed in Theorem 5 is now justified a posteriori by the above oscillatory behavior of $\partial_t E_\omega$: in order to obtain global results (as e.g Theorem 6, see below), it is indeed necessary to consider general oscillating E_1^ω in iterating the former local results from non-zero instants t_0 such as $\partial_t E_\omega(t_0) = E_1^\omega(t_0) + E_1'^\omega(t_0)e^{2i\omega^2 t_0}$. As will be seen further on (see proposition 2), the conditions on ∇E_1^ω and $\nabla E_1'^\omega$ will also be necessary for the same reason in finding accurate results of convergence on $\partial_t E_\omega$ in H^1.

Proof: We differentiate (INT_ω) with respect to t and we obtain the relation

$$\begin{aligned} \partial_t \mathcal{F}(E_\omega) = \partial_t \mathcal{F}(S_0^\omega(t)E_0^\omega) + \partial_t \mathcal{F}(S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega)) \\ +\omega^2 \textstyle\int_0^t \partial_t \mathcal{F}(S_1^\omega(t-s) f(|E_\omega|^2)E_\omega(s))ds, \end{aligned} \tag{33}$$

each term of which is now analyzed.

a) Since

$$\partial_t \mathcal{F}(S_0^\omega(t)E_0^\omega) = -i\xi^2 \frac{e^{i\omega^2(1-\sqrt{1+\xi^2/\omega^2})t} - e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t}}{2\sqrt{1+\xi^2/\omega^2}} \mathcal{F}(E_0^\omega),$$

it is easy to see that

$$\partial_t \mathcal{F}(S_0^\omega(t)E_0^\omega) - \frac{i}{2}\xi^2 \mathcal{F}(E_0) e^{i\xi^2 t/2} e^{2i\omega^2 t} + \frac{i}{2}\xi^2 \mathcal{F}(E_0) e^{-i\xi^2 t/2} \to_{\omega\to\infty} 0 \tag{34}$$

in $L^\infty(0,T,L^2)$.

b)On the other hand

$$\partial_t \mathcal{F}(S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega))$$

$$= \frac{(1+\sqrt{1+\xi^2/\omega^2})e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t} - (1-\sqrt{1+\xi^2/\omega^2})e^{i\omega^2(1-\sqrt{1+\xi^2/\omega^2})t}}{2\sqrt{1+\xi^2/\omega^2}}\mathcal{F}(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega).$$

Therefore

$$\partial_t \mathcal{F}(S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega)) - e^{i\xi^2 t/2}e^{2i\omega^2 t}(\mathcal{F}(E_1 + e^{i\omega^2\lambda}E_1')) \rightarrow_{\omega\rightarrow\infty} 0 \qquad (35)$$

in $L^\infty(0,T,L^2)$.

c) The nonlinear part of (33) is less straightforward: we begin to develop the expression

$$\omega^2 \int_0^t \partial_t \mathcal{F}(S_1^\omega(t-s) f(|E_\omega|^2)E_\omega(s))ds =$$

$$-\int_0^t \partial_s\{\frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}}\}\mathcal{F}(f(|E_\omega|^2)E_\omega(s))ds$$

$$-\int_0^t \frac{\omega^2(1-\sqrt{1+\xi^2/\omega^2})e^{i\omega^2(1-\sqrt{1+\xi^2/\omega^2})(t-s)}}{2\sqrt{1+\xi^2/\omega^2}}\mathcal{F}(f(|E_\omega|^2)E_\omega(s))ds$$

$$= I_1(\xi,t) + I_2(\xi,t),$$

whose second integral vanishes as $\omega \rightarrow \infty$. Indeed, it is clear that

$$I_2(\xi,t) \rightarrow_{\omega\rightarrow\infty} \int_0^t \frac{\xi^2}{4} e^{-i\xi^2(t-s)/2}\mathcal{F}(f(|E|^2)E(s))ds \qquad (36)$$

in $L^\infty(0,T_1,L^2)$ since $E_\omega \rightarrow_{\omega\rightarrow\infty} E$ in $L^\infty(0,T_1,H^2)$ by theorem 4. Besides, an integration by parts yields

$$I_1(\xi,t) = \int_0^t (\frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}}\partial_s\mathcal{F}(f(|E_\omega|^2)E_\omega(s))ds$$

$$-[\frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}}\mathcal{F}(f(|E_\omega|^2)E_\omega(s))]_0^t.$$

By expanding the previous relation, we thus obtain

$$I_1(\xi,t)=\int_0^t(\frac{1}{2i\sqrt{1+\xi^2/\omega^2}}e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}$$

$$\mathcal{F}((f'(|E_\omega|^2)|E_\omega|^2+f(|E_\omega|^2))\partial_s E_\omega+f'(|E_\omega|^2)E_\omega^2\partial_s\bar{E}_\omega)ds$$

$$-\frac{1}{2i\sqrt{1+\xi^2/\omega^2}}\mathcal{F}(f(|E_\omega|^2)E_\omega(t))+\frac{1}{2i\sqrt{1+\xi^2/\omega^2}}e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t}\mathcal{F}(f(|E_0^\omega|^2)E_0^\omega). \tag{37}$$

Let us now define F_ω by

$$\partial_t E_\omega=\partial_t F_\omega+\partial_t E+e^{2i\omega^2 t}S(-t)(E_1+e^{i\omega^2\lambda}E_1'-\frac{i}{2}\Delta E_0+\frac{1}{2i}f(|E_0|^2)E_0)+e^{2i\omega^2 t}g(x,t). \tag{38}$$

We rewrite (33) by using the previous expression and by taking into account the fact that E satisfies (INT):

$$\mathcal{F}(\partial_t F_\omega)=$$

$$\Big\{\mathcal{F}(\partial_t S_0^\omega(t)E_0^\omega-\partial_t S(t)E_0+\frac{i}{2}e^{2i\omega^2 t}S(-t)\Delta E_0)\Big\}$$

$$+\Big\{\mathcal{F}(\partial_t S_1^\omega(t)(E_1^\omega+e^{i\omega^2\lambda}E_1'^\omega)-e^{2i\omega^2 t}S(-t)(E_1+e^{i\omega^2\lambda}E_1'))\Big\}$$

$$+\Big\{I_2(\xi,t)-\int_0^t\frac{\xi^2}{4}e^{-i\xi^2(t-s)/2}\mathcal{F}(f(|E|^2)E(s))ds\Big\}$$

$$+\Big\{\frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t}}{2i\sqrt{1+\xi^2/\omega^2}}\mathcal{F}(f(|E_0^\omega|^2)E_0^\omega)-\frac{1}{2i}e^{2i\omega^2 t}e^{i\xi^2 t/2}\mathcal{F}(f(|E_0^\omega|^2)E_0^\omega)\Big\}$$

$$+\Big\{\frac{i\mathcal{F}(f(|E_\omega|^2)E_\omega(t))}{2\sqrt{1+\xi^2/\omega^2}}-\frac{i}{2}\mathcal{F}(f(|E|^2)E(t))\Big\}$$

$$+\Big\{\int_0^t(\frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}}\mathcal{F}((f'(|E_\omega|^2)|E_\omega|^2+f(|E_\omega|^2))\partial_s E_\omega$$

$$+f'(|E_\omega|^2)E_\omega^2\partial_s\bar{E}_\omega)ds-e^{2i\omega^2 t}\mathcal{F}(g(.,t))\Big\},$$

$$=J_1+J_2+J_3+J_4+J_5+J_6.$$

We see that $J_1,\ J_2,\ J_3\ \rightarrow_{\omega\to\infty} 0$ in $L^\infty(0,T_1,L^2)$ through (34), (35), (36). On the other hand $J_4\rightarrow_{\omega\to\infty} 0$ in $L^\infty(0,T_1,L^2)$ since $E_0^\omega\rightarrow_{\omega\to\infty} E_0$ in $H^2(\mathbf{R}^n)$,

and $J_5 \to_{\omega\to\infty} 0$ in $L^\infty(0, T_1, L^2)$ since $E_\omega \to_{\omega\to\infty} E$ in $L^\infty(0, T_1, H^2)$.
Let us now write J_6 using (38); one has

$$J_6 = \int_0^t \frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}} \mathcal{F}\Big((f'(|E_\omega|^2)|E_\omega|^2 + f(|E_\omega|^2))\times$$

$$\{\partial_s F_\omega + \partial_s E + e^{2i\omega^2 s} S(-s)(E_1 + e^{i\omega^2\lambda} E_1' - \frac{i}{2}\Delta E_0 + \frac{1}{2i} f(|E_0|^2)E_0) + e^{2i\omega^2 s} g(.,s)\}\Big) ds$$

$$+ \int_0^t \frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}} \mathcal{F}\Big((f'(|E_\omega|^2)E_\omega^2\times$$

$$\{\partial_s \bar{F}_\omega + \partial_s \bar{E} + e^{-2i\omega^2 s} S(s)(\bar{E}_1 + e^{-i\omega^2\lambda} \bar{E}_1' + \frac{i}{2}\Delta \bar{E}_0 - \frac{1}{2i} f(|E_0|^2)\bar{E}_0) + e^{-2i\omega^2 s} \bar{g}(.,s)\}\Big) ds$$
$$-e^{2i\omega^2 t}\mathcal{F}(g(.,t)).$$

We get

$$J_6 = \Big\{ \int_0^t \frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})(t-s)}}{2i\sqrt{1+\xi^2/\omega^2}} \mathcal{F}[(f'(|E_\omega|^2)|E_\omega|^2 + f(|E_\omega|^2))\partial_s F_\omega + f'(|E_\omega|^2)E_\omega^2 \partial_s \bar{F}_\omega] ds\Big\}$$

$$+\Big\{\frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t}}{2i\sqrt{1+\xi^2/\omega^2}} \int_0^t e^{i\omega^2(1-\sqrt{1+\xi^2/\omega^2})s}\times$$

$$\mathcal{F}[(f'(|E_\omega|^2)|E_\omega|^2 + f(|E_\omega|^2))(S(-s)(E_1 + e^{i\omega^2\lambda}E_1' - \frac{i}{2}\Delta E_0 + \frac{1}{2i} f(|E_0|^2)E_0) + g(.,s))] ds\Big\}$$

$$+\Big\{\frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t}}{2i\sqrt{1+\xi^2/\omega^2}} \int_0^t e^{-i\omega^2(3+\sqrt{1+\xi^2/\omega^2})s}\times$$

$$\mathcal{F}[f'(|E_\omega|^2)E_\omega^2(S(s)(\bar{E}_1 + e^{-i\omega^2\lambda}\bar{E}_1' + \frac{i}{2}\Delta\bar{E}_0 - \frac{1}{2i} f(|E_0|^2)\bar{E}_0) + \bar{g}(.,s))] ds\Big\}$$

$$+\Big\{\frac{e^{i\omega^2(1+\sqrt{1+\xi^2/\omega^2})t}}{2i\sqrt{1+\xi^2/\omega^2}} \int_0^t e^{-i\omega^2(1+\sqrt{1+\xi^2/\omega^2})s}\times$$

$$\mathcal{F}[(f'(|E_\omega|^2)|E_\omega|^2 + f(|E_\omega|^2))\partial_s E + f'(|E_\omega|^2)E_\omega^2 \partial_s \bar{E}] ds\Big\}$$
$$-e^{2i\omega^2 t}\mathcal{F}(g(.,t))$$
$$= T_1 + T_2 + T_3 + T_4 - e^{2i\omega^2 t}\mathcal{F}(g(.,t))$$

Using the same technics as for the proof of iii) lemma 2, one has $T_3,\ T_4\ \to_{\omega\to\infty} 0$ in $L^\infty(0,T_1,L^2)$.
Moreover, we remark that

$$T_2+\frac{i}{2}e^{2i\omega^2 t}e^{i\xi^2 t/2}\int_0^t e^{-i\xi^2 s/2}\times$$

$$\mathcal{F}[(f'(|E|^2)|E|^2+f(|E|^2))(S(-s)(E_1+e^{i\omega^2\lambda}E_1'-\frac{i}{2}\Delta E_0+\frac{1}{2i}f(|E_0|^2)E_0)+g(.,s))]ds\to 0$$

when $\omega\to\infty$ in $L^\infty(0,T_1,L^2)$.
On the other hand, one can estimate

$$|T_1|_{L^\infty(0,T,L^2)}\leq KT|E_\omega|^{2\sigma}_{L^\infty(0,T_1,H^2)}|\partial_t F_\omega|_{L^\infty(0,T,L^2)},$$
$$\leq TK'(|E_0|_{H^2})|\partial_t F_\omega|_{L^\infty(0,T,L^2)},$$

and since

$$\mathcal{F}(g(.,t))=-\frac{i}{2}\int_0^t e^{i\xi^2(t-s)/2}\mathcal{F}[(f'(|E|^2)|E|^2+f(|E|^2))\times$$

$$(S(-s)(E_1+e^{i\omega^2 t}E_1'-\frac{i}{2}\Delta E_0+\frac{1}{2i}f(|E_0|^2)E_0)+g(.,s))]ds,$$

we obtain

$$|J_6|_{L^\infty(0,T,L^2)}-TK'(|E_0|_{H^2})|\partial_t F_\omega|_{L^\infty(0,T,L^2)}\to_{\omega\to\infty} 0.$$

It thus follows that

$$|\partial_t F_\omega|_{L^\infty(0,T,L^2)}(1-TK'(|E_0|_{H^2}))\to_{\omega\to\infty} 0.$$

If we take T_2 so that $0<1-TK'(|E_0|_{H^2})$, we obtain the Theorem. ■

Remark: The meaning of the corrector for $\partial_t E_\omega$, as defined at the end of Theorem 5, can be explained by the following: as solutions to (NLW_ω) contain by themselves both wave contributions propagating along the temporal lines $t>0$ and $t<0$, only one of the latter corresponding to $S(t)$ converges towards the (NLS) component related to (9) in taking the limit $\omega\to\infty$ in (33). This justifies the existence of a remaining part corresponding to the group $S(-t)$ which cannot vanish in the previous expansion of $\partial_t E_\omega$, except whenever the compatibility relation of (NLS) initial data, namely $E_1-(i/2)\Delta E_0-(i/2)f(|E_0|^2)E_0=0$, is satisfied. In this case, one finds $g(x,t)=0$ by means of the Gronwall's lemma (note that the function $g(x,t)$ in this corrector results from the fact that $\partial_t E_\omega$ obeys a linear wave equation with the potential $f(|E_\omega|^2).+2f'(|E_\omega|^2)E_\omega Re(\bar{E}_\omega.))$. When assuming this compatibility condition, E_ω as well as $\partial_t E_\omega$ thus converge towards their respective (NLS) limits as $\omega\to\infty$. We can here recall that similar results of convergence were already established under the latter compatibility hypothesis, first by Tsutsumi [22], then detailed more recently by means of $L^p - L^q$ estimates by Najman [23] in the peculiar case of a power nonlinearity $f(|E_\omega|^2) = \lambda |E_\omega|^{2\sigma}$ with $\lambda > 0$, i.e. when the solution exists globally.

In the opposite situation, i.e. when the compatibility relation remains unsatisfied, Theorem 5 shows that even if E_ω converges to E as $\omega \to \infty$, the time derivative $\partial_t E_\omega$ never reaches the single-valued limit $\partial_t E$, but it oscillates inside a uniform band whose thickness gets all the wider as the function $g(x,t)$ together with the (NLS) compatibility relation differ from zero.

Besides, since E_0 lies in H^2 only, contributions such as $\frac{i}{2}\Delta E_0$ in the expansion of $\partial_t E_\omega$ belongs to L^2. Therefore it is not reasonable to obtain the latter expansion in $L^\infty(0,T,H^1)$; nevertheless we can determine the limit of $\frac{1}{\omega}\partial_t E_\omega$ as $\omega \to \infty$ in $L^\infty(0,T,H^1)$, as shown by the

Proposition 2 *Under the same asumptions as for Theorem 5, there exists T_3 depending only on $|E_0|_{H^2}$ such that*

$$\frac{1}{\omega}\partial_t E_\omega \to_{\omega\to\infty} 0$$

in $L^\infty(0,T_3,H^1)$.

Proof: We differentiate (INT_ω) with respect to t:

$$\frac{1}{\omega}\partial_t E_\omega = \frac{1}{\omega}\partial_t S_0^\omega(t)E_0^\omega + \frac{1}{\omega}\partial_t S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega)$$
$$+\omega^2 \int_0^t \frac{1}{\omega}\partial_t S_1^\omega(t-s) f(|E_\omega|^2)E_\omega(s)ds.$$

Since $\forall \xi$, $|\frac{\xi}{\omega}\partial_t \mathcal{F}(S_0^\omega(t)E_0^\omega)| \leq |\xi|^2 |\mathcal{F}(E_0^\omega)|$, we see that

$$\frac{1}{\omega}\partial_t S_0^\omega(t)E_0^\omega \to_{\omega\to\infty} 0$$

in $L^\infty(0,T,H^1)$ by Lebesgue's convergence theorem.
We also obtain

$$\frac{1}{\omega}\partial_t S_1^\omega(t)(E_1^\omega + e^{i\omega^2\lambda}E_1'^\omega) \to_{\omega\to\infty} 0$$

in $L^\infty(0,T,H^1)$ since $\nabla E_1^\omega/\omega$, $\nabla E_1'^\omega/\omega \to_{\omega\to\infty} 0$ in L^2.
For the nonlinear part, we integrate by parts:

$$\omega^2 \int_0^t \frac{1}{\omega}\partial_t S_1^\omega(t-s) f(|E_\omega|^2)E_\omega(s)ds$$
$$= \int_0^t \{\omega S_1^\omega(t-s)(f(|E_\omega|^2)\partial_s E_\omega + 2f'(|E_\omega|^2)E_\omega Re(\bar{E}_\omega \partial_s E_\omega))$$
$$+\omega S_1^\omega(t) f(|E_0^\omega|^2)E_0^\omega\}ds,$$

so that

$$|\omega^2 \int_0^t \frac{1}{\omega}\partial_t S_1^\omega(t-s) f(|E_\omega|^2)E_\omega(s)ds|_{L^\infty(0,T,H^1)}$$
$$\leq TK'(|E_\omega|_{L^\infty(0,T,H^2)})|\partial_t E_\omega|_{L^\infty(0,T,H^1)}/\omega + |\omega S_1^\omega(t) f(|E_0^\omega|^2)E_0^\omega|_{L^\infty(0,T,H^1)}.$$

Taking T_3 such as $T_3 K'(|E_\omega|_{L^\infty(0,T,H^2)}) < 1$ achieves to prove the result. ■

3.3 Existence time.

In the so-called supercritical and critical cases $\sigma n \geq 2$, some solutions to (NLS) blow up in a finite time (see [14]). On the other hand, we have constructed solutions to (NLW_ω) that also blow up. The aim of this section is to investigate the relationships between the maximal existence times of solutions to both equations. The following Theorem gives a partial result.

Theorem 6 *Under the same asumptions as for Theorem 5, let $T_\omega = T_\omega(E_0^\omega, E_1^\omega, E_1'^\omega)$ be the existence time of E_ω and $T(E_0)$ the existence time of the solution E to (NLS) with the initial datum $E(0) = E_0$. Then*

$$\liminf_{\omega\to\infty} T_\omega \geq T(E_0).$$

Moreover

$$\forall T < T(E_0),\ \ E_\omega \to_{\omega\to\infty} E \ \textit{in}\ L^\infty(0,T,H^2),$$

$$\partial_t E_\omega - \partial_t E - e^{2i\omega^2 t} S(-t)(E_1 + e^{i\omega^2\lambda}E_1' - \frac{i}{2}\Delta E_0 + \frac{1}{2i} f(|E_0|^2)E_0) - e^{2i\omega^2 t} g(x,t) \to_{\omega\to\infty} 0$$

in $L^\infty(0,T,L^2)$, where g is defined as in Theorem 5, and

$$\frac{1}{\omega}\partial_t E_\omega \to_{\omega\to\infty} 0$$

in $L^\infty(0,T,H^1)$.

Proof: We argue by contradiction and we suppose that $\tilde{T} \equiv \liminf_{\omega\to\infty} T_\omega < T(E_0)$. We consider $\eta > 0$ and we claim the

Proposition 3 *If ω is sufficiently large, then E_ω exists on $[0, \tilde{T} - \eta]$ and*

$$E_\omega \to_{\omega\to\infty} E \ \textit{in}\ L^\infty(0, \tilde{T} - \eta, H^2).$$

Furthermore,

$$\partial_t E_\omega - \partial_t E - e^{2i\omega^2 t} S(-t)(E_1 + e^{i\omega^2\lambda}E_1' - \frac{i}{2}\Delta E_0 + \frac{1}{2i} f(|E_0|^2)E_0) - e^{2i\omega^2 t} g(x,t) \to_{\omega\to\infty} 0$$

in $L^\infty(0, \tilde{T} - \eta, L^2)$, and

$$\frac{1}{\omega}\partial_t E_\omega \to_{\omega\to\infty} 0$$

in $L^\infty(0, \tilde{T} - \eta, H^1)$.

Proof: Let $\tilde{\tilde{T}}$ be the largest time before which the conclusions of the proposition are true. If $\tilde{\tilde{T}} < \tilde{T} - \eta$, we apply Theorem 4, Theorem 5 and proposition 2 to (NLW_ω) with the initial data

$$\tilde{E}_\omega(0) = E_\omega(\tilde{\tilde{T}} - \epsilon),\ \ \partial_t \tilde{E}_\omega(0) = \partial_t E_\omega(\tilde{\tilde{T}} - \epsilon),$$

where $\epsilon > 0$ will be chosen later on. We have $\tilde{E}_\omega(0) \to_{\omega\to\infty} E(\tilde{\tilde{T}} - \epsilon)$ in H^2 and

$$\partial_t \tilde{E}_\omega(0) = \tilde{E}^1_\omega + \tilde{E}'^1_\omega e^{2i\omega^2(\tilde{\tilde{T}}-\epsilon)} + \tilde{E}''^1_\omega e^{2i\omega^2(\tilde{\tilde{T}}-\epsilon)} e^{i\omega^2\lambda},$$

$$\tilde{E}^1_\omega \to_{\omega\to\infty} \partial_t E(\tilde{\tilde{T}} - \epsilon)$$

$$\tilde{E}'^1_\omega \to_{\omega\to\infty} S(-\tilde{\tilde{T}} + \epsilon)(E_1 - \frac{i}{2}\Delta E_0 + \frac{1}{2i} f(|E_0|^2)E_0) + g(x,t),$$

$$\tilde{E}''^1_\omega \to_{\omega\to\infty} S(-\tilde{\tilde{T}} + \epsilon)E'_1$$

in L^2 together with $\frac{1}{\omega}\partial_t \nabla \tilde{E}_\omega(0) \to_{\omega\to\infty} 0$ in L^2. Thus, there exists a time τ depending only on $|E|_{L^\infty(0,\tilde{T},H^2)}$ such that the conclusions of Theorem 4, Theorem 5 and proposition 2 remain valid on $[0, \tilde{\tilde{T}} - \epsilon + \tau]$. Since τ does not depend on ϵ, we can take ϵ such that $\tau - \epsilon > 0$, this leads to a contradiction. ■

To prove the theorem, we adopt the same technics:
starting from $\tilde{T} - \eta$, we can construct a solution to (NLW_ω) for ω sufficiently large on an interval $[\tilde{T} - \eta, \tilde{T} - \eta + \tau]$ with τ depending only on $|E|_{L^\infty(0,\tilde{T},H^2)}$. If we take $\tau - \eta > 0$, we obtain a contradiction and the theorem is proved. ■

Corollary 1 *Suppose that for sufficiently large ω, E_ω exists on $[0, T(E_0)]$ with $T(E_0) < \infty$. Then*

$$\lim_{\omega\to\infty} |E_\omega|_{L^\infty(0,T(E_0),H^2)} = +\infty$$

Proof: Let us suppose that

$$\lim_{\omega\to\infty} |E_\omega|_{L^\infty(0,T(E_0),H^2)} < +\infty,$$

then $E_\omega \to_{\omega\to\infty} \tilde{E}$ in $L^\infty(0, T(E_0), H^2)$ weakly. Now $\tilde{E}$ satisfies (NLS) on $[0, T(E_0)[$ and $\tilde{E} \in L^\infty(0, T(E_0), H^2)$, which is a contradiction. ■

4 Complements and final remarks.

4.1 Case of unbounded initial data.

In the case when $\partial_t E_\omega(0)$ is not bounded, we have an equivalent of Theorem 6, namely

Theorem 7 *Let $E_0^\omega \to_{\omega\to\infty} E_0$ in H^2 and $E_1^\omega \to_{\omega\to\infty} E_1$ in H^1 and E_ω the solution to (NLW_ω) with initial data $E_\omega(0) = E_0^\omega$ and $\partial_t E_\omega(0) = \omega e^{i\omega^2\lambda} E_1^\omega$. Let $\tilde{T}_\omega = T_\omega(E_0^\omega, E_1^\omega)$ the existence time of E_ω and $T(E_0)$ the existence time of the solution E to (NLS) with the initial datum $E(0) = E_0$. Then*

$$\liminf_{\omega\to\infty} \tilde{T}_\omega \geq T(E_0).$$

Moreover

$$\forall T < T(E_0) \;\; E_\omega \to_{\omega\to\infty} E \text{ in } L^\infty(0,T,H^2),$$

and

$$\frac{1}{\omega}\partial_t E_\omega - e^{i\omega^2\lambda} e^{2i\omega^2 t}[S(-t)E_1 + \phi(x,t)] \to_{\omega\to\infty} 0$$

in $L^\infty(0,T,H^1)$, where ϕ is the solution to

$$\begin{cases} 2i\partial_t\phi - \Delta\phi = [f'(|E|^2)|E|^2 + f(|E|^2)][S(-t)E_1 + \phi(x,t)], \\ \phi(x,0) = 0. \end{cases}$$

The scheme of the proof is the same as for Theorem 6, i.e. we have to prove the equivalents of Theorems 4, 5 and propositions 2, 3. We here omit the details.

4.2 Remarks on the finite time blow-up.

In the context of Theorem 2, it can be checked that, when analyzing the properties of the blowing-up solutions, not only $|E_\omega|_{L^2}$ tends to diverge, but also the time derivative of this norm is positive and also diverges as $t \to T_\omega$ (for instance, one finds $I'_\omega(t) \geq 2I_\omega(t)\omega^2/(\sqrt{\alpha} th(2\sqrt{\alpha}\omega^2(T^*(0) - t)))$ starting with $I'_\omega(0) > 0$ in the peculiar situations (C1)-(C2) of Theorem 2). This behavior explicitly shows that the mass associated with $|E_\omega|_{L^2}$ increases drastically until blow-up, which illustrates the dynamics of the collapsing solutions to (NLW_ω). When comparing this kind of blow-up with a (NLS) collapse preserving the L^2 norm, the latter phenomenon may be considered from a physical viewpoint as a strong increase of the mass induced by the small-scale wave fluctuations which acts locally in space at the locus where the waveform flattens due to the nonlinearity dominating over the dispersion. Consequently, this mass growth superimposes upon the nonlinear effects and contributes to strengthen the blow-up dynamics, which may explain why a blow up associated with (NLW_ω) can possibly occur in the subcritical case $n\sigma < 2$, by contrast with a (NLS) solution that never collapses.

Returning now to the physical problem of strong Langmuir turbulence, as introduced in section 1, one can justify the meaning of the previous mass growth by invoking the following: first, let us here recall that the Zakharov equations result from deriving fluid equations describing the high-frequency particle motions and the low-frequency ones separately (see e.g. [4]), in such a way that making the time-envelope hypothesis is nothing else but describing the low-frequency motions and thus dropping the rapid electron oscillations. When retaining the second-order in time derivative of E in (6) - i.e. for not too large ω - one necessarily takes into account the latter rapid fluctuations whose corresponding velocity field v_{HF} is linked to the amplitude of the electric field E_ω through the dynamics equation $m_e dv_{HF}/dt \approx q_e E_\omega$. As E_ω tends to diverge when blow-up happens, the high-frequency particles oscillate more and more rapidly, and participate to a Langmuir collapse by carrying an important amount of kinetic energy which *must be* included in the collapse process. This phenomenon is thus classically ignored when the time-envelope approximation is imposed in the description of the strong Langmuir turbulence.

Besides, Theorem 2 dealing with blowing-up solutions only applies on solutions E_ω for which ω belongs to a restrained range of values depending on the initial data. In particular, this theorem predicts the non-existence of solutions to (NLW_ω) for every $t \geq 0$, as long as ω does not increase above a threshold value ω_{lim}. Such a restriction on the ω-range could be explained by the fact that the way to blow-up differs between solutions to (NLW_ω) and the ones of (NLS) for what concerns their respective L^2 norm, as discussed above. We can thus expect that the ω-dependent quantity $|E_\omega|_{L^2}$ should behave as a plateau function for $\omega >> \omega_{lim}$. In order to solve this open problem, it would be interesting to investigate initial data leading to collapsing solutions and valid for all ω.

In conclusion, we finally emphasize that as its amplitude singularily grows up in time, a Langmuir wave is expected to drive the low-frequency fluctuations $\delta n_{LF}/N_0$ into the so-called supersonic regime for which the time-derivatives of (4) -reflecting the ion dynamics- cannot be ignored any longer. For a full understanding of the limitations imposed by the time-envelope approximation in strong Langmuir turbulence, it would therefore be of outmost importance to repeat the above analysis in the framework of the complete set of the Zakharov equations.

References

[1] **L.I.Rudakov and V.N.Tsytovitch,** *Strong Langmuir turbulence*, **Phys. Rep. 40C (1978) pp1-73.**

[2] **M.V.Goldman,** *Strong turbulence of plasma waves*, **Rev. Mod. Phys. 56 (1984) pp709-735.**

[3] **J.Briand, L.Bergé, A.Gomes, Y.Quemener, C.Arnas, M.Armengaud, J.P.Dinguirard and D.Pesme,** *Strong Langmuir turbulence and second harmonic spectra in a 1 μm laser-produced plasma*, **Phys. Fluids B 2 (1990) pp160-165.**

[4] **D.R.Nicholson,** *Introduction to Plasma Theory* **, J.Wiley and Sons Eds., New-York (1983) pp177-181.**

[5] **V.E.Zakharov,** *Collapse of Langmuir waves*, **Zh. Eksp. Teor. Fiz. 62 (1972) pp1745-1759 [Sov. Phys. JETP 35 (1972) pp908-914].**

[6] **O.B.Budneva, V.E.Zakharov and V.S.Synakh,** *Certain models of wave collapse*, **Fiz. Plasmy 1 (1975) pp606-613 [Sov. J. Plasma Phys. 1 (1975) pp335-338.].**

[7] **V.E.Zakharov and L.N.Shur,** *Self-similar regimes of wave collapse*, **Zh. Eksp. Teor. Fiz. 81 (1981) pp2019-2031 [Sov. Phys. JETP 54 (1981) pp1064-1070].**

[8] **V.M.Malkin and V.N.Khudik,** *Self-similar Langmuir collapse*, **Physica D 52 (1991) pp55-58.**

[9] **L.Bergé, Ph.Dousseau, G.Pelletier and D.Pesme,** *Scalar wave collapse at critical dimension*, **Phys. Rev. A 42 (1990) pp4952-4961;** *ibidem*, *Self-similar Langmuir collapse at critical dimension*, **Laser and Part. Beams 9 (1991) pp363-370.**

[10] **P.Kaw, G.Schmidt and T.Wilcox,** *Filamentation and trapping of electromagnetic radiations in plasmas*, **Phys. Fluids 16 (1973) pp1522-1525.**

[11] **D.Anderson and M.Bonnedal,** *Variational approach to nonlinear self-focusing of Gaussian laser beams*, **Phys. Fluids 22 (1979) pp105-109.**

[12] **J.J.Rasmussen and K.Rypdal,** *Blow-up in nonlinear Schrödinger Equations -I: A general review*, **Physica Scripta 33 (1986) pp481-497.**

[13] **J.Ginibre and G.Velo,** *On a class of nonlinear Schrödinger equations Part I and II,* **J. Funct. Anal. 32 (1979) pp1-32 and pp33-71;** *part III* **Ann. Inst. H Poincaré, A 28 (1978) pp287-316;** *The global Cauchy problem for the nonlinear Schrödinger equation revisited,* **Ann. Inst. Henri Poincaré, Anal. Non Linéaire 2 (1985) pp309-402.**

[14] **R.T.Glassey,** *On the blowing up solutions to the Cauchy problem for nonlinear Schrödinger equations,* **J. Math. Phys. 18 (1977) pp1794-1797.**

[15] **T.Kato,** *Nonlinear Schrödinger equations,* **Lecture note in Physics 345 (1988).**

[16] **T.Colin** *On the Cauchy problem for a nonlocal, nonlinear Schrödinger equation occuring in plasma physics.* **Differential and Integral equation 6 (1993) pp1431-1450.**

[17] **T.Colin,** *On the standing waves solution to a nonlocal, nonlinear Schrödinger equation occuring in plasma physics,* **Physica D 64 (1993) pp215-236.**

[18] **L.Bergé et T. Colin,** *Un problème de perturbation singulière pour une équation d'enveloppe en Physique des plasmas* **to appear in Comptes Rendus de l'Académie des Sciences de Paris, Série I (1994).**

[19] **I.Segal,** *Nonlinear semi-groups.* **Annals of Maths 78 (1963).**

[20] **H.A.Levine,** *Instability and nonexistence of global solutions to nonlinear wave equations of the form* $Pu_{tt} = -Au + f(u)$, **Trans. Amer. Math. Soc. 192 (1974) pp1-21.**

[21] **A.Friedman,** *Partial differential equation of parabolic type,* **Prentice Hall (1964).**

[22] **M.Tsutsumi,** *Nonrelativistic approximation of nonlinear Klein-Gordon equations in two space dimensions,* **Nonlinear Analysis, T.M.A 8 (1984) pp637-643.**

[23] **B.Najman,** *The nonrelativistic limit of the nonlinear Klein-Gordon equation,* **Nonlinear Analysis, T.M.A 15 (1990) pp217-228.**